Course 2BA1: Academic Year 2000–1.
Assignment III.

To be handed in by Friday 2nd February, 2001.
Please include both name and student number on any work handed in.

1. Let \( E \) denote the set

\[
\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}
\]

of even integers, let + and \( \times \) denote the usual arithmetic operations of addition and multiplication respectively, and let \( # \) denote the binary operation on \( E \) defined by \( x \# y = \frac{1}{2}xy \) for all even integers \( x \) and \( y \).

(a) Is \((E, +)\) a monoid?

(b) Is \((E, \times)\) a monoid?

(c) Is \((E, \#)\) a monoid?

[Briefly justify your answers.]

2. Let \( q \) and \( r \) be the quaternions given by \( q = 1 - i \) and \( r = 2i - j - k \).

Calculate the quaternion products \( q \times r \) and \( r \times q \) in the form \( w + xi + yj + zk \) for appropriate real numbers \( w, x, y \) and \( z \).

3. Let \((A, \ast)\) be a monoid, let \( s \) be an invertible element of \( A \), and let \( f: A \to A \) be the function from \( A \) to itself defined by \( f(x) = (s \ast x) \ast s^{-1} \) for all elements \( x \) of \( A \) (where \( s^{-1} \) denotes the inverse of \( s \)).

(a) Prove that the function \( f \) is a homomorphism.

(b) Let \( g: A \to A \) be the function defined by \( g(x) = (s^{-1} \ast x) \ast s \) for all elements \( x \) of \( A \). Prove that the function \( g \) is the inverse of the function \( f \).

(c) Is the function \( f \) an isomorphism?