Course 221: Hilary Term 2007
Section 8: The Lebesgue Integral

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Contents

8 The Lebesgue Integral

8.1 Measurable Functions

Definition Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let $f\colon X \to [-\infty, +\infty]$ be a function on X with values in the set $[-\infty, +\infty]$ of extended real numbers. The function f is said to be *measurable* with respect to the σ-algebra $\mathcal{A}$ if $\{x \in X : f(x) < c\} \in \mathcal{A}$ for all real numbers c.

Definition Let $(X, \mathcal{A}, \mu)$ be a measure space. A function $f\colon X \to [-\infty, \infty]$ defined on X is said to be *measurable* if it is measurable with respect to the σ-algebra $\mathcal{A}$ of measurable subsets of X.

It follows from these definitions that a function $f\colon X \to [-\infty, \infty]$ defined on a measure space $(X, \mathcal{A}, \mu)$ is measurable if and only if $\{x \in X : f(x) < c\}$ is a measurable set for all real numbers c.

Proposition 8.1 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, let $f\colon X \to [-\infty, \infty]$ be a function on X, with values in the set $[-\infty, \infty]$ of extended real numbers, which is measurable with respect to the σ-algebra $\mathcal{A}$, and let a, b and c be real numbers, where $a \leq b$. Then the following sets also belong to the σ-algebra $\mathcal{A}$:*

(i) $\{x \in X : f(x) \geq c\}$;

(ii) $\{x \in X : f(x) \leq c\}$;

(iii) $\{x \in X : f(x) > c\}$;

(iv) $\{x \in X : a \leq f(x) \leq b\}$;

(v) $\{x \in X : a < f(x) < b\}$;

(vi) $\{x \in X : a \leq f(x) < b\}$;

(vii) $\{x \in X : a < f(x) \leq b\}$;

(viii) $\{x \in X : f(x) = c\}$;

(ix) $\{x \in X : f(x) = -\infty\}$;

(x) $\{x \in X : f(x) = +\infty\}$;

(xi) $\{x \in X : f(x) < +\infty\}$;

(xii) $\{x \in X : f(x) > -\infty\}$;

(xiii) $\{x \in X : f(x) \in \mathbb{R}\}$.

Proof The set $\{x \in X : f(x) \geq c\}$ is the complement of a set $\{x \in X : f(x) < c\}$ belonging to the σ-algebra $\mathcal{A}$, and must therefore itself belong to this σ-algebra. This proves (i).

The set $\{x \in X : f(x) \leq c\}$ may be represented as a countable intersection

$$\bigcap_{n=1}^{+\infty} \left\{ x \in X : f(x) < c + \frac{1}{n} \right\}$$

of sets that are of the form $\{x \in X : f(x) < c + n^{-1}\}$ for some natural number n. These sets belong to the σ-algebra $\mathcal{A}$, and any countable intersection of sets belonging to $\mathcal{A}$ must itself belong to this σ-algebra. Therefore $\{x \in X : f(x) \leq c\}$ belongs to the σ-algebra. This proves (ii).

The set $\{x \in X : f(x) > c\}$ is the complement of a set $\{x \in X : f(x) \leq c\}$ which belongs to the σ-algebra $\mathcal{A}$, and must therefore itself belong to $\mathcal{A}$. This proves (iii).

The set $\{x \in X : a \leq f(x) \leq b\}$ is the intersection of sets $\{x \in X : f(x) \geq a\}$ and $\{x \in X : f(x) \leq b\}$ that belong to the σ-algebra $\mathcal{A}$. It follows that $\{x \in X : a \leq f(x) \leq b\}$ must itself belong to $\mathcal{A}$. Similarly $\{x \in X : a < f(x) < b\}$ is the intersection of sets $\{x \in X : f(x) > a\}$ and $\{x \in X : f(x) < b\}$, $\{x \in X : a \leq f(x) < b\}$ is the intersection of sets $\{x \in X : f(x) \geq a\}$ and $\{x \in X : f(x) < b\}$, and $\{x \in X : a < f(x) \leq b\}$ is the intersection of sets $\{x \in X : f(x) > a\}$ and $\{x \in X : f(x) \leq b\}$, and therefore $\{x \in X : a < f(x) < b\}$, $\{x \in X : a \leq f(x) < b\}$ and $\{x \in X : a < f(x) \leq b\}$ belong to $\mathcal{A}$. This proves (iv), (v), (vi) and (vii). Moreover (viii) is a special case of (iv).

The set $\{x \in X : f(x) = -\infty\}$ may be represented as a countable intersection

$$\bigcap_{n=1}^{+\infty} \{x \in X : f(x) < -n\}$$

of sets belonging to $\mathcal{A}$, and must therefore itself belong to $\mathcal{A}$. This proves (ix).

Similarly the set $\{x \in X : f(x) = +\infty\}$ may be represented as a countable intersection

$$\bigcap_{n=1}^{+\infty} \{x \in X : f(x) \geq n\}$$

of sets belonging to $\mathcal{A}$, and must therefore itself belong to $\mathcal{A}$. This proves (x).

The set $\{x \in X : f(x) < +\infty\}$ is the complement of the set specified in (x), and must therefore belong to $\mathcal{A}$. Similarly the set $\{x \in X : f(x) > -\infty\}$ is the complement of the set specified in (ix), and must therefore belong to $\mathcal{A}$. This proves (xi) and (xii).

Finally we note that $\{x \in X : f(x) \in \mathbb{R}\}$ is the intersection of the sets $\{x \in X : f(x) < +\infty\}$ and $\{x \in X : f(x) > -\infty\}$ specified in (xi) and (xii), and must therefore belong to $\mathcal{A}$, as required. ∎

Corollary 8.2 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, let $f\colon X \to [-\infty, \infty]$ be a function on X, with values in the set $[-\infty, \infty]$ of extended real numbers, which is measurable with respect to the σ-algebra $\mathcal{A}$, and let m be a real number. Then mf is measurable with respect to $\mathcal{A}$.*

Proof The result is immediate when $m = 0$. Let c be a real number. If $m > 0$ then

$$\{x \in X : mf(x) < c\} = \{x \in X : f(x) < c/m\},$$

and if $m < 0$ then

$$\{x \in X : mf(x) < c\} = \{x \in X : f(x) > c/m\}.$$

It then follows immediately from Proposition 8.1 and the definition of measurable functions that $\{x \in X : mf(x) < c\} \in \mathcal{A}$. Therefore mf is measurable, as required. ∎

Proposition 8.3 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, let $f\colon X \to [-\infty, \infty]$ and $g\colon X \to [-\infty, \infty]$ be functions on X, with values in the set $[-\infty, \infty]$ of extended real numbers, which are measurable with respect to the σ-algebra $\mathcal{A}$. Suppose that $f(x) + g(x)$ is defined for all $x \in X$. Then $f + g$ is measurable with respect to $\mathcal{A}$.*

Proof Let c be a real number, and let x be a point of X. Suppose that $f(x) + g(x) < c$. Then there exists some real number δ satisfying $\delta > 0$ for which $f(x) + g(x) < c - \delta$, and there exists a rational number q satisfying $f(x) < q < f(x) + \delta$. Then $f(x) > q - \delta$, and $g(x) < (c - \delta) - (q - \delta) = c - q$. We conclude therefore that, given any point x of X for which $f(x) + g(x) < c$, there exists some rational number q such that $f(x) < q$ and $g(x) < c - q$. It follows from this that

$$\{x \in X : f(x) + g(x) < c\} = \bigcup_{q \in \mathbb{Q}} \{x \in X : f(x) < q \text{ and } g(x) < c - q\}.$$

Now $\{x \in X : f(x) < q \text{ and } g(x) < c - q\}$ is the intersection of the sets $\{x \in X : f(x) < q\}$ and $\{x \in X : g(x) < c - q\}$. Also $\{x \in X : f(x) < q\} \in \mathcal{A}$ and $\{x \in X : g(x) < c - q\} \in \mathcal{A}$, because the functions f and g are measurable. It follows that $\{x \in X : f(x) < q \text{ and } g(x) < c - q\} \in \mathcal{A}$ for each rational number q. Also the set $\mathbb{Q}$ of rational numbers is countable. We conclude therefore that the set $\{x \in X : f(x) + g(x) < c\}$ can be represented as a countable union of sets belonging to the σ-algebra $\mathcal{A}$, and must therefore itself belong to $\mathcal{A}$. We conclude that the function $f + g$ is measurable, as required. ∎

We recall that the sum of two extended real numbers is not defined when one has the value $+\infty$ and the other has the value $-\infty$.

Corollary 8.4 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, let $f: X \to [-\infty, \infty]$ and $g: X \to [-\infty, \infty]$ be functions on X, with values in the set $[-\infty, \infty]$ of extended real numbers, which are measurable with respect to the σ-algebra $\mathcal{A}$. Then*

$$\{x \in X : f(x) + g(x) \textit{ is defined and } f(x) + g(x) < c\} \in \mathcal{A}$$

for all real numbers c.

Proof Let $X_0 = \{x \in X : f(x) + g(x) \text{ is defined}\}$. Then $X \setminus X_0$ is the union of the sets

$$\{x \in X : f(x) = +\infty\} \cap \{x \in X : g(x) = -\infty\}$$

and

$$\{x \in X : f(x) = -\infty\} \cap \{x \in X : g(x) = +\infty\},$$

and it follows from Proposition 8.1 that both these sets belong to $\mathcal{A}$. It follows that $X \setminus X_0 \in \mathcal{A}$, and therefore $X_0 \in \mathcal{A}$.

Now let $\mathcal{A}_0$ be the collection of subsets of X_0 consisting of all such subsets that are of the form $X_0 \cap E$ for some $E \in \mathcal{A}$. It is a straightforward exercise to verify that $\mathcal{A}_0$ is a σ-algebra of subsets of X_0. The restrictions of the functions f and g to X_0 are measurable with respect to the σ-algebra $\mathcal{A}_0$. It follows from Proposition 8.3 that the restriction of $f + g$ to X_0 is measurable with respect to $\mathcal{A}_0$, and therefore $\{x \in X_0 : f(x) + g(x) < c\} \in \mathcal{A}_0$ for all real numbers c. But $X_0 \in \mathcal{A}$, and therefore every set belonging to $\mathcal{A}_0$ is the intersection of two sets belonging to $\mathcal{A}$, and must therefore itself belong to $\mathcal{A}$. Thus $\mathcal{A}_0 \subset \mathcal{A}$. We conclude therefore that

$$\{x \in X : f(x) + g(x) \text{ is defined and } f(x) + g(x) < c\} \in \mathcal{A}$$

for all real numbers c, as required. ∎

Corollary 8.5 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, let $f\colon X \to [-\infty, \infty]$ and $g\colon X \to [-\infty, \infty]$ be functions on X, with values in the set $[-\infty, \infty]$ of extended real numbers, which are measurable with respect to the σ-algebra $\mathcal{A}$. Then $f \cdot g$ is measurable with respect to $\mathcal{A}$, where $(f \cdot g)(x) = f(x)g(x)$ for all $x \in X$.*

Proof Let $X_0 = \{x \in X : f(x) \in \mathbb{R} \text{ and } g(x) \in \mathbb{R}\}$. It follows from a straightforward application of Proposition 8.1 that $X_0 \in \mathcal{A}$. If we then define $\mathcal{A}_0 = \{X_0 \cap E : E \in \mathcal{A}\}$, then $\mathcal{A}_0$ is a σ-algebra of subsets of X_0, and $\mathcal{A}_0 \subset \mathcal{A}$.

Now if $c > 0$ then

$$\{x \in X_0 : f(x)^2 < c\} = X_0 \cap \{x \in X : -\sqrt{c} < f(x) < \sqrt{c}\}.$$

It follows from Proposition 8.1 that $\{x \in X_0 : f(x)^2 < c\} \in \mathcal{A}_0$ for all positive real numbers c. Also $\{x \in X_0 : f(x)^2 < c\} = \emptyset$ when $c \leq 0$. It follows from this that the restriction of the function f^2 to X_0 is measurable with respect to the σ-algebra $\mathcal{A}_0$. Similarly the restrictions of the functions g^2 and $(f + g)^2$ to X_0 are measurable with respect to $\mathcal{A}_0$. Now

$$f(x)g(x) = \tfrac{1}{2}((f(x) + g(x))^2 - f(x)^2 - g(x)^2)$$

for all $x \in X_0$. It therefore follows from a straightforward appplication of Proposition 8.3 that the restriction of $f \cdot g$ to X_0 is measurable with respect to the σ-algebra $\mathcal{A}_0$. Thus $\{x \in X_0 : f(x)g(x) < c\} \in \mathcal{A}$ for all real numbers c.

Now if $x \in X \setminus X_0$ then either $f(x) \in \{-\infty, +\infty\}$ or $g(x) \in \{-\infty, +\infty\}$. It follows easily from this that if $x \in X \setminus X_0$ then $f(x)g(x) \in \{-\infty, 0, +\infty\}$, and, by straightforward applications of the results in Proposition 8.1, it is easy to show that the sets $\{x \in X \setminus X_0 : f(x)g(x) = -\infty\}$, $\{x \in X \setminus X_0 : f(x)g(x) = 0\}$ and $\{x \in X \setminus X_0 : f(x)g(x) = +\infty\}$ all belong to $\mathcal{A}$. We conclude that the intersection of $\{x \in X : f(x)g(x) < c\}$ with the sets X_0 and $X \setminus X_0$ belongs to $\mathcal{A}$ for all real numbers c, and therefore $\{x \in X : f(x)g(x) < c\} \in \mathcal{A}$ for all real numbers c. Therefore $f \cdot g$ is measurable with respect to the σ-algebra $\mathcal{A}$, as required. ∎

Lemma 8.6 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let $f_1, f_2, \ldots, f_m$ be functions on X with values in the set $[-\infty, +\infty]$ of extended real numbers. Suppose that each of the functions $f_1, f_2, \ldots, f_m$ is measurable with respect to the σ-algebra $\mathcal{A}$. Then so are*

$$\max(f_1, f_2, \ldots, f_m) \quad \text{and} \quad \min(f_1, f_2, \ldots, f_m).$$

Proof Let c be a real number. Then

$$\{x \in X : \max(f_1, f_2, \ldots, f_m) < c\} = \bigcap_{i=1}^{m} \{x \in X : f_i(x) < c\}$$

and

$$\{x \in X : \min(f_1, f_2, \ldots, f_m) < c\} = \bigcup_{i=1}^{m} \{x \in X : f_i(x) < c\}.$$

It follows that $\{x \in X : \max(f_1, f_2, \ldots, f_m) < c\}$ is a finite intersection of sets belonging to $\mathcal{A}$, and must therefore itself belong to $\mathcal{A}$. Similarly $\{x \in X : \min(f_1, f_2, \ldots, f_m) < c\}$ is a finite union of sets belonging to $\mathcal{A}$, and must therefore itself belong to $\mathcal{A}$. The result follows. ∎

Proposition 8.7 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let $f_1, f_2, f_3, \ldots$ be an infinite sequence of functions on X with values in the set $[-\infty, +\infty]$ of extended real numbers. Suppose that each of the functions $f_1, f_2, f_3, \ldots$ is measurable with respect to the σ-algebra $\mathcal{A}$. Then so are g and h, where*

$$g(x) = \sup\{f_i(x) : i \in \mathbb{N}\}, \quad h(x) = \inf\{f_i(x) : i \in \mathbb{N}\}$$

for all $x \in X$.

Proof Let c be a real number, and let x be a point of X. Then $g(x) < c$ if and only if there exists some natural number n such that $f_i(x) < c - n^{-1}$ for all natural numbers i. Therefore

$$\{x \in X : g(x) < c\} = \bigcup_{n=1}^{+\infty} \bigcap_{i=1}^{+\infty} \left\{x \in X : f_i(x) < c - \frac{1}{n}\right\}$$

for all real numbers c. Now $\displaystyle\bigcap_{i=1}^{+\infty} \left\{x \in X : f_i(x) < c - \frac{1}{n}\right\} \in \mathcal{A}$ for each natural number n. It follows that $\{x \in X : g(x) < c\} \in \mathcal{A}$ for all real numbers c. Thus the function g is measurable with respect to $\mathcal{A}$.

Also

$$\{x \in X : h(x) < c\} = \bigcup_{i=1}^{+\infty} \{x \in X : f_i(x) < c\}.$$

It follows that $\{x \in X : h(x) < c\} \in \mathcal{A}$ for all real numbers c. Thus the function h is measurable with respect to $\mathcal{A}$.

Corollary 8.8 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let $f_1, f_2, f_3, \ldots$ be an infinite sequence of functions on X with values in the set $[-\infty, +\infty]$ of extended real numbers. Suppose that each of the functions $f_1, f_2, f_3, \ldots$ is measurable with respect to the σ-algebra $\mathcal{A}$. Then so are f^* and f_*, where*

$$f^*(x) = \limsup_{n \to +\infty} f_n(x), \quad f_*(x) = \liminf_{n \to +\infty} f_n(x)$$

for all $x \in X$.

Proof It follows from the definition of the upper and lower limits that

$$f^*(x) = \lim_{n \to +\infty} g_n(x) = \inf\{g_n(x) : n \in \mathbb{N}\},$$

where $g_n(x) = \sup\{f_i(x) : i \geq n\}$. The measurability of f^* therefore follows directly on applying Proposition 8.7. Similarly

$$f_*(x) = \lim_{n \to +\infty} h_n(x) = \sup\{h_n(x) : n \in \mathbb{N}\},$$

where $h_n(x) = \inf\{f_i(x) : i \geq n\}$, and therefore f_* is also measurable. ∎

Let $f_1, f_2, f_3, \ldots$ be an infinite sequence of functions defined on some set X with values in the set $[-\infty, +\infty]$ of extended real numbers, and let $x \in X$. Then $\lim_{n \to +\infty} f_n(x)$ is defined, and belongs to the set $[-\infty, +\infty]$ of extended real numbers if and only if $\limsup_{n \to +\infty} f_n(x) = \liminf_{n \to +\infty} f_n(x)$.

Corollary 8.9 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let $f_1, f_2, f_3, \ldots$ be an infinite sequence of functions on X with values in the set $[-\infty, +\infty]$ of extended real numbers. Suppose that each of the functions $f_1, f_2, f_3, \ldots$ is measurable with respect to the σ-algebra $\mathcal{A}$. Let*

$$X_0 = \{x \in X : \lim_{n \to +\infty} f_n(x) \text{ is defined}\}$$

Then $X_0 \in \mathcal{A}$. Moreover if $f(x) = \lim_{n \to +\infty} f_n(x)$ for all $x \in X_0$, then f is a measurable function on X_0.

Proof Note that

$$X_0 = \{x \in X : \limsup_{n \to +\infty} f_n(x) - \liminf_{n \to +\infty} f_n(x) = 0\}.$$

It follows from Proposition 8.1 that $X_0 \in \mathcal{A}$. Moreover the function f coincides with the measurable functions f^* on X_0, where $f^*(x) = \limsup_{n \to +\infty} f_n(x)$, and must therefore be a measurable function on X_0, as required. ∎

We see therefore that if $(X, \mathcal{A}, \mu)$ is a measure space then the limit of any convergent sequence of measurable functions on X must itself be measurable.

8.2 Integrals of Measurable Simple Functions

Lemma 8.10 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $E_1, E_2, \ldots, E_m$ be a finite collection of measurable subsets of X. Then there exists a finite collection $G_1, G_2, \ldots, G_r$ of pairwise disjoint measurable subsets of X such that $E_j = \bigcup_{k \in K(j)} G_k$ for $j = 1, 2, \ldots, m$, where $K(j)$ denotes the set of all integers k between 1 and r for which $G_k \subset E_j$.*

Proof For each subset S of $\{1, 2, \ldots, m\}$ let

$$F_S = \left(\bigcap_{j \in S} E_j\right) \setminus \left(\bigcup_{j \notin S} E_j\right).$$

(Thus F_S is defined to be the set of all elements x of X that satisfy $x \in E_j$ for all $j \in S$ and $x \notin E_j$ for all $j \in \{1, 2, \ldots, m\} \setminus S$.) Then each set F_S is measurable.

Given a point $x \in X$ let $S(x)$ denote the set of all integers j between 1 and m for which $x \in E_j$. Then $S(x)$ is the unique subset of $\{1, 2, \ldots, m\}$ for which $x \in F_{S(x)}$. It follows that if S' and S'' are distinct subsets of $\{1, 2, \ldots, m\}$ then $S' \cap S'' = \emptyset$. Let $S_1, S_2, \ldots, S_r$ be a list of subsets of $\{1, 2, \ldots, m\}$ with the property that every subset S of $\{1, 2, \ldots, m\}$ for which F_S is non-empty occurs exactly once in the list, and let $G_k = F_{S_k}$ for $k = 1, 2, \ldots, r$. Then the sets $G_1, G_2, \ldots, G_r$ have the required properties. ∎

Let X be a set, and let E be a subset of X. The *characteristic function* of E is defined to be the function $\chi_E \colon X \to \mathbb{R}$ defined so that

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E; \\ 0 & \text{if } x \notin E. \end{cases}$$

Lemma 8.11 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be a real-valued function on X. Suppose that there exist measurable sets $E_1, E_2, \ldots, E_m$ and real numbers $c_1, c_2, \ldots, c_m$ such that*

$$f(x) = \sum_{j=1}^{m} c_j \chi_{E_j}(x)$$

for all $x \in X$, where χ_{E_j} denotes the characteristic function of the set E_j. Then f is a measurable function on X, and $\{x \in X : f(x) = c\}$ is a measurable set for all real numbers c.

Proof Lemma 8.10 ensures that there exists a finite collection $G_1, G_2, \ldots, G_r$ of pairwise disjoint measurable subsets of X such that $E_j = \bigcup_{k \in K(j)} G_k$ for

$j = 1, 2, \ldots, m$, where $K(j)$ denotes the set of all integers k between 1 and r for which $G_k \subset E_j$. Then $f = \sum_{k=1}^{r} d_k \chi_{G_k}$, where each real number d_k denotes the sum of those real numbers c_j for which $G_k \subset E_j$. Then, given any real number b, the set $\{x \in X : f(x) = b\}$ is the union of those sets G_k for which $d_k = b$, and is therefore a measurable set. It follows that, for each real number c, the set $\{x \in X : f(x) < c\}$ is a finite union of measurable sets, and is therefore a measurable set. Thus the function f is measurable, as required. ∎

Remark The above result also follows immediately on noting that the characteristic function of a measurable set is measurable, any scalar multiple of a measurable function is measurable, and any finite sum of measurable functions is measurable. However the proof given above is more elementary than the proof of the result that finite sums of measurable functions are measurable.

Definition Let $(X, \mathcal{A}, \mu)$ be a measure space. A function $f\colon X \to \mathbb{R}$ is said to be a *measurable simple function* on X if there exist measurable sets $E_1, E_2, \ldots, E_m$ and real numbers $c_1, c_2, \ldots, c_m$ such that

$$f(x) = \sum_{j=1}^{m} c_j \chi_{E_j}(x)$$

for all $x \in X$, where χ_{E_j} denotes the characteristic function of the set E_j.

Note that Lemma 8.11 guarantees that any real-valued function on X that satisfies the definition of a measurable simple function on X is guaranteed to be measurable.

It follows directly from the definition of measurable simple functions that any constant multiple of a measurable simple function is a measurable simple function, and the sum of any finite number of measurable simple functions is a measurable simple function.

Lemma 8.12 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f\colon X \to \mathbb{R}$ be a measurable simple function on X. Suppose that*

$$f(x) = \sum_{j=1}^{m} c_j \chi_{E_j}(x) = \sum_{k=1}^{n} d_k \chi_{F_k}(x)$$

for all $x \in X$, where $E_1, E_2, \ldots, E_m$ and $F_1, F_2, \ldots, F_n$ are measurable sets, and $c_1, c_2, \ldots, c_m$ and $d_1, d_2, \ldots, d_n$ are real numbers. Then

$$\sum_{j=1}^{m} c_j \mu(E_j) = \sum_{k=1}^{n} d_k \mu(F_k).$$

Proof It follows on applying Lemma 8.10 to the finite collection consisting of all the sets $E_1, E_2, \ldots, E_m, F_1, F_2, \ldots, F_n$ that there exists a that there exists a finite collection $G_1, G_2, \ldots, G_r$ of pairwise disjoint measurable subsets of X such that $E_j = \bigcup_{s \in S(j)} G_s$ for $j = 1, 2, \ldots, m$ and $F_k = \bigcup_{s \in T(k)} G_s$ for $k = 1, 2, \ldots, m$, where $S(j)$ denotes the set of all integers s between 1 and r for which $G_s \subset E_j$ and $T(k)$ denotes the set of all integers s between 1 and r for which $G_s \subset F_k$. Then the additivity of the measure μ ensures that

$$\mu(E_j) = \sum_{s \in S(j)} \mu(G_s).$$

It follows that

$$\sum_{j=1}^{m} c_j \mu(E_j) = \sum_{j=1}^{m} \sum_{s \in S(j)} c_j \mu(G_s) = \sum_{s=1}^{r} \sum_{j \in U(s)} c_j \mu(G_s),$$

where $U(s)$ denotes the set of all integers j between 1 and m for which $G_s \subset E_j$. But clearly $\sum\limits_{j \in U(s)} c_j = g_s$, where g_s denotes the value of the function f on G_s for $s = 1, 2, \ldots, r$. It follows that

$$\sum_{j=1}^{m} c_j \mu(E_j) = \sum_{s=1}^{r} g_s \mu(G_s).$$

Similarly

$$\sum_{k=1}^{n} d_k \mu(F_k) = \sum_{s=1}^{r} g_s \mu(G_s).$$

It follows that

$$\sum_{j=1}^{m} c_j \mu(E_j) = \sum_{k=1}^{n} d_k \mu(F_k),$$

as required. ▮

Definition Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f \colon X \to \mathbb{R}$ be a measurable simple function on X. Then there exist measurable sets $E_1, E_2, \ldots, E_m$ and real numbers $c_1, c_2, \ldots, c_m$ such that

$$f(x) = \sum_{j=1}^{m} c_j \chi_{E_j}(x)$$

for all $x \in X$, where χ_{E_j} denotes the characteristic function of the set E_j. We define

$$\int_X f \, d\mu = \sum_{j=1}^{m} c_j \mu(E_j).$$

This quantity $\int_X f\,d\mu$ is referred to as the *integral* of f on X with respect to the measure μ.

Lemma 8.12 ensures that the integral $\int_X f\,d\mu$ of a measurable function is well-defined, and does not depend on the choice of measurable sets $E_1, E_2, \ldots, E_m$ and real numbers $c_1, c_2, \ldots, c_m$ for which $f = \sum_{j=1}^{m} c_j \chi_{E_j}$.

Proposition 8.13 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let f and g be measurable simple functions on X, and let c be a real number. Then*

$$\int_X cf\,d\mu = c\int_X f\,d\mu,$$

and

$$\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu.$$

Proof The result follows immediately from the definition of the integral of a measurable simple function. ∎

Lemma 8.14 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be measurable simple functions on X. Suppose that $f(x) \leq g(x)$ for all $x \in X$. Then*

$$\int_X f\,d\mu \leq \int_X g\,d\mu.$$

Proof Let

$$f(x) = \sum_{j=1}^{m} c_j \chi_{E_j}(x) \quad \text{and} \quad g(x) = \sum_{k=1}^{n} d_k \chi_{F_k}(x)$$

for all $x \in X$, where $E_1, E_2, \ldots, E_m$ and $F_1, F_2, \ldots, F_n$ are measurable sets, and $c_1, c_2, \ldots, c_m$ and $d_1, d_2, \ldots, d_n$ are real numbers. Then there exists a finite collection of pairwise disjoint measurable sets $G_1, G_2, \ldots, G_r$ such that each of the sets $E_1, E_2, \ldots, E_m, F_1, F_2, \ldots, F_n$ is a union of finitely many of the pairwise disjoint sets $G_1, G_2, \ldots, G_r$. Then

$$f(x) = \sum_{j=1}^{r} a_j \chi_{G_j}(x) \quad \text{and} \quad g(x) = \sum_{j=1}^{r} b_j \chi_{G_j}(x),$$

for all $x \in X$, where a_j and b_j denote the values of the functions f and g on G_j for $j = 1, 2, \ldots, r$. Then $a_j \leq b_j$ for each j. It follows that

$$\int_X f\,d\mu = \sum_{j=1}^{r} a_j \mu(G_j) \leq \sum_{j=1}^{r} b_j \mu(G_j) = \int_X g\,d\mu,$$

as required. ∎

Definition Let $(X, \mathcal{A}, \mu)$ be a measure space, let E be a measurable subset of X, and let $s\colon X \to [0, +\infty)$ be a non-negative measurable simple function on X. The integral $\int_E s\,d\mu$ of s over E is defined by the formula

$$\int_E s\,d\mu = \int_X s \cdot \chi_E \,d\mu,$$

where χ_E denotes the characteristic function of E and $s \cdot \chi_E$ is the product of the functions s and χ_E (so that $(s \cdot \chi_E)(x) = s(x)\chi_E(x)$ for all $x \in X$).

Proposition 8.15 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $s\colon X \to [0, +\infty)$ be a non-negative measurable simple function on X, and let $\nu(E) = \int_E s\,d\mu$ for all measurable sets E. Then ν is a measure defined on the σ-algebra $\mathcal{A}$ of measurable subsets of X.*

Proof The function s is a non-negative measurable simple function on X, and therefore there exist non-negative real numbers $c_1, c_2, \ldots, c_m$ and measurable sets $F_1, F_2, \ldots, F_m$ such that $s(x) = \sum_{j=1}^{m} c_j \chi_{F_j}(x)$ for all $x \in X$. Let E be a measurable set in X. Then $s(x)\chi_E(x) = \sum_{j=1}^{m} c_j \chi_{F_j \cap E}(x)$ for all $x \in X$, and therefore

$$\nu(E) = \int_E s\,d\mu = \int_X s \cdot \chi_E \,d\mu = \sum_{j=1}^{m} c_j \mu(F_j \cap E).$$

Let $\mathcal{E}$ be a countable collection of pairwise disjoint measurable sets. It follows from the countable additivity of the measure μ that

$$\nu\left(\bigcup_{E \in \mathcal{E}} E\right) = \sum_{j=1}^{m} c_j \mu\left(\bigcup_{E \in \mathcal{E}} (F_j \cap E)\right) = \sum_{j=1}^{m} c_j \sum_{E \in \mathcal{E}} \mu(F_j \cap E) = \sum_{E \in \mathcal{E}} \nu(E).$$

Thus the function ν is countably additive, and is therefore a measure defined on $\mathcal{A}$, as required. ∎

Corollary 8.16 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $s\colon X \to [0, +\infty)$ be a non-negative measurable simple function on X, and let $E_1, E_2, E_3, \ldots$ be an infinite sequence of measurable subsets of X, where $E_j \subset E_{j+1}$ for all positive integers j. Then*

$$\lim_{j \to +\infty} \int_{E_j} f\,d\mu = \int_E f\,d\mu,$$

where $E = \bigcup_{j=1}^{+\infty} E_j$.

Proof Let $\nu(F) = \int_F s\,d\mu$ for all measurable sets F. Then ν is a measure on X. It follows that

$$\nu\left(\bigcup_{j=1}^{+\infty} E_j\right) = \lim_{j\to+\infty} \nu(E_j)$$

(Lemma 7.20). The result follows. ∎

We shall extend the definition of the integral to non-negative measurable functions that are not necessarily simple. In developing the properties of this integral, we shall need the result that a non-negative measurable function on a measure set is the limit of a non-decreasing sequence of measurable simple functions. We now proceed to prove this result.

Proposition 8.17 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f\colon X \to [0, +\infty]$ be a non-negative measurable function on X. Then there exists an infinite sequence $s_1, s_2, s_3, \ldots$ of non-negative measurable simple functions with the following properties:*

(i) $0 \leq s_j(x) \leq s_{j+1}(x)$ *for all* $j \in \mathbb{N}$ *and* $x \in X$*;*

(ii) $\lim_{j\to+\infty} s_j(x) = f(x)$ *for all* $x \in X$*.*

Proof For each positive integer j let

$$F_j = \{x \in X : f(x) \geq j\},$$

and for each integer k satisfying $1 \leq k \leq 2^j j$, let

$$E_{j,k} = \left\{x \in X : \frac{k-1}{2^j} \leq f(x) < \frac{k}{2^j}\right\}.$$

Then the sets F_j and $E_{j,k}$ are measurable sets. Let

$$s_j(x) = j\chi_{F_j}(x) + \sum_{k=1}^{2^j j} \frac{k-1}{2^j}\chi_{E_{j,k}}(x)$$

for all $j \in \mathbb{N}$ and $x \in X$. Then s_j is a measurable simple function on X which takes the value $2^{-j}(k-1)$ when $2^{-j}(k-1) \leq f(x) < 2^{-j}k$ for some integer k between 1 and $2^j j$, and takes the value j when $f(x) \geq j$. One can readily verify that $0 \leq s_j(x) \leq s_{j+1}(x)$ for all $j \in \mathbb{N}$ and $x \in X$. If $f(x) < +\infty$ and $j \geq f(x)$ then $0 \leq f(x) - s_j(x) < 2^{-j}$. It follows that if $f(x) < +\infty$ then $\lim_{j\to+\infty} s_j(x) = f(x)$. If $f(x) = +\infty$ then $s_j(x) = j$ for all positive integers j, and therefore $\lim_{j\to+\infty} s_j(x) = f(x)$ in this case as well. The result is thus established. ∎

8.3 Integrals of Non-Negative Measurable Functions

Definition Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f\colon X \to [0, +\infty]$ be a measurable function on X taking values in the set $[0, +\infty]$ of non-negative extended real numbers. The *integral* $\int_X f \, d\mu$ of f over X is defined to be the supremum of the integrals $\int_X s \, d\mu$ as s ranges over all non-negative measurable simple functions on X that satisfy $s(x) \leq f(x)$ for all $x \in X$.

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f\colon X \to [0, +\infty]$ be a measurable function on X taking values in the set $[0, +\infty]$ of non-negative extended real numbers. It follows from the above definition that $\int_X f \, d\mu = C$ for some non-negative extended real number C if and only if the following two conditions are satisfied:

(i) $\int_X s \, d\mu \leq C$ for all non-negative measurable simple functions s on X that satisfy $s(x) \leq f(x)$ for all $x \in X$.

(ii) given any non-negative real number c satisfying $c < C$, there exists some non-negative measurable simple function s on X such that $s(x) \leq f(x)$ for all $x \in X$ and $\int_X s \, d\mu > c$.

It follows directly from Lemma 8.14 that the definition of the integral for non-negative measurable functions is consistent with that previously given for measurable simple functions.

Lemma 8.18 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f\colon X \to [0, +\infty]$ and $g\colon X \to [0, +\infty]$ be measurable functions on X with values in the set $[0, +\infty]$ of non-negative extended real numbers. Suppose that $f(x) \leq g(x)$ for all $x \in X$. Then*

$$\int_X f \, d\mu \leq \int_X g \, d\mu$$

Proof This follows immediately from the definition of the integral, since any non-negative measurable simple function s on X satisfying $s(x) \leq f(x)$ for all $x \in X$ will also satisfy $s(x) \leq g(x)$ for all $x \in X$. ∎

The Monotone Convergence Theorem

We now prove an important theorem which states that the integral of the limit of a non-decreasing sequence of measurable functions is equal to the limit of the integrals of those functions. A number of other important results follow as consequences of this basic theorem.

Theorem 8.19 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable functions on X with values in the set $[0, +\infty]$ of non-negative extended real numbers, and let $f\colon X \to [0, +\infty]$ be defined such that $f(x) = \lim\limits_{j\to+\infty} f_j(x)$ for all $x \in X$. Suppose that $0 \leq f_j(x) \leq f_{j+1}(x)$ for all $j \in \mathbb{N}$ and $x \in X$. Then*

$$\lim_{j\to+\infty} \int_X f_j \, d\mu = \int_X f \, d\mu$$

Proof It follows from Corollary 8.9 that the limit function f is measurable. Moreover $\int_X f_j \, d\mu \leq \int_X f \, d\mu$, and therefore $\lim\limits_{j\to+\infty} \int_X f_j \, d\mu \leq \int_X f \, d\mu$.

Let s be a non-negative measurable simple function on X which satisfies $s(x) \leq f(x)$ for all $x \in X$, and let c be a real number satisfying $0 < c < 1$. If $f(x) > 0$ then $f(x) > cs(x)$ and therefore there exists some positive integer j such that $f_j(x) \geq cs(x)$. If $f(x) = 0$ then $s(x) = 0$, and therefore $f_j(x) \geq cs(x)$ for all positive integers j. It follows that $\bigcup\limits_{j=1}^{+\infty} E_j = X$, where

$$E_j = \{x \in X : f_j(x) \geq cs(x)\}.$$

Now

$$c \int_{E_j} s \, d\mu \leq \int_{E_j} f_j \, d\mu \leq \int_X f_j \, d\mu \leq \lim_{j\to+\infty} \int_X f_j \, d\mu.$$

Also $E_j \subset E_{j+1}$ for all positive integers j. It therefore follows from Corollary 8.16 that

$$c \int_X s \, d\mu = \lim_{j\to+\infty} c \int_{E_j} s \, d\mu \leq \lim_{j\to+\infty} \int_X f_j \, d\mu.$$

Moreover this inequality holds for all real numbers c satisfying $0 < c < 1$, and therefore

$$\int_X s \, d\mu \leq \lim_{j\to+\infty} \int_X f_j \, d\mu.$$

This inequality holds for all non-negative measurable simple functions s satisfying $s(x) \leq f(x)$ for all $x \in X$. It now follows from the definition of the integral of a measurable function that $\int_X f \, d\mu \leq \lim\limits_{j\to+\infty} \int_X f_j \, d\mu$, and therefore $\int_X f \, d\mu = \lim\limits_{j\to+\infty} \int_X f_j \, d\mu$, as required. ∎

Proposition 8.20 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f\colon X \to [0, +\infty]$ and $g\colon X \to [0, +\infty]$ be non-negative measurable functions on X. Then*

$$\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu.$$

Proof It follows from Proposition 8.17 that there exist infinite sequences $s_1, s_2, s_3, \ldots$ and $t_1, t_2, t_3, \ldots$ of non-negative measurable simple functions such that $0 \leq s_j(x) \leq s_{j+1}(x)$ and $0 \leq t_j(x) \leq t_{j+1}(x)$ for all $j \in \mathbb{N}$ and $x \in X$, $\lim_{j\to+\infty} s_j(x) = f(x)$ and $\lim_{j\to+\infty} t_j(x) = g(x)$. Then $\lim_{j\to+\infty} (s_j(x) + t_j(x)) = f(x) + g(x)$. It therefore follows from Proposition 8.13 and the Monotone Convergence Theorem (Theorem 8.19) that

$$\begin{aligned}\int_X (f+g)\,d\mu &= \lim_{j\to+\infty} \int_X (s_j + t_j)\,d\mu = \lim_{j\to+\infty} \left(\int_X s_j\,d\mu + \int_X t_j\,d\mu\right) \\ &= \lim_{j\to+\infty} \int_X s_j\,d\mu + \lim_{j\to+\infty} \int_X t_j\,d\mu \\ &= \int_X f\,d\mu + \int_X g\,d\mu,\end{aligned}$$

as required. ∎

Proposition 8.21 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f_1, f_2, f_3, \ldots$ be an infinite sequence of non-negative measurable functions on X. Then*

$$\int_X \left(\sum_{j=1}^{+\infty} f_j\right) d\mu = \sum_{j=1}^{+\infty} \int_X f_j\,d\mu.$$

Proof It follows from Proposition 8.20 that

$$\int_X \left(\sum_{j=1}^{N} f_j\right) d\mu = \sum_{j=1}^{N} \int_X f_j\,d\mu$$

for all positive integers N. It then follows from the Monotone Convergence Theorem (Theorem 8.19) that

$$\int_X \left(\sum_{j=1}^{+\infty} f_j\right) d\mu = \lim_{N\to+\infty} \int_X \left(\sum_{j=1}^{N} f_j\right) d\mu = \sum_{j=1}^{+\infty} \int_X f_j\,d\mu,$$

as required. ∎

Lemma 8.22 (Fatou's Lemma) *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of non-negative measurable functions on X, and let $f_*(x) = \liminf_{j\to+\infty} f_j(x)$ for all $x \in X$. Then*

$$\int_X f_*\,d\mu \leq \liminf_{j\to+\infty} \int_X f_j\,d\mu.$$

Proof Let $g_j(x) = \inf\{f_k(x) : k \geq j\}$ for all $j \in \mathbb{N}$ and $x \in X$. Then the functions $g_1, g_2, g_3, \ldots$ are measurable (Proposition 8.7), and $f_*(x) = \lim_{j\to+\infty} g_j(x)$ for all $x \in X$. Also $0 \leq g_j(x) \leq g_{j+1}(x)$ for all $j \in \mathbb{N}$ and $x \in X$. It follows from the Monotone Convergence Theorem (Theorem 8.19) that

$$\int_X f_*\, d\mu = \lim_{j\to+\infty} \int_X g_j\, d\mu.$$

Now $g_j(x) \leq f_k(x)$ for all $x \in X$ and for all positive integers j and k satisfying $j \leq k$. It follows that

$$\int_X g_j\, d\mu \leq \int_X f_k\, d\mu \quad \text{whenever } j \leq k,$$

and therefore

$$\int_X g_j\, d\mu \leq \inf\left\{\int_X f_k\, d\mu : k \geq j\right\}.$$

It follows that

$$\int_X f_*\, d\mu = \lim_{j\to+\infty} \int_X g_j\, d\mu \leq \lim_{j\to+\infty} \inf\left\{\int_X f_k\, d\mu : k \geq j\right\} = \liminf_{j\to+\infty} \int_X f_j\, d\mu,$$

as required. ∎

8.4 Integration of Functions with Positive and Negative Values

Definition Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f \colon X \to [-\infty, +\infty]$ be a measurable function on X. The function f is said to be *integrable* if $\int_X |f|\, dx < +\infty$.

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f \colon X \to [-\infty, +\infty]$ be a measurable function on X. Then f gives rise to non-negative measurable functions f_+ and f_- on X, where $f_+(x) = \max(f(x), 0)$ and $f_-(x) = \max(-f(x), 0)$ for all $x \in X$. Moreover $f(x) = f_+(x) - f_-(x)$ and $|f(x)| = f_+(x) + f_-(x)$ for all $x \in X$. Now $\int_X f_+\, d\mu \leq \int_X |f|\, d\mu$, $\int_X f_-\, d\mu \leq \int_X |f|\, d\mu$ and $\int_X |f|\, d\mu = \int_X f_+\, d\mu + \int_X f_- d\mu$. It follows that $\int_X |f|\, d\mu < +\infty$ if and only if $\int_X f_+\, d\mu < +\infty$ and $\int_X f_-\, d\mu < +\infty$.

Definition Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f \colon X \to [-\infty, +\infty]$ be an integrable function on X. The *integral* $\int_X f\, d\mu$ of f on X is defined by the identity

$$\int_X f\, d\mu = \int_X f_+\, d\mu - \int_X f_-\, d\mu,$$

where $f_+(x) = \max(f(x), 0)$ and $f_-(x) = \max(-f(x), 0)$ for all $x \in X$.

Lemma 8.23 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f: X \to [-\infty, +\infty]$ be an integrable function on X, and let $u: X \to [0, +\infty]$ and $v: X \to [0, +\infty]$ be non-negative integrable functions on X such that $f(x) = u(x) - v(x)$ for all $x \in X$. Then*

$$\int_X f \, d\mu = \int_X u \, d\mu - \int_X v \, d\mu.$$

Proof Let $f_+(x) = \max(f(x), 0)$ and $f_-(x) = \max(-f(x), 0)$ for all $x \in X$. Then $f(x) = f_+(x) - f_-(x) = u(x) - v(x)$ for all $x \in X$, and therefore $f_+(x) + v(x) = f_-(x) + u(x)$ for all $x \in X$. It follows from Proposition 8.20 that

$$\int_X f_+ \, d\mu + \int_X v \, d\mu = \int_X f_- \, d\mu + \int_X u \, d\mu.$$

But then

$$\int_X f \, d\mu = \int_X f_+ \, d\mu - \int_X f_- \, d\mu = \int_X u \, d\mu - \int_X v \, d\mu,$$

as required. ∎

Lemma 8.24 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f: X \to [-\infty, +\infty]$ and $g: X \to [-\infty, +\infty]$ be integrable functions on X. Then*

$$\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu,$$

and

$$\int_X cf \, d\mu = c \int_X f \, d\mu$$

for all real numbers c.

Proof Let

$$f_+(x) = \max(f(x), 0), \quad f_-(x) = \max(-f(x), 0),$$
$$g_+(x) = \max(f(x), 0), \quad g_-(x) = \max(-f(x), 0),$$
$$u(x) = f_+(x) + g_+(x) \quad \text{and} \quad v(x) = f_-(x) + g_-(x)$$

for all $x \in X$. Then the functions u and v are integrable, and $f(x) + g(x) = u(x) - v(x)$ for all $x \in X$. It follows from Lemma 8.23 that

$$\begin{aligned}
\int_X (f + g) \, d\mu &= \int_X u \, d\mu - \int_X v \, d\mu \\
&= \int_X f_+ \, d\mu + \int_X g_+ \, d\mu - \int_X f_- \, d\mu - \int_X g_- \, d\mu \\
&= \int_X f \, d\mu + \int_X g \, d\mu.
\end{aligned}$$

The identity $\int_X cf\,d\mu = c\int_X f\,d\mu$ follows directly from the definition of the integral, on considering separately the cases when $c > 0$, $c = 0$ and $c < 0$. ∎

Integrals of Complex-Valued Functions

We can define the integral of a complex-valued function by splitting the integrand into its real and imaginary parts.

Definition Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f\colon X \to \mathbb{C}$ be a function on X with values in the field $\mathbb{C}$ of complex numbers, and $u\colon X \to \mathbb{R}$ and $v\colon X \to IR$ be the real-valued functions that determine the real and imaginary parts of the function f, so that $f(x) = u(x) + iv(x)$ for all $x \in X$, where $i = \sqrt{-1}$. The function f is said to be *measurable* if the functions u and v are both measurable; and the function f is said to be *integrable* if the functions u and v are both integrable. If the function f is integrable, then the *integral* of f over X is defined by the formula

$$\int_X f\,d\mu = \int_X u\,d\mu + i\int_X v\,d\mu.$$

Lemma 8.25 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f\colon X \to \mathbb{C}$ and $g\colon X \to \mathbb{C}$ be integrable functions on X with values in the field $\mathbb{C}$ of complex numbers. Then*

$$\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu,$$

and

$$\int_X cf\,d\mu = c\int_X f\,d\mu$$

for all complex numbers c.

Proof The identity $\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu$ follows directly on decomposing the functions f and g into their real and imaginary parts.

Let $f(x) = u(x) + iv(x)$ where $u\colon X \to \mathbb{R}$ and $v\colon X \to \mathbb{R}$ are integrable real-valued functions on X. Also let $c = a + ib$, where a and b are real numbers. Then

$$cf(x) = au(x) - bv(x) + i(av(x) + bu(x)),$$

for all $x \in X$. It follows from Lemma 8.24 that

$$\int_X cf\,d\mu \quad = \quad \int_X (au - bv)\,d\mu + i\int_X (av + bu)\,d\mu$$

$$\begin{aligned} &= a\int_X u\,d\mu - b\int_X v\,d\mu + i\left(a\int_X v\,d\mu + b\int_X u\,d\mu\right) \\ &= (a+ib)\left(\int_X u\,d\mu + i\int_X v\,d\mu\right) \\ &= c\int_X f\,d\mu \end{aligned}$$

as required. ∎

Proposition 8.26 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f\colon X \to \mathbb{C}$ be a measurable function on X with values in the field $\mathbb{C}$ of complex numbers, and let $|f|$ be the real-valued function on X defined such that $|f|(x) = |f(x)|$ for all $x \in X$. Then the function $|f|$ is measurable. Moreover the measurable function f is integrable if and only if $\int_X |f|\,d\mu < +\infty$, in which case*

$$\left|\int_X f\,d\mu\right| \leq \int_X |f|\,d\mu.$$

Proof Let $f(x) = u(x) + iv(x)$ for all $x \in X$, where u and v are real-valued functions on X. Then the functions u and v are measurable. Now

$$\{x \in X : |f|(x) < r\} = \{x \in X : u(x)^2 + v(x)^2 < r^2\}$$

for all real numbers r. Moreover the function that sends points x of X to $u(x)^2 + v(x)^2$ is a measurable function on X, as sums and products of measurable functions are measurable (Proposition 8.3 and Corollary 8.5). Therefore $\{x \in X : |f|(x) < r\}$ is a measurable set for all positive real numbers r. This shows that $|f|$ is a measurable function on X.

Now $|u(x)| \leq |f(x)|$, $|v(x)| \leq |f(x)|$ and $|f(x)| \leq |u(x)| + |v(x)|$ for all $x \in X$, and therefore

$$\int_X |u|\,d\mu \leq \int_X |f|\,d\mu, \quad \int_X |v|\,d\mu \leq \int_X |f|\,d\mu$$

and

$$\int_X |f|\,d\mu \leq \int_X |u|\,d\mu + \int_X |v|\,d\mu.$$

Thus if $\int_X |f|\,d\mu < +\infty$ then u and v are integrable functions on X, and thus f is an integrable function on X. Conversely, if f is an integrable function on X then $\int_X |u|\,d\mu < +\infty$ and $\int_X |v|\,d\mu < +\infty$, and therefore $\int_X |f|\,d\mu < +\infty$. Thus the measurable function f is integrable if and only if $\int_X |f|\,d\mu < +\infty$.

Now suppose that the function f is integrable. There exists a complex number w satisfying $|w| = 1$ for which $w\int_X f\,d\mu$ is a positive real number.

Let $wf(x) = u_1(x) + iv_1(x)$ for all $x \in X$, where u_1 and v_1 are real-valued functions on X. Then the functions u_1 and v_1 are measurable functions on X, and $u_1(x) \leq |f(x)|$ for all $x \in X$. Also $w \int_X f\,d\mu = \int_X u_1\,d\mu$ because $w \int_X f\,d\mu$ is a non-negative real number. It follows that

$$\left|\int_X f\,d\mu\right| = w \int_X f\,d\mu = \int_X u_1\,d\mu \leq \int_X |f|\,d\mu,$$

as required. ∎

8.5 Lebesgue's Dominated Convergence Theorem

Theorem 8.27 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable complex-valued functions on X, and let f be a measurable complex-valued function on X, where $f(x) = \lim_{j\to+\infty} f_j(x)$ for all $x \in X$. Suppose that there exists a non-negative integrable function $g\colon X \to [0, +\infty]$ such that $|f_j(x)| \leq g(x)$ for all $j \in \mathbb{N}$ and $x \in X$. Then the function f is integrable,*

$$\lim_{j\to+\infty} \int_X |f_j - f|\,d\mu = 0$$

and

$$\lim_{j\to+\infty} \int_X f_j\,d\mu = \int_X f\,d\mu.$$

Proof The function $|f|$ satisfies $|f|(x) \leq g(x)$ for all $x \in X$, and therefore $\int_X |f|\,d\mu \leq \int_X g\,d\mu < +\infty$. It follows from Proposition 8.26 that the function f is integrable on X.

Now $|f_j(x) - f(x)| \leq 2g(x)$ for all $j \in \mathbb{N}$ and $x \in X$. Moreover $\lim_{j\to+\infty} |f_j(x) - f(x)| = 0$ for all $x \in X$, and therefore

$$\lim_{j\to+\infty} (2g(x) - |f_j(x) - f(x)|) = 2g(x)$$

for all $x \in X$. It follows from Fatou's Lemma (Lemma 8.22) that

$$\begin{aligned} 2\int_X g\,d\mu &= \int_X \lim_{j\to+\infty} (2g - |f_j - f|)\,d\mu \\ &\leq \liminf_{j\to+\infty} \int_X (2g - |f_j - f|)\,d\mu \\ &= 2\int_X g\,d\mu - \limsup_{j\to+\infty} \int_X |f_j - f|\,d\mu, \end{aligned}$$

and therefore

$$\limsup_{j\to+\infty}\int_X |f_j - f|\,d\mu \leq 0.$$

But $\int_X |f_j - f|\,d\mu \geq 0$ for all positive integers j, as the integrand is non-negative everywhere on X. It follows that

$$\lim_{j\to+\infty}\int_X |f_j - f|\,d\mu = \limsup_{j\to+\infty}\int_X |f_j - f|\,d\mu = 0.$$

Now, on applying Proposition 8.26 and Lemma 8.25, we find that

$$\left|\int_X f_j\,d\mu - \int_X f\,d\mu\right| = \left|\int_X (f_j - f)\,d\mu\right| \leq \int_X |f_j - f|\,d\mu.$$

for all positive integers j. It follows that

$$\lim_{j\to+\infty}\left|\int_X f_j\,d\mu - \int_X f\,d\mu\right| = 0,$$

and therefore

$$\lim_{j\to+\infty}\int_X f_j\,d\mu = \int_X f\,d\mu,$$

as required. ∎

Corollary 8.28 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f\colon X \to \mathbb{C}$ be a function on X, let u be a positive real number, and, for each real number h satisfying $0 < h < u$, let $f_h\colon X \to \mathbb{C}$ be a measurable function on X, where the functions f and f_h take values in the field of complex numbers. Suppose that $\lim_{h\to 0} f_h(x) = f(x)$ for all $x \in X$. Suppose also that there exists a non-negative integrable function $g\colon X \to [0, +\infty]$ on X such that $|f_h(x)| \leq g(x)$ for all $x \in X$ and $h \in (0, u)$. Then the function f is integrable,*

$$\lim_{h\to 0}\int_X |f_h - f|\,d\mu = 0$$

and

$$\lim_{h\to 0}\int_X f_h\,d\mu = \int_X f\,d\mu.$$

Proof Let $F(h) = \int_X f_h\,d\mu$ for all $h \in (0, u)$, and let $l = \int_X f\,d\mu$. It follows from Lebesgue's Dominated Convergence Theorem that the function f is integrable, and $\lim_{j\to+\infty} F(a_j) = l$ for all infinite sequences $a_1, a_2, a_3, \ldots$ of real numbers in $(0, u)$ for which $\lim_{j \to +\infty} a_j = 0$. A standard result of analysis then ensures that $\lim_{h \to 0} F(h) = l$, and thus

$$\lim_{h \to 0} \int_X f_h \, d\mu = \int_X f \, d\mu.$$

Indeed, suppose it were the case that $\lim_{h \to 0} F(h)$ did not exist, or were not equal to the value l. Then there would exist a positive real number ε_0 with the property that, given any positive real number δ there would exist some $h \in (0, u)$ satisfying $0 < h < \delta$ for which $|F(h) - l| \geq \varepsilon_0$. It follows that that there would exist an infinite sequence $a_1, a_2, a_3, \ldots$ of elements of $(0, u)$ for which $0 < a_j < 1/j$ and $|F(a_j) - l| \geq \varepsilon_0$, and thus the sequence $F(a_1), F(a_2), F(a_3), \ldots$ would not converge to the limit l.

Similar reasoning shows that

$$\lim_{h \to 0} \int_X |f_h - f| \, d\mu = 0,$$

as required. ∎

The theory of integration provided by the theory of Lebesgue is both more general and more powerful than that of the Riemann integral. Consider bounded real-valued functions defined on a bounded interval in the real line. Any such interval may be regarded as a measure space, the measure being one-dimensional Lebesgue measure. On examining the definition of the Riemann integral, one can establish that those bounded real-valued function on the interval with well-defined Riemann integrals are also integrable with respect to Lebesgue measure, and moreover the value of the Lebesgue integral coincides with that of the Riemann integral. In particular the Lebesgue integrals of standard functions are those that can be computed by the usual techniques of Calculus. Indeed one can easily see that the standard proof of the Fundamental Theorem of Calculus is valid when the theory of integration is that of Lebesgue and the measure is Lebesgue measure on the real line.

Corollary 8.29 *Let I be an interval in the real line $\mathbb{R}$, let J be an open interval in $\mathbb{R}$, and let $f: I \times J \to \mathbb{C}$ be a continuously differentiable function on $I \times J$ with values in the field $\mathbb{C}$ of complex numbers. Suppose that there exists some non-negative integrable function $g: I \to [0, +\infty]$ such that*

$$\left| \frac{\partial f(x, y)}{\partial y} \right| \leq g(x)$$

for all $x \in I$ and $y \in J$. Then

$$\frac{d}{dy}\int_I f(x,y)\,dx = \int_I \frac{\partial f(x,y)}{\partial y}\,dx.$$

Proof We apply the theory of the Lebesgue integral, where the relevant measure is Lebesgue measure on the real line. Continuous functions are measurable. Moreover it follows directly from the Mean Value Theorem that

$$\left|\frac{f(x,y+h)-f(x,y)}{h}\right| \leq g(x)$$

whenever $x \in I$ and $y, y+h \in J$. It now follows from Corollary 8.28 that

$$\begin{aligned}\frac{d}{dy}\int_I f(x,y)\,dx &= \lim_{h\to 0}\int_I \frac{f(x,y+h)-f(x,y)}{h}\,dx \\ &= \int_I \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}\,dx \\ &= \int_I \frac{\partial f(x,y)}{\partial y}\,dx,\end{aligned}$$

as required. ∎

We now give an example to demonstrate that it is not always possible to interchanges integrals and limits, when conditions such as those in the statement of Lebesgue's Dominated Convergence Theorem are not satisfied.

Example Let $f_1, f_2, f_3, \ldots$ be the sequence of continuous functions on the interval $[0,1]$ defined by $f_n(x) = n(x^n - x^{2n})$. Now

$$\lim_{n\to+\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to+\infty}\left(\frac{n}{n+1} - \frac{n}{2n+1}\right) = \frac{1}{2}.$$

On the other hand, we shall show that $\lim_{n\to+\infty} f_n(x) = 0$ for all $x \in [0,1]$. Thus one cannot interchange limits and integrals in this case.

Suppose that $0 \leq x < 1$. We claim that $nx^n \to 0$ as $n \to +\infty$. To verify this, choose u satisfying $x < u < 1$. Then $0 \leq (n+1)u^{n+1} \leq nu^n$ for all n satisfying $n > u/(1-u)$. Therefore there exists some constant B with the property that $0 \leq nu^n \leq B$ for all n. But then $0 \leq nx^n \leq B(x/u)^n$ for all n, and $(x/u)^n \to 0$ as $n \to +\infty$. Therefore $nx^n \to 0$ as $n \to +\infty$, as claimed. It follows that

$$\lim_{n\to+\infty} f_n(x) = \left(\lim_{n\to+\infty} nx^n\right)\left(\lim_{n\to+\infty}(1-x^n)\right) = 0$$

for all x satisfying $0 \leq x < 1$. Also $f_n(1) = 0$ for all n. We conclude that $\lim_{n\to+\infty} f_n(x) = 0$ for all $x \in [0,1]$, which is what we set out to show.

8.6 Comparison with the Riemann integral

The theory of integration developed by Lebesgue is both more general and more powerful than that developed earlier in the nineteenth century by mathematicians such as Cauchy and Riemann. In order to compare the two theories of integration, we must first review the basic principles of the earlier theory of integration, that gives rise to the concept of the *Riemann integral* of a bounded function on an interval in the real line.

A *partition* P of an interval $[a, b]$ is a set $\{x_0, x_1, x_2, \ldots, x_n\}$ of real numbers satisfying $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.

Given any bounded real-valued function f on $[a, b]$, the *lower sum* $L(P, f)$ and the *upper sum* $U(P, f)$ of f for the partition P of $[a, b]$ are defined by

$$L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \qquad U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$. Clearly $L(P, f) \leq U(P, f)$. Moreover $\sum\limits_{i=1}^{n}(x_i - x_{i-1}) = b - a$, and therefore

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a),$$

for any real numbers m and M satisfying $m \leq f(x) \leq M$ for all $x \in [a, b]$.

Definition Let f be a bounded real-valued function on the interval $[a, b]$, where $a < b$. The *upper Riemann integral* $\mathcal{U}\int_a^b f(x)\,dx$ and the *lower Riemann integral* $\mathcal{L}\int_a^b f(x)\,dx$ of the function f on $[a, b]$ are defined by

$$\begin{aligned} \mathcal{U}\int_a^b f(x)\,dx &\equiv \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}, \\ \mathcal{L}\int_a^b f(x)\,dx &\equiv \sup\{L(P, f) : P \text{ is a partition of } [a, b]\} \end{aligned}$$

(i.e., $\mathcal{U}\int_a^b f(x)\,dx$ is the infimum of the values of $U(P, f)$ and $\mathcal{L}\int_a^b f(x)\,dx$ is the supremum of the values of $L(P, f)$ as P ranges over all possible partitions of the interval $[a, b]$). If

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx$$

then the function f is said to be *Riemann-integrable* on $[a, b]$, and the *Riemann integral* of f on $[a, b]$ is defined to be the common value of $\mathcal{U}\int_a^b f(x)\,dx$ and $\mathcal{L}\int_a^b f(x)\,dx$.

In developing the theory of the Riemann integral, one makes use of the notion of refinements of partitions. Let P and R be partitions of $[a, b]$, given by $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$. We say that the partition R is a *refinement* of P if $P \subset R$, so that, for each x_i in P, there is some u_j in R with $x_i = u_j$.

Let R be a refinement of some partition P of $[a, b]$. It is not difficult to show that Then $L(R, f) \geq L(P, f)$ and $U(R, f) \leq U(P, f)$ for any bounded function $f\colon [a, b] \to \mathbb{R}$.

Given any two partitions P and Q of $[a, b]$ there exists a partition R of $[a, b]$ which is a refinement of both P and Q. For example, we can take $R = P \cup Q$. Such a partition is said to be a *common refinement* of the partitions P and Q.

Let P and Q be partitions of $[a, b]$, and let R be a common refinement of P and Q. Then $L(P, f) \leq L(R, f) \leq U(R, f) \leq U(Q, f)$. Thus, on taking the supremum of the left hand side of the inequality $L(P, f) \leq U(Q, f)$ as P ranges over all possible partitions of the interval $[a, b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq U(Q, f)$ for all partitions Q of $[a, b]$. But then, taking the infimum of the right hand side of this inequality as Q ranges over all possible partitions of $[a, b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx$.

We now show that, if a bounded measurable function on a bounded interval is Riemann-integrable, then the value of the Riemann integral coincides with the value obtained on integrating the function with respect to Lebesgue measure on the real line, in accordance with the theory developed by Lebesgue.

Let $f\colon [a, b]$ be a bounded measurable function on a closed bounded interval $[a, b]$, and let P be a partition of $[a, b]$. Then the values of lower sum $L(P, f)$ and upper sum $U(P, f)$ are given by the formulae

$$L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \qquad U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$. Let $s\colon [a, b] \to \mathbb{R}$ be the function defined such that $s(a) = m_1$, $s(b) = m_n$, $s(x) = m_i$ when $x_{i-1} < x < x_i$ for some integer i between 1 and n, and $s(x_i) = \max(m_i, m_{i+1})$ for $i = 1, 2, \ldots, n-1$. Similarly let $t\colon [a, b] \to \mathbb{R}$ be the function defined such that $t(a) = M_1$, $T(b) = M_n$, $t(x) = M_i$ when $x_{i-1} < x < x_i$ for some integer i between 1 and n, and $t(x_i) = \min(M_i, M_{i+1})$ for $i = 1, 2, \ldots, n-1$.

We regard the interval $[a, b]$ as a measure space, where the measurable sets are the Lebesgue-measurable subsets of $[a, b]$, and the measure on $[a, b]$ is Lebesgue measure μ. Then the functions s and t are measurable simple functions on $[a,b]$. Moreover the integral of the functions s and t over the one-point set $\{t_i\}$ is zero for $i = 1, 2, \ldots, n$, and therefore the Lebesgue integrals of the functions s and t satisfy

$$\begin{aligned}\int_{[a,b]} s\,d\mu &= \sum_{i=1}^{n} \int_{(x_{i-1},x_i)} s\,d\mu = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = L(P,f),\\ \int_{[a,b]} t\,d\mu &= \sum_{i=1}^{n} \int_{(x_{i-1},x_i)} t\,d\mu = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = U(P,f).\end{aligned}$$

But $s(x) \leq f(x) \leq t(x)$ for all $x \in [a,b]$, and therefore

$$\int_{[a,b]} s\,d\mu \leq \int_{[a,b]} f\,d\mu \leq \int_{[a,b]} t\,d\mu,$$

where $\int_{[a,b]} f\,d\mu$ denotes the integral of the the function f over the interval $[a,b]$, taken with respect to Lebesgue measure on $[a,b]$ in accordance with the theory of Lebesgue. Thus $L(P,f) \leq \int_{[a,b]} f\,d\mu \leq U(P,f)$ for all partitions P of $[a,b]$. It follows from the definitions of the lower and upper Riemann integrals that

$$\mathcal{L}\int_a^b f(x)\,dx \leq \int_{[a,b]} f\,d\mu \leq \mathcal{U}\int_a^b f(x)\,dx.$$

Thus, if the function f is Riemann-integrable, the common value of its lower and upper Riemann integrals on the interval $[a,b]$ must coincide with the value of the Lebesgue integral. Thus every measurable Riemann-integrable function on $[a,b]$ is also Lebesgue-integrable, and the two theories of integration give the same value for the integral of such a function over a bounded interval in the real line.

One can also show that any bounded Riemann-integrable function on a bounded interval is guaranteed to be measurable on $[a,b]$. Moreover it can also be shown that a bounded measurable real-valued function on $[a,b]$ is Riemann-integrable if and only if there exists a subset E of $[a,b]$ such that E has Lebesgue measure zero and the restriction of f to $[a,b] \setminus E$ is continuous.