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Section 3: Complete Metric Spaces, Normed Vector Spaces and Banach Spaces

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3 Complete Metric Spaces, Normed Vector Spaces and Banach Spaces

3.1 The Least Upper Bound Principle

A set S of real numbers is said to be *bounded above* if there exists some real number B such $x \leq B$ for all $x \in S$. Similarly a set S of real numbers is said to be *bounded below* if there exists some real number A such that $x \geq A$ for all $x \in S$. A set S of real numbers is said to be *bounded* if it is bounded above and below. Thus a set S of real numbers is bounded if and only if there exist real numbers A and B such that $A \leq x \leq B$ for all $x \in S$.

Any bounded non-decreasing sequence of real numbers is convergent. This result can be proved using the *Least Upper Bound Principle*. The Least Upper Bound Principle expresses a basic property of the real number system. It states that, given any non-empty set S of real numbers that is bounded above, there exists a *least upper bound* (or *supremum*) for the set S. We shall denote the least upper bound of such a set S by $\sup S$. It is the least real number with the property that $s \leq \sup S$ for all $s \in S$.

The Least Upper Bound Principle also guarantees that, given any non-empty set S of real numbers that is bounded below, there exists a *greatest lower bound* (or *infimum*) for the set S. We shall denote the greatest lower bound of such a set S by $\inf S$. It is the greatest real number with the property that $s \geq \inf S$ for all $s \in S$. One can readily verify that

$$\inf S = -\sup\{x \in \mathbb{R} : -x \in S\}$$

for any set S of real numbers that is bounded below.

3.2 Monotonic Sequences of Real Numbers

An infinite sequence $a_1, a_2, a_3, \ldots$ of real numbers is said to be *bounded above* if the corresponding set $\{a_1, a_2, a_3, \ldots\}$ of values of the sequence is bounded above. Similarly an infinite sequence $a_1, a_2, a_3, \ldots$ of real numbers is said to be *bounded below* if the set $\{a_1, a_2, a_3, \ldots\}$ is bounded below. An infinite sequence is said to be *bounded* if it is bounded above and below. Thus an infinite sequence $a_1, a_2, a_3, \ldots$ of real numbers is bounded if and only if there exist real numbers A and B such that $A \leq a_j \leq B$ for all positive integers j.

An infinite sequence $a_1, a_2, a_3, \ldots$ is said to be *non-decreasing* if $a_{j+1} \geq a_j$ for all positive integers j. Similarly an infinite sequence $a_1, a_2, a_3, \ldots$ is said to be *non-increasing* if $a_{j+1} \leq a_j$ for all positive integers j. A sequence is said to be *monotonic* if it is non-increasing, or it is non-decreasing.

Theorem 3.1 *Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.*

Proof Let $a_1, a_2, a_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Principle that there exists a least upper bound l for the set $\{a_j : j \in \mathbb{N}\}$. We claim that the sequence converges to l.

Let $\varepsilon > 0$ be given. We must show that there exists some positive integer N such that $|a_j - l| < \varepsilon$ whenever $j \geq N$. Now $l - \varepsilon$ is not an upper bound for the set $\{a_j : j \in \mathbb{N}\}$ (since l is the least upper bound), and therefore there must exist some positive integer N such that $a_N > l - \varepsilon$. But then $l - \varepsilon < a_j \leq l$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by l. Thus $|a_j - l| < \varepsilon$ whenever $j \geq N$. Therefore $a_j \to l$ as $j \to +\infty$, as required.

If the sequence $a_1, a_2, a_3, \ldots$ is non-increasing and bounded below then the sequence $-a_1, -a_2, -a_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $a_1, a_2, a_3, \ldots$ is also convergent. ∎

3.3 Upper and Lower Limits of Bounded Sequences of Real Numbers

Let $a_1, a_2, a_3, \ldots$ be a bounded infinite sequence of real numbers, and, for each positive integer j, let

$$S_j = \{a_j, a_{j+1}, a_{j+2}, \ldots\} = \{a_k : k \geq j\}.$$

The sets $S_1, S_2, S_3, \ldots$ are all bounded. It follows that there exist well-defined infinite sequences $u_1, u_2, u_3, \ldots$ and $l_1, l_2, l_3, \ldots$ of real numbers, where $u_j = \sup S_j$ and $l_j = \inf S_j$ for all positive integers j. Now S_{j+1} is a subset of S_j for each positive integer j, and therefore $u_{j+1} \leq u_j$ and $l_{j+1} \geq l_j$ for each positive integer j. It follows that the bounded infinite sequence $(u_j : j \in \mathbb{N})$ is a non-increasing sequence, and is therefore convergent (Theorem 3.1). Similarly the bounded infinite sequence $(l_j : j \in \mathbb{N})$ is a non-decreasing sequence, and is therefore convergent. We define

$$\begin{aligned}
\limsup_{j \to +\infty} a_j &= \lim_{j \to +\infty} u_j = \lim_{j \to +\infty} \sup\{a_j, a_{j+1}, a_{j+2}, \ldots\}, \\
\liminf_{j \to +\infty} a_j &= \lim_{j \to +\infty} l_j = \lim_{j \to +\infty} \inf\{a_j, a_{j+1}, a_{j+2}, \ldots\}.
\end{aligned}$$

The quantity $\limsup_{j\to+\infty} a_j$ is referred to as the *upper limit* of the sequence $a_1, a_2, a_3, \ldots$. The quantity $\liminf_{j\to+\infty} a_j$ is referred to as the *lower limit* of the sequence $a_1, a_2, a_3, \ldots$.

Note that every bounded infinite sequence $a_1, a_2, a_3, \ldots$. of real numbers has a well-defined upper limit $\limsup_{j\to+\infty} a_j$ and a well-defined lower limit $\liminf_{j\to+\infty} a_j$.

Proposition 3.2 *A bounded infinite sequence $a_1, a_2, a_3, \ldots$. of real numbers is convergent if and only if $\liminf_{j\to+\infty} a_j = \limsup_{j\to+\infty} a_j$, in which case the limit of the sequence is equal to the common value of its upper and lower limits.*

Proof For each positive integer j, let $u_j = \sup S_j$ and $l_j = \inf S_j$, where

$$S_j = \{a_j, a_{j+1}, a_{j+2}, \ldots\} = \{a_k : k \geq j\}.$$

Then $\liminf_{j\to+\infty} a_j = \lim_{j\to+\infty} l_j$ and $\limsup_{j\to+\infty} a_j = \lim_{j\to+\infty} u_j$.

Suppose that $\liminf_{j\to+\infty} a_j = \limsup_{j\to+\infty} a_j = c$ for some real number c. Then, given any positive real number ε, there exist natural numbers N_1 and N_2 such that $c - \varepsilon < l_j \leq c$ whenever $j \geq N_1$, and $c \leq u_j \leq c + \varepsilon$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then $a_j \in S_N$, and therefore

$$c - \varepsilon < l_N \leq a_j \leq u_N < c + \varepsilon.$$

Thus $|a_j - c| < \varepsilon$ whenever $j \geq N$. This proves that the infinite sequence $a_1, a_2, a_3, \ldots$ converges to the limit c.

Conversely let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers that converges to some value c. Let $\varepsilon > 0$ be given. Then there exists some natural number N such that $c - \frac{1}{2}\varepsilon < a_j < c + \frac{1}{2}\varepsilon$ whenever $j \geq N$. It follows that $S_j \subset (c - \frac{1}{2}\varepsilon, c + \frac{1}{2}\varepsilon)$ whenever $j \geq N$. But then

$$c - \tfrac{1}{2}\varepsilon \leq l_j \leq u_j \leq c + \tfrac{1}{2}\varepsilon$$

whenever $j \geq N$, where $u_j = \sup S_j$ and $l_j = \inf S_j$. We see from this that, given any positive real number ε, there exists some natural number N such that $|l_j - c| < \varepsilon$ and $|u_j - c| < \varepsilon$ whenever $j \geq N$. It follows from this that

$$\limsup_{j\to+\infty} a_j = \lim_{j\to+\infty} u_j = c \text{ and } \liminf_{j\to+\infty} a_j = \lim_{j\to+\infty} l_j = c,$$

as required. ∎

3.4 Convergence of Sequences in Euclidean Space

Lemma 3.3 *Let* $\mathbf{p}$ *be a point of* $\mathbb{R}^n$*, where* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$*. Then a sequence* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *of points in* $\mathbb{R}^n$ *converges to* $\mathbf{p}$ *if and only if the* i*th components of the elements of this sequence converge to* p_i *for* $i = 1, 2, \ldots, n$*.*

Proof Let x_{ji} and p_i denote the ith components of $\mathbf{x}_j$ and $\mathbf{p}$. Then $|x_{ji} - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$ for all j. It follows directly from the definition of convergence that if $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ then $x_{ji} \to p_i$ as $j \to +\infty$.

Conversely suppose that, for each i, $x_{ji} \to p_i$ as $j \to +\infty$. Let $\varepsilon > 0$ be given. Then there exist natural numbers $N_1, N_2, \ldots, N_n$ such that $|x_{ji} - p_i| < \varepsilon/\sqrt{n}$ whenever $j \geq N_i$. Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^{n} (x_{ji} - p_i)^2 < n(\varepsilon/\sqrt{n})^2 = \varepsilon^2,$$

so that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. ∎

3.5 Cauchy's Criterion for Convergence

Definition An infinite sequence $a_1, a_2, a_3, \ldots$ of real numbers said to be a *Cauchy sequence* if, given any positive real number ε, there exists some positive integer N such that $|a_j - a_k| < \varepsilon$ for all j and k satisfying $j \geq N$ and $k \geq N$.

Theorem 3.4 (Cauchy's Criterion for Convergence) *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Proof Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers. Suppose that this sequence converges to some limit c. Let some positive real number ε be given. Then there exists some natural number N such that $|a_j - c| < \frac{1}{2}\varepsilon$ whenever $j \geq N$. If j and k are positive integers satisfying $j \geq N$ and $k \geq N$ then

$$|a_j - a_k| \leq |a_j - c| + |c - a_k| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

This shows that any convergent sequence of real numbers is a Cauchy sequence.

Next let $a_1, a_2, a_3, \ldots$ be a Cauchy sequence of real numbers. We must prove that this sequence is convergent. First we show that it is bounded. Now there exists some natural number M such that $|a_j - a_k| < 1$ for all positive integers j and k satisfying $j > M$ and $k > M$. Let R be the maximum of the real numbers

$$|a_1|, |a_2|, \ldots, |a_{M-1}|, |a_M| + 1.$$

It is clear that $|a_j| \leq R$ when $j < M$. If $j \geq M$ then $|a_j - a_M| < 1$, and therefore $|a_j| < |a_M| + 1 \leq R$. Thus $|a_j| \leq R$ for all positive integers j. This proves that the Cauchy sequence is bounded.

For each positive integer j, let

$$u_j = \sup\{a_k : k \geq j\} \text{ and } l_j = \inf\{a_k : k \geq j\}.$$

Then $u_1, u_2, u_3, \ldots$ is a non-increasing sequence which converges to $\limsup_{j \to +\infty} a_j$, and $l_1, l_2, l_3, \ldots$ is a non-decreasing sequence which converges to $\liminf_{j \to +\infty} a_j$.

Let ε be some given positive real number. Then there exists some natural number N such that $|a_j - a_k| < \varepsilon$ for all positive integers j and k satisfying $j \geq N$ and $k \geq N$. It follows from this that $a_N - \varepsilon < a_j < a_N + \varepsilon$ for all positive integers j satisfying $j \geq N$. It then follow from the definitions of u_N and l_N that $a_N - \varepsilon \leq l_N \leq u_N \leq a_N + \varepsilon$. Now $0 \leq u_j - l_j \leq u_N - l_N$ whenever $j \geq N$. It follows that

$$\limsup_{j \to +\infty} a_j - \liminf_{j \to +\infty} a_j = \lim_{j \to +\infty} (u_j - l_j) \leq u_N - l_N \leq 2\varepsilon.$$

Thus if $d = \limsup_{j \to +\infty} a_j - \liminf_{j \to +\infty} a_j$ then $0 \leq d \leq 2\varepsilon$ for all positive real numbers ε. It must therefore be the case that $d = 0$. Thus $\limsup_{j \to +\infty} a_j = \liminf_{j \to +\infty} a_j$. It now follows from Proposition 3.2 that the Cauchy sequence $a_1, a_2, a_3, \ldots$ is convergent, as required. ∎

An infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of n-dimensional Euclidean space $\mathbb{R}^n$ is said to be a *Cauchy sequence* if, given any positive real number ε, there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < \varepsilon$ for all j and k satisfying $j \geq N$ and $k \geq N$.

Corollary 3.5 *Every Cauchy sequence of points of n-dimensional Euclidean space $\mathbb{R}^n$ is convergent.*

Proof If an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2, \ldots$ of points in $\mathbb{R}^n$ is a Cauchy sequence, then, for each integer i between 1 and n, the ith components of those points constitute a Cauchy sequence of real numbers. But every Cauchy sequence of real numbers is convergent (Theorem 3.4). Therefore the ith components of the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2, \ldots$ converge. It then follows from Lemma 3.3 that the Cauchy sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to some point of $\mathbb{R}^n$, as required. ∎

3.6 The Bolzano-Weierstrass Theorem

Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of real numbers. A *subsequence* of this sequence is a sequence that is of the form $a_{m_1}, a_{m_2}, a_{m_3}, \ldots$, where $m_1, m_2, m_3, \ldots$ are positive integers satisfying $m_1 < m_2 < m_3 < \cdots$. Thus, for example, $a_2, a_4, a_6, \ldots$ and $a_1, a_4, a_9, \ldots$ are subsequences of the given sequence.

Lemma 3.6 *Let $a_1, a_2, a_3, \ldots$ be a bounded infinite sequence of real numbers, and let c be a real number satisfying $c < \limsup_{j \to +\infty} a_j$. Then there exist infinitely many positive integers j such that $a_j > c$.*

Proof Let N be a positive integer. Then

$$c < \limsup_{j \to +\infty} a_j \leq \sup\{a_j : j \geq N\},$$

It follows that c is not an upper bound for the set $\{a_j : j \geq N\}$, and therefore there exists some positive integer satisfying $j \geq N$ for which $a_j > c$. We conclude from this that there does not exist any positive integer N with the property that $a_j \leq c$ whenever $j \geq N$. Therefore $\{j \in \mathbb{N} : a_j > c\}$ is not a finite set. The result follows. ∎

Proposition 3.7 *Any bounded infinite sequence $a_1, a_2, a_3, \ldots$ of real numbers has a subsequence which converges to the upper limit $\limsup_{j \to +\infty} a_j$ of the given sequence.*

Proof Let $s = \limsup_{j \to +\infty} a_j$, and let

$$u_N = \sup\{a_N, a_{N+1}, a_{N+2}, \ldots\} = \sup\{a_j : j \geq N\}$$

for all positive integers N. The upper limit s of the sequence $a_1, a_2, a_3, \ldots$ is then the limit of the non-increasing sequence $u_1, u_2, u_3, \ldots$.

Let ε be positive real number. The convergence of the infinite sequence $u_1, u_2, u_3, \ldots$ to s ensures that there exists some positive integer N such that $u_N < s + \varepsilon$. But then $a_j < s + \varepsilon$ whenever $j \geq N$. It follows that the number of positive integers j for which $a_j \geq s + \varepsilon$ is finite. Also it follows from Lemma 3.6 that the number of positive integers j for which $a_j > s - \varepsilon$ is infinite. Putting these two facts together, we see that the number of positive integers j for which $s - \varepsilon < a_j < s + \varepsilon$ is infinite. (Indeed let $S_1 = \{j \in \mathbb{N} : a_j > s - \varepsilon\}$ and $S_2 = \{j \in \mathbb{N} : a_j \geq s + \varepsilon\}$. Then S_1 is an infinite set, S_2 is a finite set, and therefore $S_1 \setminus S_2$ is an infinite set. Moreover $s - \varepsilon < a_j < s + \varepsilon$ for all $j \in S_1 \setminus S_2$.)

Now given any positive integer j, and given any positive number m_j such that $|a_{m_j} - s| < j^{-1}$, there exists some positive integer m_{j+1} such that $m_{j+1} > m_j$ and $|a_{m_{j+1}} - s| < (j+1)^{-1}$. It follow from this that there exists a subsequence $a_{m_1}, a_{m_2}, a_{m_3}, \ldots$ of the infinite sequence $a_1, a_2, a_3, \ldots$, where $m_1 < m_2 < m_3 < \cdots$, which has the property that $|a_{m_j} - s| < j^{-1}$ for all positive integers j. This subsequence converges to s as required. ∎

The following theorem, known as the *Bolzano-Weierstrass Theorem*, is an immediate consequence of Proposition 3.7.

Theorem 3.8 (Bolzano-Weierstrass) *Every bounded sequence of real numbers has a convergent subsequence.*

The following result is the analogue of the Bolzano-Weierstrass Theorem for sequences in n-dimensional Euclidean space.

Corollary 3.9 *Every bounded sequence of points in $\mathbb{R}^n$ has a convergent subsequence.*

Proof Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a bounded sequence of points in $\mathbb{R}^n$. Let us denote by $x_j^{(i)}$ the ith component of the point $\mathbf{x}_j$, so that

$$\mathbf{x}_j = (x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(n)})$$

for all positive integers j. Suppose that, for some integer s between 1 and $n-1$, the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ has a subsequence $\mathbf{x}_{p_1}, \mathbf{x}_{p_2}, \mathbf{x}_{p_3}, \ldots$ with the property that, for each integer i satisfying $1 \leq i \leq s$, the ith components of the members of this subsequence constitute a convergent sequence $x_{p_1}^{(i)}, x_{p_2}^{(i)}, x_{p_3}^{(i)}, \ldots$ of real numbers. Let $a_j = x_{p_j}^{(s+1)}$ for each positive integer j. Then $a_1, a_2, a_3, \ldots$ is a bounded sequence of real numbers. It follows from the Bolzano-Weierstrass Theorem (Theorem 3.8) that this sequence has a convergent subsequence $a_{m_1}, a_{m_2}, a_{m_3}, \ldots$, where $m_1 < m_2 < m_3 < \cdots$. Let $q_j = p_{m_j}$ for each positive integer j. Then $\mathbf{x}_{q_1}, \mathbf{x}_{q_2}, \mathbf{x}_{q_3}, \ldots$ is a subsequence of the original bounded sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ which has the property that, for each integer i satisfying $1 \leq i \leq s+1$, the ith components of the members of the subsequence constitute a convergent sequence $x_{q_1}^{(i)}, x_{q_2}^{(i)}, x_{q_3}^{(i)}, \ldots$ of real numbers.

Repeated applications of this result show that the bounded sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ has a subsequence $\mathbf{x}_{r_1}, \mathbf{x}_{r_2}, \mathbf{x}_{r_3}, \ldots$ with the property that, for each integer i satisfying $1 \leq i \leq n$, the ith components of the members of the subsequence constitute a convergent sequence of real numbers. Let $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ where, for each value of i between 1 and n, the ith component z_i of $\mathbf{z}$ is the limit of the sequence $x_{r_1}^{(i)}, x_{r_2}^{(i)}, x_{r_3}^{(i)}, \ldots$ of ith components of the members of the subsequence $\mathbf{x}_{r_1}, \mathbf{x}_{r_2}, \mathbf{x}_{r_3}, \ldots$. Then this subsequence converges to the point $\mathbf{z}$, as required. ∎

3.7 Complete Metric Spaces

Definition Let X be a metric space with distance function d. A sequence $x_1, x_2, x_3, \ldots$ of points of X is said to be a *Cauchy sequence* in X if and only if, given any $\varepsilon > 0$, there exists some positive integer N such that $d(x_j, x_k) < \varepsilon$ for all j and k satisfying $j \geq N$ and $k \geq N$.

Every convergent sequence in a metric space is a Cauchy sequence. Indeed let X be a metric space with distance function d, and let $x_1, x_2, x_3, \ldots$ be a sequence of points in X which converges to some point p of X. Given any positive real number ε, there exists some positive integer N such that $d(x_n, p) < \varepsilon/2$ whenever $n \geq N$. But then it follows from the Triangle Inequality that

$$d(x_j, x_k) \leq d(x_j, p) + d(p, x_k) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

whenever $j \geq N$ and $k \geq N$.

Definition A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges to some point of X.

The spaces $\mathbb{R}$ and $\mathbb{C}$ are complete metric spaces with respect to the distance function given by $d(z, w) = |z - w|$. Indeed this result is *Cauchy's Criterion for Convergence*. However the space $\mathbb{Q}$ of rational numbers (with distance function $d(q, r) = |q - r|$) is not complete. Indeed one can construct an infinite sequence $q_1, q_2, q_3, \ldots$ of rational numbers which converges (in $\mathbb{R}$) to $\sqrt{2}$. Such a sequence of rational numbers is a Cauchy sequence in both $\mathbb{R}$ and $\mathbb{Q}$. However this Cauchy sequence does not converge to an point of the metric space $\mathbb{Q}$ (since $\sqrt{2}$ is an irrational number). Thus the metric space $\mathbb{Q}$ is not complete.

It follows immediately from Corollary 3.5 that n-dimensional Euclidean space $\mathbb{R}^n$ is a complete metric space.

Lemma 3.10 *Let X be a complete metric space, and let A be a subset of X. Then A is complete if and only if A is closed in X.*

Proof Suppose that A is closed in X. Let $a_1, a_2, a_3, \ldots$ be a Cauchy sequence in A. This Cauchy sequence must converge to some point p of X, since X is complete. But the limit of every sequence of points of A must belong to A, since A is closed. In particular $p \in A$. We deduce that A is complete.

Conversely, suppose that A is complete. Suppose that A were not closed. Then the complement $X \setminus A$ of A would not be open, and therefore there would exist a point p of $X \setminus A$ with the property that $B_X(p, \delta) \cap A$ is non-empty for all $\delta > 0$, where $B_X(p, \delta)$ denotes the open ball in X of radius δ centred at p. We could then find a sequence $a_1, a_2, a_3, \ldots$ of points of A satisfying $d(a_j, p) < 1/j$ for all positive integers j. This sequence would be a Cauchy sequence in A which did not converge to a point of A, contradicting the completeness of A. Thus A must be closed, as required. ∎

Theorem 3.11 *The metric space $\mathbb{R}^n$ (with the Euclidean distance function) is a complete metric space.*

Proof Let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots$ be a Cauchy sequence in $\mathbb{R}^n$. Then for each integer m between 1 and n, the sequence $(\mathbf{p}_1)_m, (\mathbf{p}_2)_m, (\mathbf{p}_3)_m, \ldots$ is a Cauchy sequence of real numbers, where $(\mathbf{p}_j)_m$ denotes the mth component of $\mathbf{p}_j$. But every Cauchy sequence of real numbers is convergent (Cauchy's criterion for convergence). Let $q_m = \lim_{j \to +\infty} (\mathbf{p}_j)_m$ for $m = 1, 2, \ldots, n$, and let $\mathbf{q} = (q_1, q_2, \ldots, q_n)$. We claim that $\mathbf{p}_j \to \mathbf{q}$ as $j \to +\infty$.

Let $\varepsilon > 0$ be given. Then there exist positive integers $N_1, N_2, \ldots, N_n$ such that $|(\mathbf{p}_j)_m - q_m| < \varepsilon/\sqrt{n}$ whenever $j \geq N_m$ (where $m = 1, 2, \ldots, n$). Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then

$$|\mathbf{p}_j - \mathbf{q}|^2 = \sum_{m=1}^{n} ((\mathbf{p}_j)_m - q_m)^2 < \varepsilon^2.$$

Thus $\mathbf{p}_j \to \mathbf{q}$ as $j \to +\infty$. Thus every Cauchy sequence in $\mathbb{R}^n$ is convergent, as required. ∎

The following result follows directly from Lemma 3.10 and Theorem 3.11.

Corollary 3.12 *A subset X of $\mathbb{R}^n$ is complete if and only if it is closed.*

Example The n-sphere S^n (with the chordal distance function given by $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$) is a complete metric space, where

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}.$$

3.8 Normed Vector Spaces

A set X is a *vector space* over some field $\mathbb{F}$ if

- given any $x, y \in X$ and $\lambda \in \mathbb{F}$, there are well-defined elements $x + y$ and λx of X,
- X is an Abelian group with respect to the operation $+$ of addition,
- the identities

$$\lambda(x + y) = \lambda x + \lambda y, \qquad (\lambda + \mu)x = \lambda x + \mu x,$$

$$(\lambda\mu)x = \lambda(\mu x), \qquad 1x = x$$

  are satisfied for all $x, y \in X$ and $\lambda, \mu \in \mathbb{F}$.

Elements of the field $\mathbb{F}$ are referred to as *scalars*. We consider here only *real vector spaces* and *complex vector spaces*: these are vector spaces over the fields of real numbers and complex numbers respectively.

Definition A *norm* $\|.\|$ on a real or complex vector space X is a function, associating to each element x of X a corresponding real number $\|x\|$, such that the following conditions are satisfied:—

(i) $\|x\| \geq 0$ for all $x \in X$,

(ii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$,

(iii) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in X$ and for all scalars λ,

(iv) $\|x\| = 0$ if and only if $x = 0$.

A *normed vector space* $(X, \|.\|)$ consists of a a real or complex vector space X, together with a norm $\|.\|$ on X.

Note that any normed complex vector space can also be regarded as a normed real vector space.

Example The field $\mathbb{R}$ is a one-dimensional normed vector space over itself: the norm $|t|$ of $t \in \mathbb{R}$ is the absolute value of t.

Example The field $\mathbb{C}$ is a one-dimensional normed vector space over itself: the norm $|z|$ of $z \in \mathbb{C}$ is the modulus of z. The field $\mathbb{C}$ is also a two-dimensional normed vector space over $\mathbb{R}$.

Example Let $\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ be the real-valued functions on $\mathbb{C}^n$ defined by

$$\begin{aligned}\|\mathbf{z}\|_1 &= \sum_{j=1}^{n} |z_j|,\\ \|\mathbf{z}\|_2 &= \left(\sum_{j=1}^{n} |z_j|^2\right)^{\frac{1}{2}},\\ \|\mathbf{z}\|_\infty &= \max(|z_1|, |z_2|, \ldots, |z_n|),\end{aligned}$$

for each $\mathbf{z} \in \mathbb{C}^n$, where $\mathbf{z} = (z_1, z_2, \ldots, z_n)$. Then $\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ are norms on $\mathbb{C}^n$. In particular, if we regard $\mathbb{C}^n$ as a $2n$-dimensional real vector space naturally isomorphic to $\mathbb{R}^{2n}$ (via the isomorphism

$$(z_1, z_2, \ldots, z_n) \mapsto (x_1, y_1, x_2, y_2, \ldots, x_n, y_n),$$

where x_j and y_j are the real and imaginary parts of z_j for $j = 1, 2, \ldots, n$) then $\|.\|_2$ represents the Euclidean norm on this space. The inequality $\|\mathbf{z}+\mathbf{w}\|_2 \leq \|\mathbf{z}\|_2 + \|\mathbf{w}\|_2$ satisfied for all $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$ is therefore just the standard Triangle Inequality for the Euclidean norm.

Example The space $\mathbb{R}^n$ is also an n-dimensional real normed vector space with respect to the norms $\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ defined above. Note that $\|.\|_2$ is the standard Euclidean norm on $\mathbb{R}^n$.

Example Let

$$\begin{aligned}\ell_1 &= \{(z_1, z_2, z_3, \ldots) \in \mathbb{C}^\infty : |z_1| + |z_2| + |z_3| + \cdots \text{ converges}\},\\ \ell_2 &= \{(z_1, z_2, z_3, \ldots) \in \mathbb{C}^\infty : |z_1|^2 + |z_2|^2 + |z_3|^2 + \cdots \text{ converges}\},\\ \ell_\infty &= \{(z_1, z_2, z_3, \ldots) \in \mathbb{C}^\infty : \text{the sequence } |z_1|, |z_2|, |z_3|, \ldots \text{ is bounded}\}.\end{aligned}$$

where $\mathbb{C}^\infty$ denotes the set of all sequences $(z_1, z_2, z_3, \ldots)$ of complex numbers. Then ℓ_1, ℓ_2 and ℓ_∞ are infinite-dimensional normed vector spaces, with norms $\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ respectively, where

$$\begin{aligned}\|\mathbf{z}\|_1 &= \sum_{j=1}^{+\infty} |z_j|,\\ \|\mathbf{z}\|_2 &= \left(\sum_{j=1}^{+\infty} |z_j|^2\right)^{\frac{1}{2}},\\ \|\mathbf{z}\|_\infty &= \sup\{|z_1|, |z_2|, |z_3|, \ldots\}.\end{aligned}$$

(For example, to show that $\|\mathbf{z} + \mathbf{w}\|_2 \leq \|\mathbf{z}\|_2 + \|\mathbf{w}\|_2$ for all $\mathbf{z}, \mathbf{w} \in \ell_2$, we note that

$$\left(\sum_{j=1}^{n} |z_j + w_j|^2\right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^{n} |z_j|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} |w_j|^2\right)^{\frac{1}{2}} \leq \|\mathbf{z}\|_2 + \|\mathbf{w}\|_2$$

for all positive integers n, by the Triangle Inequality in $\mathbb{C}^n$. Taking limits as $n \to +\infty$, we deduce that $\|\mathbf{z} + \mathbf{w}\|_2 \leq \|\mathbf{z}\|_2 + \|\mathbf{w}\|_2$, as required.)

If $x_1, x_2, \ldots, x_m$ are elements of a normed vector space X then

$$\left\| \sum_{k=1}^{m} x_k \right\| \leq \sum_{k=1}^{m} \|x_k\|,$$

where $\|.\|$ denotes the norm on X. (This can be verified by induction on m, using the inequality $\|x + y\| \leq \|x\| + \|y\|$.)

A norm $\|.\|$ on a vector space X induces a corresponding distance function on X: the distance $d(x, y)$ between elements x and y of X is defined by $d(x, y) = \|x - y\|$. This distance function satisfies the metric space axioms. Thus any vector space with a given norm can be regarded as a metric space.

Lemma 3.13 *Let X be a normed vector space over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let (x_j) and (y_j) be convergent sequences in X, and let (λ_j) be a convergent sequence in $\mathbb{F}$. Then the sequences $(x_j + y_j)$ and $(\lambda_j x_j)$ are convergent in X, and*

$$\begin{aligned} \lim_{j \to +\infty} (x_j + y_j) &= \lim_{j \to +\infty} x_j + \lim_{j \to +\infty} y_j, \\ \lim_{j \to +\infty} (\lambda_j x_j) &= \left(\lim_{j \to +\infty} \lambda_j\right) \left(\lim_{j \to +\infty} x_j\right). \end{aligned}$$

Proof First we prove that $\lim_{j \to +\infty} (x_j + y_j) = x + y$, where Let $x = \lim_{j \to +\infty} x_j$, $y = \lim_{j \to +\infty} y_j$. Let $\varepsilon > 0$ be given. Then there exist natural numbers N_1 and N_2 such that $\|x_j - x\| < \frac{1}{2}\varepsilon$ whenever $j \geq N_1$ and $\|y_j - y\| < \frac{1}{2}\varepsilon$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then

$$\|(x_j + y_j) - (x + y)\| \leq \|x_j - x\| + \|y_j - y\| < \varepsilon.$$

It follows from this that $\lim_{j \to +\infty} (x_j + y_j) = x + y$.

Next we prove that $\lim_{j \to +\infty} (\lambda_j x_j) = \lambda x$, where $\lambda = \lim_{j \to +\infty} \lambda_j$. Let $\varepsilon > 0$ be given. Then there exist natural numbers N_3 and N_4 such that

$$\|x_j - x\| < \frac{\varepsilon}{2(|\lambda| + 1)}$$

whenever $j \geq N_3$, and

$$|\lambda_j - \lambda| < \frac{\varepsilon}{2(\|x\| + 1)} \text{ and } |\lambda_j - \lambda| < 1$$

whenever $j \geq N_4$. Let N be the maximum of N_3 and N_4. if $j \geq N$ then

$$\begin{aligned} \|\lambda_j x_j - \lambda x\| &= \|\lambda_j (x_j - x) + (\lambda_j - \lambda) x\| \leq |\lambda_j| \, \|x_j - x\| + |\lambda_j - \lambda| \|x\| \\ &\leq (|\lambda| + 1) \, \|x_j - x\| + |\lambda_j - \lambda| \|x\| < \varepsilon. \end{aligned}$$

It follows from this that $\lim_{j \to +\infty} (\lambda_j x_j) = \lambda x$, as required. ∎

Let X be a normed vector space, and let $x_1, x_2, x_3, \ldots$ be elements of X. The infinite series $\sum_{n=1}^{+\infty} x_n$ is said to *converge* to some element s of X if, given any positive real number ε, there exists some positive integer N such that

$$\|s - \sum_{n=1}^{m} x_n\| < \varepsilon$$

for all $m \geq N$ (where $\|.\|$ denotes the norm on X).

We say that a normed vector space X is *complete* A normed vector space is complete if and only if every Cauchy sequence in X is convergent. A complete normed vector space is referred to as a *Banach space*. (The basic theory of such spaces was extensively developed by the famous Polish mathematician Stefan Banach and his colleagues.)

Lemma 3.14 *Let X be a Banach space, and let $x_1, x_2, x_3, \ldots$ be elements of X. Suppose that $\sum_{n=1}^{+\infty} \|x_n\|$ is convergent. Then $\sum_{n=1}^{+\infty} x_n$ is convergent, and*

$$\left\| \sum_{n=1}^{+\infty} x_n \right\| \leq \sum_{n=1}^{+\infty} \|x_n\|.$$

Proof For each positive integer n, let

$$s_n = x_1 + x_2 + \cdots + x_n.$$

Let $\varepsilon > 0$ be given. We can find N such that $\sum_{n=N}^{+\infty} \|x_n\| < \varepsilon$, since $\sum_{n=1}^{+\infty} \|x_n\|$ is convergent. Let $s_n = x_1 + x_2 + \cdots + x_n$. If $j \geq N$, $k \geq N$ and $j < k$ then

$$\|s_k - s_j\| = \left\| \sum_{n=j+1}^{k} x_n \right\| \leq \sum_{n=j+1}^{k} \|x_n\| \leq \sum_{n=N}^{+\infty} \|x_n\| < \varepsilon.$$

Thus $s_1, s_2, s_3, \ldots$ is a Cauchy sequence in X, and therefore converges to some element s of X, since X is complete. But then $s = \sum_{j=1}^{+\infty} x_j$. Moreover, on choosing m large enough to ensure that $\|s - s_m\| < \varepsilon$, we deduce that

$$\|s\| \leq \left\| \sum_{n=1}^{m} x_n \right\| + \left\| s - \sum_{n=1}^{m} x_n \right\| \leq \sum_{n=1}^{m} \|x_n\| + \left\| s - \sum_{n=1}^{m} x_n \right\| < \sum_{n=1}^{+\infty} \|x_n\| + \varepsilon.$$

Since this inequality holds for all $\varepsilon > 0$, we conclude that

$$\|s\| \leq \sum_{n=1}^{+\infty} \|x_n\|,$$

as required. ∎

3.9 Bounded Linear Transformations

Let X and Y be real or complex vector spaces. A function $T\colon X \to Y$ is said to be a *linear transformation* if $T(x + y) = Tx + Ty$ and $T(\lambda x) = \lambda Tx$ for all elements x and y of X and scalars λ. A linear transformation mapping X into itself is referred to as a *linear operator* on X.

Definition Let X and Y be normed vector spaces. A linear transformation $T\colon X \to Y$ is said to be *bounded* if there exists some non-negative real number C with the property that $\|Tx\| \leq C\|x\|$ for all $x \in X$. If T is bounded, then the smallest non-negative real number C with this property is referred to as the *operator norm* of T, and is denoted by $\|T\|$.

Lemma 3.15 *Let X and Y be normed vector spaces, and let $S\colon X \to Y$ and $T\colon X \to Y$ be bounded linear transformations. Then $S + T$ and λS are bounded linear transformations for all scalars λ, and*

$$\|S + T\| \leq \|S\| + \|T\|, \qquad \|\lambda S\| = |\lambda|\|S\|.$$

Moreover $\|S\| = 0$ if and only if $S = 0$. Thus the vector space $B(X, Y)$ of bounded linear transformations from X to Y is a normed vector space (with respect to the operator norm).

Proof $\|(S+T)x\| \leq \|Sx\|+\|Tx\| \leq (\|S\|+\|T\|)\|x\|$ for all $x \in X$. Therefore $S+T$ is bounded, and $\|S+T\| \leq \|S\|+\|T\|$. Using the fact that $\|(\lambda S)x\| = |\lambda|\,\|Sx\|$ for all $x \in X$, we see that λS is bounded, and $\|\lambda S\| = |\lambda|\,\|S\|$. If $S = 0$ then $\|S\| = 0$. Conversely if $\|S\| = 0$ then $\|Sx\| \leq \|S\|\,\|x\| = 0$ for all $x \in X$, and hence $S = 0$. The result follows. ∎

Lemma 3.16 *Let X, Y and Z be normed vector spaces, and let $S: X \to Y$ and $T: Y \to Z$ be bounded linear transformations. Then the composition TS of S and T is also bounded, and $\|TS\| \leq \|T\| \, \|S\|$.*

Proof $\|TSx\| \leq \|T\| \, \|Sx\| \leq \|T\| \, \|S\| \, \|x\|$ for all $x \in X$. The result follows. ∎

Proposition 3.17 *Let X and Y be normed vector spaces, and let $T: X \to Y$ be a linear transformation from X to Y. Then the following conditions are equivalent:—*

(i) *$T: X \to Y$ is continuous,*

(ii) *$T: X \to Y$ is continuous at 0,*

(iii) *$T: X \to Y$ is bounded.*

Proof Obviously (i) implies (ii). We show that (ii) implies (iii) and (iii) implies (i). The equivalence of the three conditions then follows immediately.

Suppose that $T: X \to Y$ is continuous at 0. Then there exists $\delta > 0$ such that $\|Tx\| < 1$ for all $x \in X$ satisfying $\|x\| < \delta$. Let C be any positive real number satisfying $C > 1/\delta$. If x is any non-zero element of X then $\|\lambda x\| < \delta$, where $\lambda = 1/(C\|x\|)$, and hence

$$\|Tx\| = C\|x\| \, \|\lambda Tx\| = C\|x\| \, \|T(\lambda x)\| < C\|x\|.$$

Thus $\|Tx\| \leq C\|x\|$ for all $x \in X$, and hence $T: X \to Y$ is bounded. Thus (ii) implies (iii).

Finally suppose that $T: X \to Y$ is bounded. Let x be a point of X, and let $\varepsilon > 0$ be given. Choose $\delta > 0$ satisfying $\|T\|\delta < \varepsilon$. If $x' \in X$ satisfies $\|x' - x\| < \delta$ then

$$\|Tx' - Tx\| = \|T(x' - x)\| \leq \|T\| \, \|x' - x\| < \|T\|\delta < \varepsilon.$$

Thus $T: X \to Y$ is continuous. Thus (iii) implies (i), as required. ∎

Proposition 3.18 *Let X be a normed vector space and let Y be a Banach space. Then the space $B(X, Y)$ of bounded linear transformations from X to Y is also a Banach space.*

Proof We have already shown that $B(X, Y)$ is a normed vector space (see Lemma 3.15). Thus it only remains to show that $B(X, Y)$ is complete.

Let $S_1, S_2, S_3, \ldots$ be a Cauchy sequence in $B(X, Y)$. Let $x \in X$. We claim that $S_1 x, S_2 x, S_3 x, \ldots$ is a Cauchy sequence in Y. This result is trivial if $x = 0$. If $x \neq 0$, and if $\varepsilon > 0$ is given then there exists some positive integer N such that $\|S_j - S_k\| < \varepsilon/\|x\|$ whenever $j \geq N$ and $k \geq N$. But then $\|S_j x - S_k x\| \leq \|S_j - S_k\| \, \|x\| < \varepsilon$ whenever $j \geq N$ and $k \geq N$. This shows that $S_1 x, S_2 x, S_3 x, \ldots$ is indeed a Cauchy sequence. It therefore converges to some element of Y, since Y is a Banach space.

Let the function $S\colon X \to Y$ be defined by $Sx = \lim_{n \to +\infty} S_n x$. Then

$$S(x+y) = \lim_{n \to +\infty} (S_n x + S_n y) = \lim_{n \to +\infty} S_n x + \lim_{n \to +\infty} S_n y = Sx + Sy,$$

(see Lemma 3.13), and

$$S(\lambda x) = \lim_{n \to +\infty} S_n(\lambda x) = \lambda \lim_{n \to +\infty} S_n x = \lambda S x,$$

Thus $S\colon X \to Y$ is a linear transformation.

Next we show that $S_n \to S$ in $B(X,Y)$ as $n \to +\infty$. Let $\varepsilon > 0$ be given. Then there exists some positive integer N such that $\|S_j - S_n\| < \frac{1}{2}\varepsilon$ whenever $j \geq N$ and $n \geq N$, since the sequence $S_1, S_2, S_3, \ldots$ is a Cauchy sequence in $B(X,Y)$. But then $\|S_j x - S_n x\| \leq \frac{1}{2}\varepsilon \|x\|$ for all $j \geq N$ and $n \geq N$, and thus

$$\begin{aligned} \|Sx - S_n x\| &= \left\| \lim_{j \to +\infty} (S_j x - S_n x) \right\| \leq \lim_{j \to +\infty} \|S_j x - S_n x\| \\ &\leq \lim_{j \to +\infty} \|S_j - S_n\| \, \|x\| \leq \tfrac{1}{2}\varepsilon \|x\| \end{aligned}$$

for all $n \geq N$ (since the norm is a continuous function on Y). But then

$$\|Sx\| \leq \|S_n x\| + \|Sx - S_n x\| \leq \left(\|S_n\| + \tfrac{1}{2}\varepsilon \right) \|x\|$$

for any $n \geq N$, showing that $S\colon X \to Y$ is a bounded linear transformation, and $\|S - S_n\| \leq \frac{1}{2}\varepsilon < \varepsilon$ for all $n \geq N$, showing that $S_n \to S$ in $B(X,Y)$ as $n \to +\infty$. Thus the Cauchy sequence $S_1, S_2, S_3, \ldots$ is convergent in $B(X,Y)$, as required. ∎

Corollary 3.19 *Let X and Y be Banach spaces, and let $T_1, T_2, T_3, \ldots$ be bounded linear transformations from X to Y. Suppose that $\sum_{n=0}^{+\infty} \|T_n\|$ is convergent. Then $\sum_{n=0}^{+\infty} T_n$ is convergent, and*

$$\left\| \sum_{n=0}^{+\infty} T_n \right\| \leq \sum_{n=0}^{+\infty} \|T_n\|.$$

Proof The space $B(X, Y)$ of bounded linear maps from X to Y is a Banach space by Proposition 3.18. The result therefore follows immediately on applying Lemma 3.14. ∎

Example Let T be a bounded linear operator on a Banach space X (i.e., a bounded linear transformation from X to itself). The infinite series

$$\sum_{n=0}^{+\infty} \frac{\|T\|^n}{n!}$$

converges to $\exp(\|T\|)$. It follows immediately from Lemma 3.16 (using induction on n) that $\|T^n\| \leq \|T\|^n$ for all $n \geq 0$ (where T^0 is the identity operator on X). It therefore follows from Corollary 3.19 that there is a well-defined bounded linear operator $\exp T$ on X, defined by

$$\exp T = \sum_{n=0}^{+\infty} \frac{1}{n!} T^n$$

(where T^0 is the identity operator I on X).

Proposition 3.20 *Let T be a bounded linear operator on a Banach space X. Suppose that $\|T\| < 1$. Then the operator $I - T$ has a bounded inverse $(I - T)^{-1}$ (where I denotes the identity operator on X). Moreover*

$$(I - T)^{-1} = I + T + T^2 + T^3 + \cdots.$$

Proof $\|T^n\| \leq \|T\|^n$ for all n, and the geometric series

$$1 + \|T\| + \|T\|^2 + \|T\|^3 + \cdots$$

is convergent (since $\|T\| < 1$). It follows from Corollary 3.19 that the infinite series

$$I + T + T^2 + T^3 + \cdots$$

converges to some bounded linear operator S on X. Now

$$\begin{aligned} (I - T)S &= \lim_{n \to +\infty} (I - T)(I + T + T^2 + \cdots + T^n) = \lim_{n \to +\infty} (I - T^{n+1}) \\ &= I - \lim_{n \to +\infty} T^{n+1} = I, \end{aligned}$$

since $\|T\|^{n+1} \to 0$ and therefore $T^{n+1} \to 0$ as $n \to +\infty$. Similarly $S(I - T) = I$. This shows that $I - T$ is invertible, with inverse S, as required. ∎

3.10 Spaces of Bounded Continuous Functions on a Metric Space

Let X be a metric space. We say that a function $f\colon X \to \mathbb{R}^n$ from X to $\mathbb{R}^n$ is *bounded* if there exists some non-negative constant K such that $|f(x)| \leq K$ for all $x \in X$. If f and g are bounded continuous functions from X to $\mathbb{R}^n$, then so is $f + g$. Also λf is bounded and continuous for any real number λ. It follows from this that the space $C(X, \mathbb{R}^n)$ of bounded continuous functions from X to $\mathbb{R}^n$ is a vector space over $\mathbb{R}$. Given $f \in C(X, \mathbb{R}^n)$, we define the *supremum norm* $\|f\|$ of f by the formula

$$\|f\| = \sup_{x \in X} |f(x)|.$$

One can readily verify that $\|.\|$ is a norm on the vector space $C(X, \mathbb{R}^n)$. We shall show that $C(X, \mathbb{R}^n)$, with the supremum norm, is a Banach space (i.e., the supremum norm on $C(X, \mathbb{R}^n)$ is complete). The proof of this result will make use of the following characterization of continuity for functions whose range is $\mathbb{R}^n$.

Theorem 3.21 *The normed vector space $C(X, \mathbb{R}^n)$ of all bounded continuous functions from some metric space X to $\mathbb{R}^n$, with the supremum norm, is a Banach space.*

Proof Let $f_1, f_2, f_3, \ldots$ be a Cauchy sequence in $C(X, \mathbb{R}^n)$. Then, for each $x \in X$, the sequence $f_1(x), f_2(x), f_3(x), \ldots$ is a Cauchy sequence in $\mathbb{R}^n$ (since $|f_j(x) - f_k(x)| \leq \|f_j - f_k\|$ for all positive integers j and k), and $\mathbb{R}^n$ is a complete metric space. Thus, for each $x \in X$, the sequence $f_1(x), f_2(x), f_3(x), \ldots$ converges to some point $f(x)$ of $\mathbb{R}^n$. We must show that the limit function f defined in this way is bounded and continuous.

Let $\varepsilon > 0$ be given. Then there exists some positive integer N with the property that $\|f_j - f_k\| < \frac{1}{3}\varepsilon$ for all $j \geq N$ and $k \geq N$, since $f_1, f_2, f_3, \ldots$ is a Cauchy sequence in $C(X, \mathbb{R}^n)$. But then, on taking the limit of the left hand side of the inequality $|f_j(x) - f_k(x)| < \frac{1}{3}\varepsilon$ as $k \to +\infty$, we deduce that $|f_j(x) - f(x)| \leq \frac{1}{3}\varepsilon$ for all $x \in X$ and $j \geq N$. In particular $|f_N(x) - f(x)| \leq \frac{1}{3}\varepsilon$ for all $x \in X$. It follows that $|f(x)| \leq \|f_N\| + \frac{1}{3}\varepsilon$ for all $x \in X$, showing that the limit function f is bounded.

Next we show that the limit function f is continuous. Let $p \in X$ and $\varepsilon > 0$ be given. Let N be chosen large enough to ensure that $|f_N(x) - f(x)| \leq \frac{1}{3}\varepsilon$ for all $x \in X$. Now f_N is continuous. It follows from the definition of continuity for functions between metric spaces that there exists some real number δ satisfying $\delta > 0$ such that $|f_N(x) - f_N(p)| < \frac{1}{3}\varepsilon$ for all elements x of X

satisfying $d_X(x,p) < \delta$, where d_X denotes the distance function on X. Thus if $x \in X$ satisfies $d_X(x,p) < \delta$ then

$$\begin{aligned} |f(x) - f(p)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(p)| + |f_N(p) - f(p)| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon. \end{aligned}$$

Therefore the limit function f is continuous. Thus $f \in C(X, \mathbb{R}^n)$.

Finally we observe that $f_j \to f$ in $C(X, \mathbb{R}^n)$ as $j \to +\infty$. Indeed we have already seen that, given $\varepsilon > 0$ there exists some positive integer N such that $|f_j(x) - f(x)| \leq \frac{1}{3}\varepsilon$ for all $x \in X$ and for all $j \geq N$. Thus $\|f_j - f\| \leq \frac{1}{3}\varepsilon < \varepsilon$ for all $j \geq N$, showing that $f_j \to f$ in $C(X, \mathbb{R}^n)$ as $j \to +\infty$. This shows that $C(X, \mathbb{R}^n)$ is a complete metric space, as required. ∎

Corollary 3.22 *Let X be a metric space and let F be a closed subset of $\mathbb{R}^n$. Then the space $C(X, F)$ of bounded continuous functions from X to F is a complete metric space with respect to the distance function ρ, where*

$$\rho(f, g) = \|f - g\| = \sup_{x \in X} |f(x) - g(x)|$$

for all $f, g \in C(X, F)$.

Proof Let $f_1, f_2, f_3, \ldots$ be a Cauchy sequence in $C(X, F)$. Then $f_1, f_2, f_3, \ldots$ is a Cauchy sequence in $C(X, \mathbb{R}^n)$ and therefore converges in $C(X, \mathbb{R}^n)$ to some function $f\colon X \to \mathbb{R}^n$. Let x be some point of X. Then $f_j(x) \to f(x)$ as $j \to +\infty$. But then $f(x) \in F$, since $f_j(x) \in F$ for all j, and F is closed in $\mathbb{R}^n$. This shows that $f \in C(X, F)$, and thus the Cauchy sequence $f_1, f_2, f_3, \ldots$ converges in $C(X, F)$. We conclude that $C(X, F)$ is a complete metric space, as required. ∎

3.11 The Contraction Mapping Theorem and Picard's Theorem

Let X be a metric space with distance function d. A function $T\colon X \to X$ mapping X to itself is said to be a *contraction mapping* if there exists some constant λ satisfying $0 \leq \lambda < 1$ with the property that $d(T(x), T(x')) \leq \lambda d(x, x')$ for all $x, x' \in X$.

One can readily check that any contraction map $T\colon X \to X$ on a metric space (X, d) is continuous. Indeed let x be a point of X, and let $\varepsilon > 0$ be given. Then $d(T(x), T(x')) < \varepsilon$ for all points x' of X satisfying $d(x, x') < \varepsilon$.

Theorem 3.23 (Contraction Mapping Theorem) *Let X be a complete metric space, and let $T\colon X \to X$ be a contraction mapping defined on X. Then T has a unique fixed point in X (i.e., there exists a unique point x of X for which $T(x) = x$).*

Proof Let λ be chosen such that $0 \leq \lambda < 1$ and $d(T(u), T(u')) \leq \lambda d(u, u')$ for all $u, u' \in X$, where d is the distance function on X. First we show the existence of the fixed point x. Let x_0 be any point of X, and define a sequence $x_0, x_1, x_2, x_3, x_4, \ldots$ of points of X by the condition that $x_n = T(x_{n-1})$ for all positive integers n. It follows by induction on n that $d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0)$. Using the Triangle Inequality, we deduce that if j and k are positive integers satisfying $k > j$ then

$$d(x_k, x_j) \leq \sum_{n=j}^{k-1} d(x_{n+1}, x_n) \leq \frac{\lambda^j - \lambda^k}{1-\lambda} d(x_1, x_0) \leq \frac{\lambda^j}{1-\lambda} d(x_1, x_0).$$

(Here we have used the identity

$$\lambda^j + \lambda^{j+1} + \cdots + \lambda^{k-1} = \frac{\lambda^j - \lambda^k}{1-\lambda}.)$$

Using the fact that $0 \leq \lambda < 1$, we deduce that the sequence (x_n) is a Cauchy sequence in X. This Cauchy sequence must converge to some point x of X, since X is complete. But then we see that

$$T(x) = T\left(\lim_{n\to+\infty} x_n\right) = \lim_{n\to+\infty} T(x_n) = \lim_{n\to+\infty} x_{n+1} = x,$$

since $T\colon X \to X$ is a continuous function, and thus x is a fixed point of T.

If x' were another fixed point of T then we would have

$$d(x', x) = d(T(x'), T(x)) \leq \lambda d(x', x).$$

But this is impossible unless $x' = x$, since $\lambda < 1$. Thus the fixed point x of the contraction map T is unique. ∎

We use the Contraction Mapping Theorem in order to prove the following existence theorem for solutions of ordinary differential equations.

Theorem 3.24 (Picard's Theorem) *Let $F\colon U \to \mathbb{R}$ be a continuous function defined over some open set U in the plane $\mathbb{R}^2$, and let (x_0, t_0) be an element of U. Suppose that there exists some non-negative constant M such that*

$$|F(u,t) - F(v,t)| \leq M|u - v| \text{ for all } (u,t) \in U \text{ and } (v,t) \in U.$$

Then there exists a continuous function $\varphi\colon [t_0 - \delta, t_0 + \delta] \to \mathbb{R}$ defined on the interval $[t_0 - \delta, t_0 + \delta]$ for some $\delta > 0$ such that $x = \varphi(t)$ is a solution to the differential equation

$$\frac{dx(t)}{dt} = F(x(t), t)$$

with initial condition $x(t_0) = x_0$.

Proof Solving the differential equation with the initial condition $x(t_0) = x_0$ is equivalent to finding a continuous function $\varphi\colon I \to \mathbb{R}$ satisfying the integral equation

$$\varphi(t) = x_0 + \int_{t_0}^{t} F(\varphi(s), s)\, ds.$$

where I denotes the closed interval $[t_0 - \delta, t_0 + \delta]$. (Note that any continuous function φ satisfying this integral equation is automatically differentiable, since the indefinite integral of a continuous function is always differentiable.)

Let $K = |F(x_0, t_0)| + 1$. Using the continuity of the function F, together with the fact that U is open in $\mathbb{R}^2$, one can find some $\delta_0 > 0$ such that the open disk of radius δ_0 about (x_0, t_0) is contained in U and $|F(x, t)| \leq K$ for all points (x, t) in this open disk. Now choose $\delta > 0$ such that

$$\delta\sqrt{1 + K^2} < \delta_0 \text{ and } M\delta < 1.$$

Note that if $|t - t_0| \leq \delta$ and $|x - x_0| \leq K\delta$ then (x, t) belongs to the open disk of radius δ_0 about (x_0, t_0), and hence $(x, t) \in U$ and $|F(x, t)| \leq K$.

Let J denote the closed interval $[x_0 - K\delta, x_0 + K\delta]$. The space $C(I, J)$ of continuous functions from the interval I to the interval J is a complete metric space, by Corollary 3.22. Define $T\colon C(I, J) \to C(I, J)$ by

$$T(\varphi)(t) = x_0 + \int_{t_0}^{t} F(\varphi(s), s)\, ds.$$

We claim that T does indeed map $C(I, J)$ into itself and is a contraction mapping.

Let $\varphi\colon I \to J$ be an element of $C(I, J)$. Note that if $|t - t_0| \leq \delta$ then

$$|(\varphi(t), t) - (x_0, t_0)|^2 = (\varphi(t) - x_0)^2 + (t - t_0)^2 \leq \delta^2 + K^2\delta^2 < \delta_0^2,$$

hence $|F(\varphi(t), t)| \leq K$. It follows from this that

$$|T(\varphi)(t) - x_0| \leq K\delta$$

for all t satisfying $|t - t_0| < \delta$. The function $T(\varphi)$ is continuous, and is therefore a well-defined element of $C(I, J)$ for all $\varphi \in C(I, J)$.

We now show that T is a contraction mapping on $C(I, J)$. Let φ and ψ be elements of $C(I, J)$. The hypotheses of the theorem ensure that

$$|F(\varphi(t), t) - F(\psi(t), t)| \leq M|\varphi(t) - \psi(t)| \leq M\rho(\varphi, \psi)$$

for all $t \in I$, where $\rho(\varphi, \psi) = \sup_{t \in I} |\varphi(t) - \psi(t)|$. Therefore

$$\begin{aligned} |T(\varphi)(t) - T(\psi)(t)| &= \left| \int_{t_0}^{t} (F(\varphi(s), s) - F(\psi(s), s))\, ds \right| \\ &\leq M|t - t_0|\rho(\varphi, \psi) \end{aligned}$$

for all t satisfying $|t - t_0| \leq \delta$. Therefore $\rho(T(\varphi), T(\psi)) \leq M\delta\rho(\varphi, \psi)$ for all $\varphi, \psi \in C(I, J)$. But δ has been chosen such that $M\delta < 1$. This shows that $T: C(I, J) \to C(I, J)$ is a contraction mapping on $C(I, J)$. It follows from the Contraction Mapping Theorem (Theorem 3.23) that there exists a unique element φ of $C(I, J)$ satisfying $T(\varphi) = \varphi$. This function φ is the required solution to the differential equation. ∎

A straightforward, but somewhat technical, least upper bound argument can be used to show that if $x = \psi(t)$ is any other continuous solution to the differential equation

$$\frac{dx}{dt} = F(x, t)$$

on the interval $[t_0 - \delta, t_0 + \delta]$ satisfying the initial condition $\psi(t_0) = x_0$, then $|\psi(t) - x_0| \leq K\delta$ for all t satisfying $|t - t_0| \leq \delta$. Thus such a solution to the differential equation must belong to the space $C(I, J)$ defined in the proof of Theorem 3.24. The uniqueness of the fixed point of the contraction mapping $T: C(I, J) \to C(I, J)$ then shows that $\psi = \varphi$, where $\varphi: [t_0 - \delta, t_0 + \delta] \to \mathbb{R}$ is the solution to the differential equation whose existence was proved in Theorem 3.24. This shows that the solution to the differential equation is in fact unique on the interval $[t_0 - \delta, t_0 + \delta]$.

3.12 The Completion of a Metric Space

We describe below a construction whereby any metric space can be embedded in a complete metric space.

Lemma 3.25 *Let X be a metric space with distance function d, let (x_j) and (y_j) be Cauchy sequences of points in X, and let $d_j = d(x_j, y_j)$ for all positive integers j. Then (d_j) is a Cauchy sequence of real numbers.*

Proof It follows from the Triangle Inequality that

$$d_j \leq d(x_j, x_k) + d_k + d(y_k, y_j)$$

and thus $d_j - d_k \leq d(x_j, x_k) + d(y_j, y_k)$ for all integers j and k. Similarly $d_k - d_j \leq d(x_j, x_k) + d(y_j, y_k)$. It follows that

$$|d_j - d_k| \leq d(x_j, x_k) + d(y_j, y_k)$$

for all integers j and k.

Let $\varepsilon > 0$ be given. Then there exists some positive integer N such that $d(x_j, x_k) < \frac{1}{2}\varepsilon$ and $d(y_j, y_k) < \frac{1}{2}\varepsilon$ whenever $j \geq N$ and $k \geq N$, since the sequences (x_j) and (y_j) are Cauchy sequences in X. But then $|d_j - d_k| < \varepsilon$ whenever $j \geq N$ and $k \geq N$. Thus the sequence (d_j) is a Cauchy sequence of real numbers, as required. ∎

Let X be a metric space with distance function d. It follows from Cauchy's Criterion for Convergence and Lemma 3.25 that $\lim_{j\to+\infty} d(x_j, y_j)$ exists for all Cauchy sequences (x_j) and (y_j) in X.

Lemma 3.26 *Let X be a metric space with distance function d, and let (x_j), (y_j) and (z_j) be Cauchy sequences of points in X. Then*

$$0 \leq \lim_{j\to+\infty} d(x_j, z_j) \leq \lim_{j\to+\infty} d(x_j, y_j) + \lim_{j\to+\infty} d(y_j, z_j).$$

Proof This follows immediately on taking limits of both sides of the Triangle Inequality. ∎

Lemma 3.27 *Let X be a metric space with distance function d, and let (x_j), (y_j) and (z_j) be Cauchy sequences of points in X. Suppose that*

$$\lim_{j\to+\infty} d(x_j, y_j) = 0 \text{ and } \lim_{j\to+\infty} d(y_j, z_j) = 0.$$

Then $\lim_{j\to+\infty} d(x_j, z_j) = 0$.

Proof This is an immediate consequence of Lemma 3.26. ∎

Lemma 3.28 *Let X be a metric space with distance function d, and let (x_j), (x'_j), (y_j) and (y'_j) be Cauchy sequences of points in X. Suppose that*

$$\lim_{j\to+\infty} d(x_j, x'_j) = 0 \text{ and } \lim_{j\to+\infty} d(y_j, y'_j) = 0.$$

Then $\lim_{j\to+\infty} d(x_j, y_j) = \lim_{j\to+\infty} d(x'_j, y'_j)$.

Proof It follows from Lemma 3.26 that

$$\begin{aligned} \lim_{j\to+\infty} d(x_j, y_j) &\leq \lim_{j\to+\infty} d(x_j, x'_j) + \lim_{j\to+\infty} d(x'_j, y'_j) + \lim_{j\to+\infty} d(y'_j, y_j) \\ &= \lim_{j\to+\infty} d(x'_j, y'_j). \end{aligned}$$

Similarly $\lim_{j\to+\infty} d(x'_j, y'_j) \leq \lim_{j\to+\infty} d(x_j, y_j)$. It follows that $\lim_{j\to+\infty} d(x_j, y_j) = \lim_{j\to+\infty} d(x'_j, y'_j)$, as required. ∎

Let X be a metric space with distance function d. Then there is an equivalence relation on the set of Cauchy sequences of points in X, where two Cauchy sequences (x_j) and (x'_j) in X are equivalent if and only if

$\lim_{j \to +\infty} d(x_j, x'_j) = 0$. Let $\tilde{X}$ denote the set of equivalence classes of Cauchy sequences in X with respect to this equivalence relation. Let $\tilde{x}$ and $\tilde{y}$ be elements of $\tilde{X}$, and let (x_j) and (y_j) be Cauchy sequences belonging to the equivalence classes represented by $\tilde{x}$ and $\tilde{y}$. We define

$$d(\tilde{x}, \tilde{y}) = \lim_{j \to +\infty} d(x_j, y_j).$$

It follows from Lemma 3.28 that the value of $d(\tilde{x}, \tilde{y})$ does not depend on the choice of Cauchy sequences (x_j) and (y_j) representing $\tilde{x}$ and $\tilde{y}$. We obtain in this way a distance function on the set $\tilde{X}$. This distance function satisfies the Triangle Inequality (Lemma 3.26) and the other metric space axioms. Therefore $\tilde{X}$ with this distance function is a metric space. We refer to the space $\tilde{X}$ as the *completion* of the metric space X.

We can regard the metric space X as being embedded in its completion $\tilde{X}$, where a point x of X is represented in $\tilde{X}$ by the equivalence class of the constant sequence $x, x, x, \ldots$.

Example The completion of the space $\mathbb{Q}$ of rational numbers is the space $\mathbb{R}$ of real numbers.

Theorem 3.29 *The completion $\tilde{X}$ of a metric space X is a complete metric space.*

Proof Let $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots$ be a Cauchy sequence in the completion $\tilde{X}$ of X. For each positive integer m let $x_{m,1}, x_{m,2}, x_{m,3}, \ldots$ be a Cauchy sequence in X belonging to the equivalence class that represents the element $\tilde{x}_m$ of $\tilde{X}$. Then, for each positive integer m there exists a positive integer $N(m)$ such that $d(x_{m,j}, x_{m,k}) < 1/m$ whenever $j \geq N(m)$ and $k \geq N(m)$. Let $y_m = x_{m,N(m)}$. We claim that the sequence $y_1, y_2, y_3, \ldots$ is a Cauchy sequence in X, and that the element $\tilde{y}$ of $\tilde{X}$ corresponding to this Cauchy sequence is the limit in $\tilde{X}$ of the sequence $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots$.

Let $\varepsilon > 0$ be given. Then there exists some positive integer M such that $M > 3/\varepsilon$ and $d(\tilde{x}_p, \tilde{x}_q) < \frac{1}{3}\varepsilon$ whenever $p \geq M$ and $q \geq M$. It follows from the definition of the distance function on $\tilde{X}$ that if $p \geq M$ and $q \geq M$ then $d(x_{p,k}, x_{q,k}) < \frac{1}{3}\varepsilon$ for all sufficiently large positive integers k. If $p \geq M$ and $k \geq N(p)$ then

$$d(y_p, x_{p,k}) = d(x_{p,N(p)}, x_{p,k}) < 1/p \leq 1/M < \tfrac{1}{3}\varepsilon$$

It follows that if $p \geq M$ and $q \geq M$, and if k is sufficiently large, then $d(y_p, x_{p,k}) < \frac{1}{3}\varepsilon$, $d(y_q, x_{q,k}) < \frac{1}{3}\varepsilon$, and $d(x_{p,k}, x_{q,k}) < \frac{1}{3}\varepsilon$, and hence $d(y_p, y_q) <$

ε. We conclude that the sequence $y_1, y_2, y_3, \ldots$ of points of X is indeed a Cauchy sequence.

Let $\tilde{y}$ be the element of $\tilde{X}$ which is represented by the Cauchy sequence $y_1, y_2, y_3, \ldots$ of points of X, and, for each positive integer m, let $\tilde{y}_m$ be the element of $\tilde{X}$ represented by the constant sequence $y_m, y_m, y_m, \ldots$ in X. Now

$$d(\tilde{y}, \tilde{y_m}) = \lim_{p \to +\infty} d(y_p, y_m),$$

and therefore $d(\tilde{y}, \tilde{y_m}) \to 0$ as $m \to +\infty$. Also

$$d(\tilde{y}_m, \tilde{x}_m) = \lim_{j \to +\infty} d(x_{m,N(m)}, x_{m,j}) \leq \frac{1}{m}$$

and hence $d(\tilde{y}_m, \tilde{x}_m) \to 0$ as $m \to +\infty$. It follows from this that $d(\tilde{y}, \tilde{x}_m) \to 0$ as $m \to +\infty$, and therefore the Cauchy sequence $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots$ in $\tilde{X}$ converges to the point $\tilde{y}$ of $\tilde{X}$. We conclude that $\tilde{X}$ is a complete metric space, since we have shown that every Cauchy sequence in $\tilde{X}$ is convergent. ∎

Remark In a paper published in 1872, Cantor gave a construction of the real number system in which real numbers are represented as Cauchy sequences of rational numbers. The real numbers represented by two Cauchy sequences of rational numbers are equal if and only if the difference of the Cauchy sequences converges to zero. Thus the construction of the completion of a metric space, described above, generalizes Cantor's construction of the system of real numbers from the system of rational numbers.