Course 221: Trinity Term 2007.

Assignment II.

To be handed in by Monday 7 May, 2007.
Please include both name and student number on any work handed in.

1. (a) Let X be a connected subset of $[-1,1]$, let $a = \inf X$ and $b = \sup X$, and let c be a real number satisfying $a < c < b$. Prove that $c \in X$.

   [Hint: were it the case that $c \notin X$, you could define a function on X which takes the value 0 for all $x \in X$ satisfying $x < c$ and takes the value 1 for all $x \in X$ satisfying $x > c$. The existence of such a function would lead to a contradiction if X is connected. Why?]

   (b) It follows from (a) that every connected subset of $[-1,1]$ is an interval. Show that the connected open subsets of $[-1,1]$ (in the subspace topology) are the intervals of the form (a,b), where $a,b \in [-1,1]$, the intervals of the form $(a,1]$, where $a \in [-1,1)$, and the intervals of the form $[-1,b)$, where $b \in (-1,1]$.

2. The set $[-\infty,+\infty]$ of extended real numbers is a topological space. Moreover every open subset of $[-\infty,+\infty]$ can be represented as a countable union of open intervals, the open intervals in $[-\infty,\infty]$ being sets of the form (a,b) where $a,b \in [-\infty,+\infty]$, sets of the form $(a,+\infty]$, where $a \in [-\infty,+\infty)$, and sets of the form $[-\infty,b)$, where $b \in (-\infty,+\infty]$.

   Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let $f\colon X \to [-\infty,+\infty]$ be a function mapping X into the set $[-\infty,+\infty]$ of extended real numbers.

   (a) Prove that the function f is measurable with respect to the σ-algebra $\mathcal{A}$ if and only if $f^{-1}(U) \in \mathcal{A}$ for all open sets U in $[-\infty,+\infty]$. [Hint: you will probably need to make use of results described above.]

   (b) Let $g\colon [-\infty,+\infty] \to [-\infty,+\infty]$ be a function mapping the set $[-\infty,+\infty]$ of extended real numbers to itself. Suppose that the function f is measurable and that the function g is continuous. Prove that the composition function $g \circ f$ is measurable. [Note: it is possible to prove this with a handful of well-chosen sentences.]

Note

[This is not part of the assignment.]

The open sets in $[-\infty, +\infty]$ correspond to those in $[-1, 1]$ under the homeomorphism $\varphi: [-\infty, +\infty] \to [-1, 1]$ from $[-\infty, \infty]$ to $[-1, 1]$ defined such that $\varphi(+\infty) = 1$, $\varphi(-\infty) = -1$ and $\varphi(x) = x(1 + |x|)^{-1}$ for all $x \in \mathbb{R}$. In particular the open intervals in $[-\infty, +\infty]$ correspond, under φ, to the intervals in $[-1, 1]$ that are open with respect to the subspace topology on $[-1, 1]$. This enables us to characterize the collection of open intervals in $[-\infty, +\infty]$ as described in question 2.

Let X be a subset of $[-1, 1]$ which is open with respect to the subspace topology on $[-1, 1]$, let X_0 be a connected component of X, and let $s \in X$. If $s \in (-1, 1)$ then there exists $\delta > 0$ such that $(s - \delta, s + \delta) \subset X$. But the interval $(s - \delta, s + \delta)$ is a connected set containing s. It follows from the definition of connected components that $(s - \delta, s + \delta) \subset X_0$. Similarly if $s = 1$ then there exists $\delta > 0$ such that $(1 - \delta, 1] \subset X_0$, and if $s = -1$ then there exists $\delta > 0$ such that $[-1, -1 + \delta) \subset X_0$. This shows that every connected component of X is open with respect to the subspace topology on $[-1, 1]$. Such a connected component is therefore an interval of the sort described in 1(b). It follows from this that every component of X contains rational numbers.

Any topological space is the disjoint union of its connected components. Thus, in particular, every subset X of $[-1, 1]$ that is open with respect to the subspace topology on $[-1, 1]$ is a disjoint union of its connected components, and these connected components are intervals of the sort described in 1(b). Moreover the number of such connected components is countable. Indeed let S be the set of connected components of S, let B be the set of rational numbers belonging to X, and let $f: B \to S$ be the function that sends each rational number in B to the connected component to which it belongs. Then B is a countable set, $f: B \to S$ is a surjection, and therefore S is a countable set. We deduce from this that every connected subset $[-1, 1]$ that is open with respect to the subspace topology can be represented as a countable union of intervals of the sort described in question 1(b). It follows from this that every subset of $[-\infty, \infty]$ can be represented as a countable union of intervals of the sort described in question 2.