UNIVERSITY OF DUBLIN

XMA2141

TRINITY COLLEGE

Faculty of Science

school of mathematics

SF Mathematics
SF Theoretical Physics
SF TSM Mathematics

Michaelmas Term 2007

Course 214

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Credit will be given for the best 4 questions answered.

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) What is meant by saying that a subset X of the complex plane is an *open set*? What is meant by saying that a subset X of the complex plane is a *closed set*?

   (b) Let w be a complex number, and let r be a positive real number. Prove that the open disk $\{z \in \mathbb{C} : |z - w| < r\}$ is an open set in the complex plane.

   (c) Determine which of the following subsets of the complex plane are open sets, and which are closed sets:

   (i) the set $\{z \in \mathbb{C} : |z - 2| < 1 \text{ or } |z - 5| \leq 1\}$;

   (ii) the set $\{z \in \mathbb{C} : |z - 2| < 1 \text{ or } |z - 5| < 1\}$;

   (iii) the set $\{z \in \mathbb{C} : |z - 2| \geq 1 \text{ and } |z - 5| \geq 1\}$.

   [Briefly justify your answers.]

2. (a) Give the definition of the *winding number* $n(\gamma, w)$ of a closed path $\gamma\colon [a, b] \to \mathbb{C}$ about some point w of the complex plane that does not lie on γ.

   (b) State and prove the *Fundamental Theorem of Algebra*.

   [You may use without proof the result that if $\gamma_s\colon [a, b] \to \mathbb{C}$ is a closed path for each real number s in some interval $[c, d]$, then the value of the winding number $n(\gamma_s, w)$ of γ_s about some complex number w is independent of the value of s, provided that $\gamma_s(t)$ is a continuous function of s and t, and provided also that none of the paths γ_s passes through w.]

3. (a) Let $f\colon D \to \mathbb{C}$ be a function defined over an open set D in the complex plane. What is meant by saying that the function f is *holomorphic*? What is the *derivative* of a holomorphic function?

(b) Let $f\colon D \to \mathbb{C}$ be a function defined on an open set D in the complex plane. Prove that f is holomorphic on D if and only if, given any complex number w belonging to D, and given any positive real number ε, there exists some real positive number δ such that $|f(z) - f(w) - (z - w)f'(w)| \leq \varepsilon|z - w|$ whenever $|z - w| < \delta$.

(c) Describe the *Cauchy-Riemann* equations satisfied by a holomorphic function, and explain why any holomorphic function satisfies these equations.

4. (a) What is meant by saying that an open set D in the complex plane is *star-shaped*?

(b) Let $f\colon D \to \mathbb{C}$ be a continuous complex-valued function defined over a star-shaped open set D in $\mathbb{C}$. Suppose that

$$\int_{\partial T} f(z)\, dz = 0$$

for all closed triangles T contained in D (where the above denotes the path integral of the function f taken around the boundary of the triangle T in the anti-clockwise direction). Prove that there exists a holomorphic function $F\colon D \to \mathbb{C}$ such that $f(z) = F'(z)$ for all $z \in D$.

(c) State and prove Cauchy's Theorem for star-shaped domains. [You may use, without proof, the fact that the path integral of a holomorphic function taken around the boundary of a triangle in the complex plane is zero, provided that the triangle is contained in the domain of the holomorphic function.]

5. Use the method of contour integration to evaluate

$$\int_{-\infty}^{+\infty} \frac{e^{isx}}{x^2 + 2x + 5}\,dx$$

and

$$\int_{-\infty}^{+\infty} \frac{e^{isx}}{x^4 + 2x^2 + 1}\,dx$$

when s is a real number satisfying $s > 0$.

[Briefly justify your answers. You may use, without proof, the result that if R is a positive real number, if f is a continuous complex-valued function defined everywhere on the semicircle S_R, where

$$S_R = \{z \in \mathbb{C} : |z| = R \text{ and } \mathrm{Im}[z] \geq 0\},$$

and if there exists a non-negative real number $M(R)$ such that $|f(z)| \leq M(R)$ for all $z \in S_R$ then

$$\left| \int_{\sigma_R} f(z) e^{isz}\,dz \right| \leq \frac{\pi M(R)}{s}$$

for all $s > 0$, where $\sigma_R\colon [0, \pi] \to \mathbb{C}$ is the path with $[\sigma_R] = S_R$ defined such that $\sigma_R(\theta) = Re^{i\theta}$ for all $\theta \in [0, \pi]$.]

6. (a) What is an *elliptic function*?

   (b) What is a *fundamental region* for an elliptic function?

   (c) Let f be an elliptic function, and let X be a fundamental region for f. Prove that the sum of the residues of f at those poles of f located in the fundamental region X is zero.