

UNIVERSITY OF DUBLIN

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TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

SF Mathematics
SF Theoretical Physics
SF TSM Mathematics
Foundation Scholarship

Hilary Term 2007

COURSE 214

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Credit will be given for the best FOUR questions attempted.

You may not start this examination until instructed to do so by the invigilator.

1. (a) Let $f: D \rightarrow \mathbb{C}$ be a complex-valued function defined on some subset D of the complex plane. Define what is meant by saying that the function f is *continuous*.
(b) Let D be a subset of the complex plane. Define what is meant by saying that D is an *open set* in the complex plane.
(c) Let $f: D \rightarrow \mathbb{C}$ be a complex-valued function defined on an open subset D of the complex plane. Prove that the function f is continuous if and only if $f^{-1}(V)$ is an open subset of \mathbb{C} for all open subsets V of \mathbb{C} . [You should use the definition of continuity, and the definition of open sets in the complex plane, but you should not assume, without proof, any theorem concerned with continuous functions or open sets.]
(d) Explain why $\{z \in \mathbb{C} : |\exp(z)^3 - \exp(z)| > |z|^3\}$ is an open set in the complex plane, fully justifying your answer.
2. (a) Prove the Path Lifting Theorem, which states that, given any path $\gamma: [a, b] \rightarrow \mathbb{C} \setminus \{0\}$ in the complex plane that does not pass through zero, there exists a path $\tilde{\gamma}: [a, b] \rightarrow \mathbb{C}$ in the complex plane which satisfies $\exp(\tilde{\gamma}(t)) = \gamma(t)$ for all $t \in [a, b]$.
(b) Give the definition of the *winding number* $n(\gamma, w)$ of a closed path $\gamma: [a, b] \rightarrow \mathbb{C}$ about some point w of the complex plane that does not lie on γ .

3. (a) What is meant by saying that an open set D in the complex plane is *star-shaped*?
 (b) Let $f: D \rightarrow \mathbb{C}$ be a continuous complex-valued function defined over a star-shaped open set D in \mathbb{C} . Suppose that

$$\int_{\partial T} f(z) dz = 0$$

for all closed triangles T contained in D (where the above denotes the path integral of the function f taken around the boundary of the triangle T in the anti-clockwise direction). Prove that there exists a holomorphic function $F: D \rightarrow \mathbb{C}$ such that $f(z) = F'(z)$ for all $z \in D$.

- (c) State and prove Cauchy's Theorem for star-shaped domains. [You may use, without proof, the fact that the path integral of a holomorphic function taken around the boundary of a triangle in the complex plane is zero, provided that the triangle is contained in the domain of the holomorphic function.]

4. Let w be a complex number, let r be a positive real number, and let $f: D_{w,r} \rightarrow \mathbb{C}$ be a holomorphic function on the open disk $D_{w,r}$ of radius r about w . Taylor's Theorem for Holomorphic Functions asserts that the function f may then be differentiated any number of times on $D_{w,r}$, there exist complex numbers a_0, a_2, a_3, \dots such that

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - w)^n,$$

and moreover

$$a_n = \frac{f^{(n)}(w)}{n!} = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{(z - w)^{n+1}} dz,$$

where γ_R is a closed path which traverses a circle of radius R centred on w once in the anti-clockwise direction, and $0 < R < r$. Prove this theorem.

5. Use the method of contour integration to evaluate

$$\int_{-\infty}^{+\infty} \frac{e^{isx}}{x^2 + x + 1} dx$$

when s is a real number satisfying $s > 0$, and also when s is a real number satisfying $s < 0$.

[Briefly justify your answers. You may use, without proof, the result that if R is a positive real number, if f is a continuous complex-valued function defined everywhere on the semicircle S_R , where

$$S_R = \{z \in \mathbb{C} : |z| = R \text{ and } \text{Im}[z] \geq 0\},$$

and if there exists a non-negative real number $M(R)$ such that $|f(z)| \leq M(R)$ for all $z \in S_R$ then

$$\left| \int_{\sigma_R} f(z) e^{isz} dz \right| \leq \frac{\pi M(R)}{s}$$

for all $s > 0$, where $\sigma_R: [0, \pi] \rightarrow \mathbb{C}$ is the path with $[\sigma_R] = S_R$ defined such that $\sigma_R(\theta) = Re^{i\theta}$ for all $\theta \in [0, \pi]$.

6. (a) State and Prove Liouville's Theorem. [You may use, without proof, Taylor's Theorem for Holomorphic Functions, whose statement is given in question 4 above.]
- (b) What is meant by saying that a meromorphic function defined throughout the complex plane is *doubly-periodic*?
- (c) Prove that any holomorphic doubly-periodic function defined throughout the complex plane is constant.
- (d) What is an *elliptic function*?