Course 212—Academic year 1991-92
Problem Set I
To be handed in by FRIDAY 15th NOVEMBER

1. Let X and Y be metric spaces with distance functions d_X and d_Y respectively, let $f\colon X \to Y$ be a function from X to Y, and let p be a point of X. Suppose that, given any sequence $x_1, x_2, x_3, \ldots$ of points of X converging to p, the sequence $f(x_1), f(x_2), f(x_3), \ldots$ converges to $f(p)$. Prove that the function f is continuous at p. [Hint: note that the function f fails to be continuous at p if and only if there exists $\varepsilon > 0$ such that, given any $\delta > 0$, there exists $x \in X$ satisfying $d_X(x, p) < \delta$ and $d_Y(f(x), f(p)) \geq \varepsilon$; apply this result with $\delta = 1/n$ for each natural number n.]

2. Let X be a set, and let d be the function on $X \times X$ defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y; \\ 0 & \text{if } x = y. \end{cases}$$

   Prove that X is a metric space with respect to the distance function d. Show also that the topology on X generated by the distance function d is the discrete topology on X (i.e., show that every subset of X is an open set).

3. Let X be a topological space whose only open sets are $\emptyset$ and X itself. Show that X is Hausdorff if and only if X consists of a single point.

4. A topological space X is said to be *sequentially compact* if every sequence of points in X has a convergent subsequence. Every closed bounded set in ^{n} is sequentially compact.

   (a) Let X be a sequentially compact topological space, and let F be a closed subset of X. Prove that F is sequentially compact.

   (b) Let $f\colon X \to Y$ be a continuous map from a between topological spaces X and Y. Suppose that X is sequentially compact. Prove that the image $f(X)$ of the map f is sequentially compact.

   (c) Let X be a sequentially compact topological space and let $f\colon X \to$ be a continuous function from X to (where is given the usual topology). Prove that the function f is bounded (i.e., there exists some constant K such that $|f(x)| \leq K$ for all $x \in X$). Show also that there exist points u and v of X such that $f(u) \leq f(x) \leq f(v)$ for all $x \in X$. [Hint: generalize corresponding results towards the end of §8 of Course 121.]

(The concept of *sequential compactness* introduced in this question should not be confused with the concept of *compactness* discussed in lectures towards the end of the Michaelmas term. It can however be proved that the concepts of *sequential compactness* and *compactness* coincide for metric spaces, though the proof is somewhat lengthy.)

Course 212—Academic year 1991-92
Problem Set II
To be handed in by FRIDAY 29th NOVEMBER

1. (a) Let $f\colon X \to Y$ be a continuous map between topological spaces X and Y. Suppose that X is path-connected. Prove that the image $f(X)$ of the map f is also path-connected.

   (b) Let X and Y be path-connected topological spaces. Prove that the Cartesian product $X \times Y$ of X and Y is path-connected.

2. Determine the connected components of the following subsets of ^{2}:

   (i) $\{(x,y) \in {}^2 : x^2 + y^2 = 1\}$;

   (ii) $\{(x,y) \in {}^2 : x^2 - y^2 = 1\}$;

   (iii) $\{(x,y) \in {}^2 : y^2 = x(x^2 - 1)\}$;

   (iv) $\{(x,y) \in {}^2 : (x-n)^2 + y^2 > \frac{1}{4} \text{ for all } n \in \}$.

   [Fully justify your answers.]

3. Let X_+, X_- and X_0 be the subsets of ^{2} defined by

$$
\begin{aligned}
X_+ &= \left\{ (x,y) \in {}^2 : x > 0 \text{ and } y = \sin\left(\frac{1}{x}\right) \right\}, \\
X_- &= \left\{ (x,y) \in {}^2 : x < 0 \text{ and } y = \sin\left(\frac{1}{x}\right) \right\}, \\
X_0 &= \left\{ (x,y) \in {}^2 : x = 0 \text{ and } -1 \le y \le 1 \right\},
\end{aligned}
$$

   and let $X = X_+ \cup X_0 \cup X_-$.

   (a) Explain why X_+, X_- and X_0 are path-connected.

   (b) Are X_+, X_- and X_0 connected? [Justify your answer.]

   (c) Show that the point $(0,0)$ of X_0 belongs to the closures of X_+ and X_- in X. [Hint: find sequences in X_+ and X_- which converge to $(0,0)$, and use a result proved in §2 of the course concerning topological spaces.]

   (d) Explain why $(0,0)$ belongs to the connected component of X containing the set X_+ and also to the connected component of X containing the set X_-. Hence prove that the sets X_+, X_- and X_0 are contained in the same connected component of X, showing that X is a connected topological space. [Hint: use results proved in §3 of the course concerning connected topological spaces.]

Let $\gamma\colon [0,1] \to X$ be a path in X, and let

$$S = \{t \in [0,1] : \gamma(t) \in X_0\}.$$

(e) Explain why S is closed in $[0,1]$. [Hint: show that S is the preimage of a closed set in X.]

(f) Suppose that $\gamma(s) \in X_0$ for some $s \in [0,1]$. Show that there exists some $\delta > 0$ such that $|\gamma(t) - \gamma(s)| < \frac{1}{2}$ for all $t \in [0,1]$ satisfying $|t - s| < \delta$. Then, by using the Intermediate Value Theorem, or otherwise, deduce that $\gamma(t) \in X_0$ for all $t \in [0,1]$ satisfying $|t - s| < \delta$, showing that S is open in $[0,1]$.

(g) By using (e), (f) and the connectedness of the interval $[0,1]$, or otherwise, show that if $\gamma(s) \in X_0$ for some $s \in [0,1]$ then $\gamma([0,1]) \subset X_0$.

(h) Is the topological space X path-connected? [Justify your answer.]

4. Let $f\colon [a,b] \to$ be a continuous function on the closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a < b$.

(a) Explain why, given any $s \in [a,b]$, there exists some $\delta(s) > 0$ such that $|f(t) - f(s)| < \frac{1}{2}\varepsilon$ for all $t \in [a,b]$ satisfying $|t - s| < 2\delta(s)$.

(b) Show that there exists an finite set $\{s_1, s_2, \ldots, s_k\}$ of real numbers belonging to $[a,b]$ such that, given any $t \in [a,b]$, there exists some s_j in this set such that $|t - s_j| < \delta(s_j)$. [Hint: apply the Heine-Borel Theorem.]

(c) Let δ be the minimum of $\delta(s_1), \delta(s_2), \ldots, \delta(s_k)$. Show that if t and u are elements of $[a,b]$ satisfying $|t - u| < \delta$ then $|f(t) - f(u)| < \varepsilon$.