Course 212: Academic Year 1991-2
Section 9: Winding Numbers

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Contents

9 Winding Numbers

9.1 Winding Numbers of Closed Curves in the Plane

Let $\gamma\colon [0,1] \to \mathbb{C}$ be a continuous closed curve in the complex plane which is defined on some closed interval $[0,1]$ (so that $\gamma(0) = \gamma(1)$), and let w be a complex number which does not belong to the image of the closed curve γ. It then follows from the Path Lifting Theorem (Theorem 8.10) that there exists a continuous path $\tilde{\gamma}\colon [0,1] \to \mathbb{C}$ in $\mathbb{C}$ such that $\gamma(t) - w = \exp(\tilde{\gamma}(t))$ for all $t \in [0,1]$. Let us define

$$n(\gamma, w) = \frac{\tilde{\gamma}(1) - \tilde{\gamma}(0)}{2\pi i}.$$

Now $\exp(\tilde{\gamma}(1)) = \gamma(1) = \gamma(0) = \exp(\tilde{\gamma}(0))$ (since γ is a closed curve). It follows from this that $n(\gamma, w)$ is an integer. This integer is known as the *winding number* of the closed curve γ about w.

Lemma 9.1 *The value of the winding number $n(\gamma, w)$ does not depend on the choice of the lift $\tilde{\gamma}$ of the curve γ.*

Proof Let $\sigma\colon [0,1] \to \mathbb{C}$ be a continuous curve in $\mathbb{C}$ with the property that $\exp(\sigma(t)) = \gamma(t) - w = \exp(\tilde{\gamma}(t))$ for all $t \in [0,1]$. Then

$$\frac{\sigma(t) - \tilde{\gamma}(t)}{2\pi i}$$

is an integer for all $t \in [0,1]$. But the map sending $t \in [0,1]$ to $\sigma(t) - \tilde{\gamma}(t)$ is continuous on $[0,1]$. This map must therefore be a constant map, since the interval $[0,1]$ is connected. Thus there exists some integer m with the property that $\sigma(t) = \tilde{\gamma}(t) + 2\pi i m$ for all $t \in [0,1]$. But then

$$\sigma(1) - \sigma(0) = \tilde{\gamma}(1) - \tilde{\gamma}(0).$$

This proves that the value of the winding number $n(\gamma, w)$ of the closed curve γ about w is indeed independent of the choice of the lift $\tilde{\gamma}$ of γ. ∎

9.2 Winding Numbers and Contour Integrals

A continuous curve is said to be *piecewise* C^1 if it is made up of a finite number of continuously differentiable segments. We now show how the winding number of a piecewise C^1 closed curve in the complex plane can be expressed as a contour integral.

Proposition 9.2 *Let $\gamma\colon [0,1] \to \mathbb{C}$ be a piecewise C^1 closed curve in the complex plane, and let w be a point of $\mathbb{C}$ that does not lie on the curve γ. Then the winding number $n(\gamma, w)$ of γ about w is given by*

$$n(\gamma, w) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - w}.$$

Proof By definition

$$n(\gamma, w) = \frac{\sigma(1) - \sigma(0)}{2\pi i},$$

where $\sigma\colon [0,1] \to \mathbb{C}$ is a path in $\mathbb{C}$ such that $\gamma(t) - w = \exp(\sigma(t))$ for all $t \in [0,1]$. Taking derivatives, we see that

$$\gamma'(t) = \exp(\sigma(t))\sigma'(t) = (\gamma(t) - w)\sigma'(t).$$

Thus

$$\begin{aligned} n(\gamma, w) &= \frac{\sigma(1) - \sigma(0)}{2\pi i} = \frac{1}{2\pi i} \int_0^1 \sigma'(t)\, dt = \frac{1}{2\pi i} \int_0^1 \frac{\gamma'(t)\, dt}{\gamma(t) - w} \\ &= \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - w}. \quad \blacksquare \end{aligned}$$

One of the most important properties of winding numbers of closed curves in the complex plane is their invariance under continuous deformations of the closed curve.

Proposition 9.3 *Let w be a complex number and, for each $\tau \in [0,1]$, let $\gamma_\tau\colon [0,1] \to \mathbb{C}$ be a closed curve in $\mathbb{C}$ which does not pass through w. Suppose that the map sending $(t, \tau) \in [0,1] \times [0,1]$ to $\gamma_\tau(t)$ is a continuous map from $[0,1] \times [0,1]$ to $\mathbb{C}$. Then $n(\gamma_\tau, w) = n(\gamma_0, w)$ for all $\tau \in [0,1]$. In particular, $n(\gamma_1, w) = n(\gamma_0, w)$.*

Proof Let $H\colon [0,1] \times [0,1] \to \mathbb{C} \setminus \{0\}$ be defined by $H(t, \tau) = \gamma_\tau(t) - w$. It follows from the Monodromy Theorem (Theorem 8.11) that there exists a continuous map $\tilde{H}\colon [0,1] \times [0,1] \to \mathbb{C}$ such that $H = \exp \circ \tilde{H}$. But then

$$\tilde{H}(1, \tau) - \tilde{H}(0, \tau) = 2\pi i n(\gamma_\tau, w)$$

for all $\tau \in [0,1]$, and therefore the function $\tau \mapsto n(\gamma_\tau, w)$ is a continuous function on the interval $[0,1]$ taking values in the set $\mathbb{Z}$ of integers. But such a function must be constant on $[0,1]$, since the interval $[0,1]$ is connected. Thus $n(\gamma_0, w) = n(\gamma_1, w)$, as required. $\blacksquare$

Corollary 9.4 (Dog-Walking Principle) *Let* $\gamma_0\colon [0,1] \to \mathbb{C}$ *and* $\gamma_1\colon [0,1] \to \mathbb{C}$ *be continuous closed curves in* $\mathbb{C}$*, and let* w *be a complex number which does not lie on the images of the closed curves* γ_0 *and* γ_1*. Suppose that* $|\gamma_1(t) - \gamma_0(t)| < |w - \gamma_0(t)|$ *for all* $t \in [0,1]$*. Then* $n(\gamma_0, w) = n(\gamma_1, w)$*.*

Proof Let $\gamma_\tau(t) = (1-\tau)\gamma_0(t) + \tau\gamma_1(t)$ for all $t \in [0,1]$ and $\tau \in [0,1]$. Then

$$|\gamma_\tau(t) - \gamma_0(t)| = \tau|\gamma_1(t) - \gamma_0(t)| < |w - \gamma_0(t)|,$$

for all $t \in [0,1]$ and $\tau \in [0,1]$, and thus the closed curve γ_τ does not pass through w. The result therefore follows from Proposition 9.3. ∎

Corollary 9.5 *Let* $\gamma\colon [0,1] \to \mathbb{C}$ *be a continuous closed curve in* $\mathbb{C}$*, and let* $\sigma\colon [0,1] \to \mathbb{C}$ *be a continuous path in* $\mathbb{C}$ *whose image does not intersect the image of* γ*. Then* $n(\gamma, \sigma(0)) = n(\gamma, \sigma(1))$*. Thus the function* $w \mapsto n(\gamma, w)$ *is constant over each path-component of the set* $\mathbb{C} \setminus \gamma([0,1])$*.*

Proof For each $\tau \in [0,1]$, let $\gamma_\tau\colon [0,1] \to \mathbb{C}$ be the closed curve given by

$$\gamma_\tau(t) = \gamma(t) - \sigma(\tau).$$

Then the closed curves γ_τ do not pass through 0 (since the curves γ and σ do not intersect), and the map from $[0,1] \times [0,1]$ to $\mathbb{C}$ sending (t,τ) to $\gamma_\tau(t)$ is continuous. It follows from Proposition 9.3 that

$$n(\gamma, \sigma(0)) = n(\gamma_0, 0) = n(\gamma_1, 0) = n(\gamma, \sigma(1)),$$

as required. ∎

9.3 The Fundamental Theorem of Algebra

Theorem 9.6 (The Fundamental Theorem of Algebra) *Let* $P\colon \mathbb{C} \to \mathbb{C}$ *be a non-constant polynomial with complex coefficients. Then there exists some complex number* z_0 *such that* $P(z_0) = 0$*.*

Proof The result is trivial if $P(0) = 0$. Thus it suffices to prove the result when $P(0) \neq 0$.

For any $r > 0$, let the closed curve σ_r denote the circle about zero of radius r, traversed once in the anticlockwise direction, given by $\sigma_r(t) = r\exp(2\pi i t)$ for all $t \in [0,1]$. Consider the winding number $n(P \circ \sigma_r, 0)$ of $P \circ \sigma_r$ about zero. We claim that this winding number is equal to m for large values of r, where m is the degree of the polynomial P.

Let $P(z) = a_0 + a_1 z + \cdots + a_m z^m$, where $a_1, a_2, \ldots, a_n$ are complex numbers, and where $a_m \neq 0$. We write $P(z) = P_m(z) + Q(z)$, where $P_m(z) = a_m z^m$ and

$$Q(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}.$$

Let

$$R = \frac{|a_0| + |a_1| + \cdots + |a_m|}{|a_m|}.$$

If $|z| > R$ then

$$\left| \frac{Q(z)}{P_m(z)} \right| = \frac{1}{|a_m z|} \left| \frac{a_0}{z^{m-1}} + \frac{a_1}{z^{m-2}} + \cdots + a_{m-1} \right| < 1,$$

since $R \geq 1$, and thus $|P(z) - P_m(z)| < |P_m(z)|$. It follows from the Dog-Walking Principle (Corollary 9.4) that $n(P \circ \sigma_r, 0) = n(P_m \circ \sigma_r, 0) = m$ for all $r > R$.

Given $r > 0$, let $\gamma_\tau = P \circ \sigma_{\tau r}$ for all $\tau \in [0, 1]$. Then $n(\gamma_0, 0) = 0$, since γ_0 is a constant curve with value $P(0)$. Thus if the polynomial P were everywhere non-zero, then it would follow from Proposition 9.3 that $n(\gamma_1, 0) = n(\gamma_0, 0) = 0$. But $n(\gamma_1, 0) = n(P \circ \sigma_r, 0) = m$ for all $r > R$, and $m > 0$. Therefore the polynomial P must have at least one zero in the complex plane. ∎

9.4 The Kronecker Principle

The proof of the Fundamental Theorem of Algebra given above depends on continuity of the polynomial P, together with the fact that the winding number $n(P \circ \sigma_r, 0)$ is non-zero for sufficiently large r, where σ_r denotes the circle of radius r about zero, described once in the anticlockwise direction. We can therefore generalize the proof of the Fundamental Theorem of Algebra in order to obtain the following result (sometimes referred to as the *Kronecker Principle*).

Proposition 9.7 *Let $f: D \to \mathbb{C}$ be a continuous map defined on the closed unit disk D in $\mathbb{C}$, and let $w \in \mathbb{C} \setminus f(D)$. Then $n(f \circ \sigma, w) = 0$, where $\sigma: [0, 1] \to \mathbb{C}$ is the parameterization of unit circle defined by $\sigma(t) = \exp(2\pi i t)$, and $n(f \circ \sigma, w)$ is the winding number of $f \circ \sigma$ about w.*

Proof Define $\gamma_\tau(t) = f(\tau \exp(2\pi i t))$ for all $t \in [0, 1]$ and $\tau \in [0, 1]$. Then none of the closed curves γ_τ passes through w, and γ_0 is the constant curve with value $f(0)$. It follows from Proposition 9.3 that

$$n(f \circ \sigma, w) = n(\gamma_1, w) = n(\gamma_0, w) = 0,$$

as required. ∎

9.5 The Brouwer Fixed Point Theorem in Two Dimensions

We now use Proposition 9.7 to show that there is no continuous 'retraction' mapping the closed unit disk onto its boundary circle.

Corollary 9.8 *There does not exist a continuous map $r: D \to \partial D$ with the property that $r(z) = z$ for all $z \in \partial D$, where ∂D denotes the boundary circle of the closed unit disk D.*

Proof Let $\sigma: [0, 1] \to \mathbb{C}$ be defined by $\sigma(t) = \exp(2\pi i t)$. If a continuous map $r: D \to \partial D$ with the required property were to exist, then $r(z) \neq 0$ for all $z \in D$ (since $r(D) \subset \partial D$), and therefore $n(\sigma, 0) = n(r \circ \sigma, 0) = 0$, by Proposition 9.7. But $\sigma = \exp \circ \tilde{\sigma}$, where $\tilde{\sigma}(t) = 2\pi i t$ for all $t \in [0, 1]$, and thus

$$n(\sigma, 0) = \frac{\tilde{\sigma}(1) - \tilde{\sigma}(0)}{2\pi i} = 1.$$

This shows that there cannot exist any continuous map r with the required property. ∎

Theorem 9.9 (The Brouwer Fixed Point Theorem) *Let $f: D \to D$ be a continuous map which maps the closed unit disk D into itself. Then there exists some $z_0 \in D$ such that $f(z_0) = z_0$.*

Proof Suppose that there did not exist any fixed point z_0 of $f: D \to D$. Then one could define a continuous map $r: D \to \partial D$ as follows: for each $z \in D$, let $r(z)$ be the point on the boundary ∂D of D obtained by continuing the line segment joining $f(z)$ to z beyond z until it intersects ∂D at the point $r(z)$. Then $r: D \to \partial D$ would be a continuous map, and moreover $r(z) = z$ for all $z \in \partial D$. But Corollary 9.8 shows that there does not exist any continuous map $r: D \to \partial D$ with this property. We conclude that $f: D \to D$ must have at least one fixed point. ∎

Remark The Brouwer Fixed Point Theorem is also valid in higher dimensions. This theorem states that any continuous map from the closed n-dimensional ball into itself must have at least one fixed point. The proof of the theorem for $n > 2$ is analogous to the proof for $n = 2$, once one has shown that there is no continuous map from the closed n-dimensional ball to its boundary which is the identity map on the boundary. However winding numbers cannot be used to prove this result, and thus more powerful topological techniques need to be employed.

9.6 The Borsuk-Ulam Theorem

Lemma 9.10 *Let $f: S^1 \to \mathbb{C}$ be a continuous function defined on S^1, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Suppose that $f(-z) = -f(z)$ for all $z \in \mathbb{C}$. Then the winding number $n(f \circ \sigma, 0)$ of $f \circ \sigma$ about 0 is odd, where $\sigma: [0,1] \to S^1$ is given by $\sigma(t) = \exp(2\pi i t)$.*

Proof It follows from the Path Lifting Theorem (Theorem 8.10) that there exists a continuous path $\tilde{\gamma}: [0,1] \to \mathbb{C}$ in $\mathbb{C}$ such that $\exp(\tilde{\gamma}(t)) = f(\sigma(t))$ for all $t \in [0,1]$. Now $f(\sigma(t+\frac{1}{2})) = -f(\sigma(t))$ for all $t \in [0, \frac{1}{2}]$, since $\sigma(t+\frac{1}{2}) = -\sigma(t)$ and $f(-z) = -f(z)$ for all $z \in \mathbb{C}$. Thus $\exp(\tilde{\gamma}(t+\frac{1}{2})) = \exp(\tilde{\gamma}(t) + \pi i)$ for all $t \in [0, \frac{1}{2}]$. It follows that $\tilde{\gamma}(t+\frac{1}{2}) = \tilde{\gamma}(t) + (2m+1)\pi i$ for some integer m. (The value of m for which this identity is valid does not depend on t, since every continuous function from $[0, \frac{1}{2}]$ to the set of integers is necessarily constant.) Hence

$$n(f \circ \sigma, 0) = \frac{\tilde{\gamma}(1) - \tilde{\gamma}(0)}{2\pi i} = \frac{\tilde{\gamma}(1) - \tilde{\gamma}(\frac{1}{2})}{2\pi i} - \frac{\tilde{\gamma}(\frac{1}{2}) - \tilde{\gamma}(0)}{2\pi i} = 2m + 1.$$

Thus $n(f \circ \sigma, 0)$ is an odd integer, as required. ∎

We shall identify the space $\mathbb{R}^2$ with $\mathbb{C}$, identifying $(x,y) \in \mathbb{R}^2$ with the complex number $x + iy \in \mathbb{C}$ for all $x, y \in \mathbb{R}$. This is permissible, since we are interested in purely topological results concerning continuous functions defined on appropriate subsets of these spaces. Under this identification the closed unit disk D is given by

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

As usual, we define

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Lemma 9.11 *Let $f: S^2 \to \mathbb{R}^2$ be a continuous map with the property that $f(-\mathbf{n}) = -f(\mathbf{n})$ for all $\mathbf{n} \in S^2$. Then there exists some point $\mathbf{n}_0$ of S^2 with the property that $f(\mathbf{n}_0) = 0$.*

Proof Let $\varphi: D \to S^2$ be the map defined by

$$\varphi(x,y) = (x, y, +\sqrt{1 - x^2 - y^2}).$$

(Thus the map φ maps the closed disk D homeomorphically onto the upper hemisphere in $\mathbb{R}^3$.) Let $\sigma: [0,1] \to S^2$ be the parameterization of the equator in S^2 defined by

$$\sigma(t) = (\cos 2\pi t, \sin 2\pi t, 0)$$

for all $t \in [0,1]$. Let $f\colon S^2 \to \mathbb{R}^2$ be a continuous map with the property that $f(-\mathbf{n}) = -f(\mathbf{n})$ for all $\mathbf{n} \in S^2$. The winding number $n(f \circ \sigma, 0)$ is an odd integer, by Lemma 9.10, and is thus non-zero. It follows from Proposition 9.7, applied to $f \circ \varphi\colon D \to \mathbb{R}^2$, that $0 \in f(\varphi(D))$, (since otherwise the winding number $n(f \circ \sigma, 0)$ would be zero). Thus $f(\mathbf{n}_0) = 0$ for some $\mathbf{n}_0 = \sigma(D)$, as required. ∎

Theorem 9.12 (Borsuk-Ulam) *Let $f\colon S^2 \to \mathbb{R}^2$ be a continuous map. Then there exists some point $\mathbf{n}$ of S^2 with the property that $f(-\mathbf{n}) = f(\mathbf{n})$.*

Proof This result follows immediately on applying Lemma 9.11 to the continuous function $g\colon S^2 \to \mathbb{R}^2$ defined by $g(\mathbf{n}) = f(\mathbf{n}) - f(-\mathbf{n})$. ∎

Remark It is possible to generalize the Borsuk-Ulam Theorem to n dimensions. Let S^n be the unit n-sphere centered on the origin in $\mathbb{R}^n$. The Borsuk-Ulam Theorem in n-dimensions states that if $f\colon S^n \to \mathbb{R}^n$ is a continuous map then there exists some point $\mathbf{x}$ of S^n with the property that $f(\mathbf{x}) = f(-\mathbf{x})$.

9.7 The Hairy Dog Theorem

We shall use winding number techniques to prove a theorem, the *Hairy Dog Theorem*, which states that any continuous vector field on a 2-dimensional sphere that is everywhere tangent to the sphere must be zero at some point of the sphere. This result is also known as the *Hairy Ball Theorem* or, in German, the *Igelsatz* ('hedgehog theorem'). The result can be generalized to higher dimensions: any continuous vector field on an even-dimensional sphere that is everywhere tangent to the sphere must be zero at some point of the sphere.

Let $\mathbf{s}$ denote the 'south pole' of the 2-dimensional sphere S^2, defined by $\mathbf{s} = (0,0,-1)$. Given any point $\mathbf{x}$ of $S^2 \setminus \{\mathbf{s}\}$, we define $\varphi(\mathbf{x}) = (y_1, y_2)$, where $(y_1, y_2, 0)$ is the point at which the line passing through the points $\mathbf{s}$ and $\mathbf{x}$ intersects the plane

$$\{(t_1, t_2, t_3) \in \mathbb{R}^3 : t_3 = 0\}.$$

Then $\varphi\colon S^2 \setminus \{\mathbf{s}\} \to \mathbb{R}^2$ is a homeomorphism. It follows directly from the definition of φ that if $\mathbf{x} = (x_1, x_2, x_3)$ and $\varphi(\mathbf{x}) = (y_1, y_2)$, then

$$y_1 = \varphi_1(\mathbf{x}) = \frac{x_1}{1 + x_3}, \quad y_2 = \varphi_2(\mathbf{x}) = \frac{x_2}{1 + x_3}.$$

The homeomorphism φ represents *stereographic projection* from the south pole $\mathbf{s}$ of the sphere S^2. We now calculate the inverse of the homeomorphism φ by solving for x_1, x_2 and x_3 in terms of y_1 and y_2. Now

$$|\mathbf{y}|^2 = y_1^2 + y_2^2 = \frac{x_1^2 + x_2^2}{(1 + x_3)^2} = \frac{1 - x_3^2}{(1 + x_3)^2} = \frac{1 - x_3}{1 + x_3}$$

(since $x_1^2 + x_2^2 + x_3^2 = 1$), and thus

$$1 + |\mathbf{y}|^2 = \frac{(1 + x_3) + (1 - x_3)}{1 + x_3} = \frac{2}{1 + x_3}.$$

Therefore

$$1 + x_3 = \frac{2}{1 + |\mathbf{y}|^2}.$$

Thus $\varphi^{-1}(y_1, y_2) = (x_1, x_2, x_3)$, where

$$x_1 = \frac{2y_1}{1 + |\mathbf{y}|^2}, \quad x_2 = \frac{2y_2}{1 + |\mathbf{y}|^2}, \quad x_3 = \frac{1 - |\mathbf{y}|^2}{1 + |\mathbf{y}|^2}.$$

Now let $\mathbf{u}$ be a vector in $\mathbb{R}^3$ which is tangent to the sphere S^2 at some point $\mathbf{x}$, where $\mathbf{x} \neq \mathbf{s}$. Let us write $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{u} = (u_1, u_2, u_3)$. Then $x_1 u_1 + x_2 u_2 + x_3 u_3 = 0$. Let $\mathbf{y} = \varphi(\mathbf{x})$. We say that a vector $\mathbf{v}$ in $\mathbb{R}^2$ based at $\mathbf{y}$ is the *push-forward* of the vector $\mathbf{u}$ under stereographic projection from $(0, 0, -1)$ if $(\varphi \circ \gamma)'(0) = \mathbf{v}$ for any differentiable curve $\gamma\colon (-\varepsilon, \varepsilon) \to S^2$ satisfying $\gamma(0) = \mathbf{x}$ and $\gamma'(0) = \mathbf{u}$.

Lemma 9.13 *Let* $\mathbf{u}$ *be a vector tangent to the sphere* S^2 *at some point* $\mathbf{x} \in S^2$*, where* $\mathbf{x} \neq (0, 0, -1)$*, and let* $\mathbf{v}$ *be the vector in* $\mathbb{R}^2$ *that is the push-forward of* $\mathbf{u}$ *under stereographic projection from* $(0, 0, -1)$*. Then*

$$\begin{aligned} \mathbf{v} &= \frac{1}{1 + x_3}\left(u_1 - \frac{x_1}{1 + x_3}u_3,\, u_2 - \frac{x_2}{1 + x_3}u_3\right) \\ &= \tfrac{1}{2}(1 + |\mathbf{y}|^2)(u_1 - y_1 u_3, u_2 - y_2 u_3). \end{aligned}$$

where $\mathbf{y}$ *is the point of* $\mathbb{R}^2$ *corresponding to* $\mathbf{x}$ *under stereographic projection from* $(0, 0, -1)$*, given by*

$$\mathbf{y} = \left(\frac{x_1}{1 + x_3}, \frac{x_2}{1 + x_3}\right).$$

Moreover

$$|\mathbf{v}|^2 = \tfrac{1}{4}(1 + |\mathbf{y}|^2)^2 |\mathbf{u}|^2,$$

and thus $\mathbf{v} = 0$ *if and only if* $\mathbf{u} = 0$*.*

Proof Let $\gamma\colon (-\varepsilon, \varepsilon) \to S^2$ be a differentiable curve on S^2 satisfying $\gamma(0) = \mathbf{x}$ and $\gamma'(0) = \mathbf{u}$, and let $\mathbf{y} = \varphi(\mathbf{x})$. Then the push-forward $\mathbf{v}$ of $\mathbf{u}$ under stereographic projection from the south pole $(0,0,-1)$ is given by

$$\begin{aligned}
\mathbf{v} &= (\varphi\circ\gamma)'(0) = \frac{d}{dt}\left(\frac{\gamma_1(t)}{1+\gamma_3(t)}, \frac{\gamma_2(t)}{1+\gamma_3(t)}\right)\Bigg|_{t=0} \\
&= \left(\frac{\gamma_1'(0)}{1+\gamma_3(0)} - \frac{\gamma_1(0)\gamma_3'(0)}{(1+\gamma_3(0))^2}, \frac{\gamma_2'(0)}{1+\gamma_3(0)} - \frac{\gamma_2(0)\gamma_3'(0)}{(1+\gamma_3(0))^2}\right) \\
&= \frac{1}{1+x_3}\left(u_1 - \frac{x_1}{1+x_3}u_3,\; u_2 - \frac{x_2}{1+x_3}u_3\right) \\
&= \tfrac{1}{2}(1+|\mathbf{y}|^2)(u_1 - y_1u_3, u_2 - y_2u_3),
\end{aligned}$$

Now

$$2(y_1u_1 + y_2u_2) + (1-|\mathbf{y}|^2)u_3 = (1+|\mathbf{y}|^2)(x_1u_1 + x_2u_2 + x_3u_3) = 0,$$

since the vector $\mathbf{u}$ is tangent to S^2 at $\mathbf{x}$, and thus $x_1u_1 + x_2u_2 + x_3u_3 = 0$. Therefore

$$\begin{aligned}
|\mathbf{v}|^2 &= \tfrac{1}{4}(1+|\mathbf{y}|^2)^2((u_1 - y_1u_3)^2 + (u_2 - y_1u_3)^2) \\
&= \tfrac{1}{4}(1+|\mathbf{y}|^2)^2(u_1^2 + u_2^2 + |\mathbf{y}|^2u_3^2 - 2(y_1u_1 + y_2u_2)u_3) \\
&= \tfrac{1}{4}(1+|\mathbf{y}|^2)^2|\mathbf{u}|^2.
\end{aligned}$$

Thus $\mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u} = \mathbf{0}$, as required. ∎

Let $\mathbf{U}$ be a continuous vector field on the 2-sphere S^2 that is everywhere tangent to S^2, and let $\mathbf{V}$ be a vector field on the plane $\mathbb{R}^2$. We say that the vector fields $\mathbf{U}$ and $\mathbf{V}$ are *related* under stereographic projection from the south pole $(0,0,-1)$ if, given any point $\mathbf{x}$ of $S^2\backslash\{(0,0,-1)\}$, the value $\mathbf{V}(\varphi(\mathbf{x})$ of $\mathbf{V}$ at $\varphi(\mathbf{x})$ is the push-forward of the value $\mathbf{U}(\mathbf{x})$ of $\mathbf{U}$ at the point $\mathbf{x}$, where φ denotes the stereographic projection map). Every continuous tangential vector field $\mathbf{U}$ on S^2 is related under stereographic projection from $(0,0,-1)$ to a unique continuous vector field $\mathbf{V}$ on $\mathbb{R}^2$: indeed it follows immediately from Lemma 9.13 that $\mathbf{V}(\mathbf{y})$ is the vector

$$\tfrac{1}{2}(1+|\mathbf{y}|^2)(U_1(\varphi^{-1}(\mathbf{y})) - y_1U_3(\varphi^{-1}(\mathbf{y})),\; U_2(\varphi^{-1}(\mathbf{y})) - y_2U_3(\varphi^{-1}(\mathbf{y})))$$

for all $\mathbf{y}\in\mathbb{R}^2$, where U_1, U_2 and U_3 are the Cartesian components of $\mathbf{U}$.

Theorem 9.14 (Hairy Dog Theorem) *Any continuous vector field on the 2-dimensional sphere S^2 that is everywhere tangent to S^2 must be zero at some point of S^2.*

Proof Let $\mathbf{U}: S^2 \to \mathbb{R}^3$ be a continuous vector field on S^2 that is everywhere tangent to S^2. We may assume that $\mathbf{U}$ is non-zero on the equator

$$\{(x_1, x_2, x_3) \in S^2 : x_3 = 0\}.$$

Let $\mathbf{V}$ be the unique vector field on $\mathbb{R}^2$ that is related to $\mathbf{U}$ under stereographic projection from $(0, 0, -1)$. It follows from Lemma 9.13 that, if the vector field $\mathbf{U}$ has no zeros in the northern hemisphere

$$\{(x_1, x_2, x_3) \in S^2 : x_3 \geq 0\}$$

then the vector field $\mathbf{V}$ has no zeros in the closed unit disk. Thus if $\alpha: [0, 1] \to \mathbb{C}$ is the closed curve defined by the formula

$$\alpha(t) = V_1(\cos 2\pi t, \sin 2\pi t) = iV_2(\cos 2\pi t, \sin 2\pi t),$$

then $n(\alpha, 0) = 0$, where $n(\alpha, 0)$ denotes the winding number of the closed curve α about 0 (Proposition 9.7). Now a point (y_1, y_2) on the unit circle $y_1^2 + y_2^2 = 1$ is the image of the point $(y_1, y_2, 0)$ under stereographic projection from $(0, 0, -1)$. It follows from Lemma 9.13 that $\mathbf{V}(y_1, y_2)$ is the vector

$$\tfrac{1}{2}(1 + |\mathbf{y}|^2)(U_1(y_1, y_2, 0) - y_1 U_3(y_1, y_2, 0),\ U_2(y_1, y_2, 0) - y_2 U_3(y_1, y_2, 0))$$

for all points (y_1, y_2) on the unit circle $y_1^2 + y_2^2 = 1$. Therefore

$$\alpha(t) = \sigma(t) - \exp(2\pi i t)\lambda(t)$$

for all $t \in [0, 1]$, where

$$\begin{aligned} \sigma(t) &= U_1(\cos 2\pi t, \sin 2\pi t, 0) + iU_2(\cos 2\pi t, \sin 2\pi t, 0), \\ \lambda(t) &= U_3(\cos 2\pi t, \sin 2\pi t, 0). \end{aligned}$$

Consider now the continuous vector field $\mathbf{W}$ on S^2 given by

$$\mathbf{W}(x_1, x_2, x_3) = (U_1(x_1, x_2, -x_3), U_2(x_1, x_2, -x_3), -U_3(x_1, x_2, -x_3))$$

for all $(x, y, z) \in S^2$. Then $\mathbf{W}$ is everywhere tangent to S^2, and the vector fields $\mathbf{U}$ and $\mathbf{W}$ correspond under reflection in the equatorial plane $x_3 = 0$. Thus the vector field $\mathbf{U}$ has no zeros in the southern hemisphere

$$\{(x_1, x_2, x_3) \in S^2 : x_3 \leq 0\}$$

if and only if the vector field $\mathbf{W}$ has no zeros in the northern hemisphere. Thus if the vector field $\mathbf{U}$ has no zeros in the southern hemisphere then $n(\beta, 0) = 0$, where $\beta: [0, 1] \to \mathbb{C}$ is the closed curve defined by

$$\beta(t) = \sigma(t) + \exp(2\pi i t)\lambda(t)$$

for all $t \in [0,1]$. (This follows directly on replacing U_3 by $-U_3$ in the definition of the closed curve α.) We conclude therefore that if the continuous vector field $\mathbf{U}$ were to have no zeros on S^2, then $n(\alpha, 0) = n(\beta, 0) = 0$.

We claim however that $n(\alpha, 0) + n(\beta, 0) = 2$. The orthogonality condition

$$x_1 U_1(x_1, x_2, 0) + x_2 u_2(x_1, x_2, 0)$$

at points $(x_1, x_2, 0)$ on the equator of the sphere S^2 implies that

$$\cos 2\pi t \operatorname{Re} \sigma(t) + \sin 2\pi t \operatorname{Im} \sigma(t) = 0$$

for all $t \in [0,1]$. We can therefore write $\sigma(t)$ in the form

$$\sigma(z) = i \exp(2\pi i t)\mu(t),$$

where $\mu: [0,1] \to \mathbb{R}$ is a real-valued function on $[0,1]$. It follows that

$$\begin{aligned} \alpha(t)\beta(t) &= (\sigma(t) - \exp(2\pi i t)\lambda(t))(\sigma(t) + \exp(2\pi i t)\lambda(t)) \\ &= \sigma(t)^2 - \exp(4\pi i t)\lambda(t)^2 \\ &= -\exp(4\pi i t)(\mu(t)^2 + \lambda(t)^2). \end{aligned}$$

Moreover $\mu(t)^2 + \lambda(t)^2 > 0$ for all $t \in [0,1]$. Indeed

$$\mu(t)^2 + \lambda(t)^2 = |\mathbf{U}(\cos 2\pi i t, \sin 2\pi t, 0)|^2$$

for all $t \in [0,1]$, and the vector field $\mathbf{U}$ has no zeros on the equator of S^2. Let

$$\gamma_\tau(t) = -\exp(4\pi i t)(1 - \tau + \tau(\mu(t)^2 + \lambda(t)^2))$$

for all $\tau \in [0,1]$ and $t \in [0,1]$. Then $\gamma_1(t) = \alpha(t)\beta(t)$, and

$$\gamma(0)(t) = -\exp(4\pi i t) = \exp(\tilde{\gamma}(t))$$

for all $t \in [0,1]$, where $\tilde{\gamma} = 4\pi i t + \frac{i\pi}{2}$, and no curve γ_τ passes through 0. It follows from Proposition 9.3 that

$$n(\gamma_1, 0) = n(\gamma_0, 0) = \frac{\tilde{\gamma}(1) - \tilde{\gamma}(0)}{2\pi i} = 2.$$

But also

$$\gamma_1(t) = \exp(\tilde{\alpha}(t) + \tilde{\beta}(t)),$$

where $\tilde{\alpha}: [0,1] \to \mathbb{C}$ and $\tilde{\beta}: [0,1] \to \mathbb{C}$ are continuous curves which satisfy $\exp(\tilde{\alpha}(t)) = \alpha(t)$ and $\exp(\tilde{\beta}(t)) = \beta(t)$, and therefore

$$n(\gamma_1, 0) = \frac{\tilde{\alpha}(1) + \tilde{\beta}(1) - \tilde{\alpha}(0) - \tilde{\beta}(0)}{2\pi i} = n(\alpha, 0) + n(\beta, 0).$$

It follows $n(\alpha, 0) + n(\beta, 0) = 2$, and therefore $n(\alpha, 0)$ and $n(\beta, 0)$ cannot both be zero. We deduce that the continuous vector field $\mathbf{U}$ cannot be non-zero everywhere on S^2, and must therefore have a zero at some point of S^2, as required. ∎