Course 212: Academic Year 1991-2
Section 7: Introduction to Functional Analysis

D. R. Wilkins

Contents

7 Introduction to Functional Analysis

7.1 Spaces of Bounded Continuous Functions

Let X be a topological space. We say that a function $f\colon X \to \mathbb{R}^n$ from X to $\mathbb{R}^n$ is *bounded* if there exists some non-negative constant K such that $|f(x)| \leq K$ for all $x \in X$. If f and g are bounded continuous functions from X to $\mathbb{R}^n$, then so is $f + g$. Also λf is bounded and continuous for any real number λ. It follows from this that the space $C(X, \mathbb{R}^n)$ of bounded continuous functions from X to $\mathbb{R}^n$ is a vector space over $\mathbb{R}$. Given $f \in C(X, \mathbb{R}^n)$, we define the *supremum norm* $\|f\|$ of f by the formula

$$\|f\| = \sup_{x \in X} |f(x)|.$$

One can readily verify that $\|.\|$ is a norm on the vector space $C(X, \mathbb{R}^n)$. We shall show that $C(X, \mathbb{R}^n)$, with the supremum norm, is a Banach space (i.e., the supremum norm on $C(X, \mathbb{R}^n)$ is complete). The proof of this result will make use of the following characterization of continuity for functions whose range is $\mathbb{R}^n$.

Lemma 7.1 *A function $f\colon X \to \mathbb{R}^n$ mapping a topological space X into $\mathbb{R}^n$ is continuous if and only if it satisfies the following criterion: given any point x of X and given any $\varepsilon > 0$, there exists some open set U_x in x such that $x \in U_x$ and $|f(u) - f(x)| < \varepsilon$ for all $u \in U_x$.*

Proof Suppose that $f\colon X \to \mathbb{R}^n$ is continuous. Let $x \in X$ and $\varepsilon > 0$ be given. Let

$$U_x = \{u \in X : |f(u) - f(x)| < \varepsilon\}.$$

Then U_x is open in X, since it is the preimage under f of the open ball of radius ε about $f(x)$ in $\mathbb{R}^n$. Thus U_x is the required open set.

Conversely suppose that $f\colon X \to \mathbb{R}^n$ is a function satisfying the given criterion. We must show that f is continuous. Let V be an open set in $\mathbb{R}^n$, and let $x \in f^{-1}(V)$. Then there exists some $\varepsilon > 0$ with the property that

$$\{\mathbf{y} \in \mathbb{R}^n : |\mathbf{y} - f(x)| < \varepsilon\} \subset V.$$

Now the criterion satisfied by f ensures the existence of some open set U_x in X such that $x \in U_x$ and $|f(u) - f(x)| < \varepsilon$ for all $u \in U_x$, and moreover the choice of ε ensures that $U_x \subset f^{-1}(V)$. Therefore the preimage $f^{-1}(V)$ of the open set V is the union of the open sets U_x as x ranges over all points of $f^{-1}(V)$, and is thus itself an open set. Thus $f\colon X \to \mathbb{R}^n$ is continuous, as required. ∎

Theorem 7.2 *The normed vector space $C(X, \mathbb{R}^n)$ of all bounded continuous functions from some topological space X to $\mathbb{R}^n$, with the supremum norm, is a Banach space.*

Proof Let $f_1, f_2, f_3, \ldots$ be a Cauchy sequence in $C(X, \mathbb{R}^n)$. Then, for each $x \in X$, the sequence $f_1(x), f_2(x), f_3(x), \ldots$ is a Cauchy sequence in $\mathbb{R}^n$ (since $|f_j(x) - f_k(x)| \leq \|f_j - f_k\|$ for all natural numbers j and k), and $\mathbb{R}^n$ is a complete metric space (Theorem 5.2). Thus, for each $x \in X$, the sequence $f_1(x), f_2(x), f_3(x), \ldots$ converges to some point $f(x)$ of $\mathbb{R}^n$. We must show that the limit function f defined in this way is bounded and continuous.

Using this inequality, one can easily deduce that $f_1(x), f_2(x), f_3(x), \ldots$ is a Cauchy sequence in $\mathbb{R}^n$. But every Cauchy sequence in $\mathbb{R}^n$ is convergent, since $\mathbb{R}^n$ is complete. Thus there exists some point $f(x)$ of $\mathbb{R}^n$ such that $f_j(x) \to f(x)$ as $j \to +\infty$. In this way we obtain a function $f\colon X \to \mathbb{R}^n$ from X to $\mathbb{R}^n$. We must show that f is both bounded and continuous.

Let $\varepsilon > 0$ be given. Then there exists some natural number N with the property that $\|f_j - f_k\| < \frac{1}{3}\varepsilon$ for all $j \geq N$ and $k \geq N$, since $f_1, f_2, f_3, \ldots$ is a Cauchy sequence in $C(X, \mathbb{R}^n)$. But then, on taking the limit of the left hand side of the inequality $|f_j(x) - f_k(x)| < \frac{1}{3}\varepsilon$ as $k \to +\infty$, we deduce that $\|f_j(x) - f(x)\| \leq \frac{1}{3}\varepsilon$ for all $x \in X$ and $j \geq N$. In particular $|f_N(x) - f(x)| < \frac{1}{3}\varepsilon$ for all $x \in X$. It follows that $|f(x)| \leq \|f_N\| + \frac{1}{3}\varepsilon$ for all $x \in X$, showing that the limit function f is bounded.

Next we show that the limit function f is continuous. Let $x \in X$ and $\varepsilon > 0$ be given. Let N be chosen large enough to ensure that $|f_N(u) - f(u)| \leq \frac{1}{3}\varepsilon$ for all $u \in X$. Now f_N is continuous. It follows from Lemma reffunanal-?C2 that there exists some open set U_x in X such that $x \in U_x$ and $|f_N(u) - f_N(x)| < \frac{1}{3}\varepsilon$ for all $u \in U_x$. Thus if $u \in U_x$ then

$$\begin{aligned} |f(u) - f(x)| &\leq |f(u) - f_N(u)| + |f_N(u) - f_N(x)| + |f_N(x) - f(x)| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon. \end{aligned}$$

It follows from Lemma 7.1 that the limit function f is continuous. Thus $f \in C(X, \mathbb{R}^n)$.

Finally we observe that $f_j \to f$ in $C(X, \mathbb{R}^n$ as $j \to +\infty$. Indeed we have already seen that, given $\varepsilon > 0$ there exists some natural number N such that $|f_j(x) - f(x)| \leq \frac{1}{3}\varepsilon$ for all $x \in X$ and for all $j \geq N$. Thus $\|f_j - f\| \leq \frac{1}{3}\varepsilon < \varepsilon$ for all $j \geq N$, showing that $f_j \to f$ in $C(X, \mathbb{R}^n)$ as $j \to +\infty$. This shows that $C(X, \mathbb{R}^n)$ is a complete metric space, as required. ∎

Corollary 7.3 *Let X be a metric space and let F be a closed subset of $\mathbb{R}^n$. Then the space $C(X, F)$ of bounded continuous functions from X to F is a* *complete metric space with respect to the distance function ρ, where*

$$\rho(f, g) = \|f - g\| = \sup_{x \in X} |f(x) - g(x)|$$

for all $f, g \in C(X, F)$.

Proof Let $f_1, f_2, f_3, \ldots$ be a Cauchy sequence in $C(X, F)$. Then $f_1, f_2, f_3, \ldots$ is a Cauchy sequence in $C(X, \mathbb{R}^n)$ and therefore converges in $C(X, \mathbb{R}^n)$ to some function $f\colon X \to \mathbb{R}^n$. Let x be some point of X. Then $f_j(x) \to f(x)$ as $j \to +\infty$. But $f_j(x) \in F$ for all j, and F is closed in $\mathbb{R}^n$. Therefore $f(x) \in F$, by Lemma 2.7. This shows that $f \in C(X, F)$, and thus the Cauchy sequence $f_1, f_2, f_3, \ldots$ converges in $C(X, F)$. We conclude that $C(X, F)$ is a complete metric space, as required. ∎

7.2 The Contraction Mapping Theorem and Picard's Theorem

Let X be a metric space with distance function d. A function $T\colon X \to X$ mapping X to itself is said to be a *contraction mapping* if there exists some constant λ satisfying $0 \leq \lambda < 1$ with the property that $d(T(x), T(x')) \leq \lambda d(x, x')$ for all $x, x' \in X$.

One can readily check that any contraction map $T\colon X \to X$ on a metric space (X, d) is continuous. Indeed let x be a point of X, and let $\varepsilon > 0$ be given. Then $d(T(x), T(x')) < \varepsilon$ for all points x' of X satisfying $d(x, x') < \varepsilon$.

Theorem 7.4 (Contraction Mapping Theorem) *Let X be a complete metric space, and let $T\colon X \to X$ be a contraction mapping defined on X. Then T has a unique fixed point in X (i.e., there exists a unique point x of X for which $T(x) = x$).*

Proof Let λ be chosen such that $0 \leq \lambda < 1$ and $d(T(u), T(u')) \leq \lambda d(u, u')$ for all $u, u' \in X$, where d is the distance function on X. First we show the existence of the fixed point x. Let x_0 be any point of X, and define a sequence $x_0, x_1, x_2, x_3, x_4, \ldots$ of points of X by the condition that $x_n = T(x_{n-1})$ for all natural numbers n. It follows by induction on n that $d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0)$. Using the Triangle Inequality, we deduce that if j and k are natural numbers satisfying $k > j$ then

$$d(x_k, x_j) \leq \sum_{n=j}^{k-1} d(x_{n+1}, x_n) \leq \frac{\lambda^j - \lambda^k}{1 - \lambda} d(x_1, x_0) \leq \frac{\lambda^j}{1 - \lambda} d(x_1, x_0).$$

(Here we have used the identity

$$\lambda^j + \lambda^{j+1} + \cdots + \lambda^{k-1} = \frac{\lambda^j - \lambda^k}{1-\lambda}.)$$

Using the fact that $0 \leq \lambda < 1$, we deduce that the sequence (x_n) is a Cauchy sequence in X. This Cauchy sequence must converge to some point x of X, since X is complete. But then, using Lemma 1.3, we see that

$$T(x) = T\left(\lim_{n\to+\infty} x_n\right) = \lim_{n\to+\infty} T(x_n) = \lim_{n\to+\infty} x_{n+1} = x,$$

so that x is a fixed point of T.

If x' were another fixed point of T then we would have

$$d(x', x) = d(T(x'), T(x)) \leq \lambda d(x', x).$$

But this is impossible unless $x' = x$, since $\lambda < 1$. Thus the fixed point x of the contraction map T is unique. ∎

We use the Contraction Mapping Theorem in order to prove the following existence theorem for solutions of ordinary differential equations.

Theorem 7.5 (Picard's Theorem) *Let $F: U \to \mathbb{R}$ be a continuous function defined over some open set U in the plane $\mathbb{R}^2$, and let (x_0, t_0) be an element of U. Suppose that there exists some non-negative constant M such that*

$$|F(u,t) - F(v,t)| \leq M|u - v| \text{ for all } (u,t) \in U \text{ and } (v,t) \in U.$$

Then there exists a continuous function $\varphi: [t_0 - \delta, t_0 + \delta] \to \mathbb{R}$ defined on the interval $[t_0 - \delta, t_0 + \delta]$ for some $\delta > 0$ such that $x = \varphi(t)$ is a solution to the differential equation

$$\frac{dx(t)}{dt} = F(x(t), t)$$

with initial condition $x(t_0) = x_0$.

Proof Solving the differential equation with the initial condition $x(t_0) = x_0$ is equivalent to finding a continuous function $\varphi: I \to \mathbb{R}$ satisfying the integral equation

$$\varphi(t) = x_0 + \int_{t_0}^{t} F(\varphi(s), s)\, ds.$$

where I denotes the closed interval $[t_0 - \delta, t_0 + \delta]$. (Note that any continuous function φ satisfying this integral equation is automatically differentiable, since the indefinite integral of a continuous function is always differentiable.)

Let $K = |F(x_0, t_0)| + 1$. Using the continuity of the function F, together with the fact that U is open in $\mathbb{R}^2$, one can find some $\delta_0 > 0$ such that the open disk of radius δ_0 about (x_0, t_0) is contained in U and $|F(x,t)| \leq K$ for all points (x, t) in this open disk. Now choose $\delta > 0$ such that

$$\delta\sqrt{1 + K^2} < \delta_0 \text{ and } M\delta < 1.$$

Note that if $|t - t_0| \leq \delta$ and $|x - x_0| \leq K\delta$ then (x, t) belongs to the open disk of radius δ_0 about (x_0, t_0), and hence $(x, t) \in U$ and $|F(x, t)| \leq K$.

Let J denote the closed interval $[x_0 - K\delta, x_0 + K\delta]$. The space $C(I, J)$ of continuous functions from the interval I to the interval J is a complete metric space, by Corollary 7.3. Define $T: C(I, J) \to C(I, J)$ by

$$T(\varphi)(t) = x_0 + \int_{t_0}^{t} F(\varphi(s), s)\, ds.$$

We claim that T does indeed map $C(I, J)$ into itself and is a contraction mapping.

Let $\varphi: I \to J$ be an element of $C(I, J)$. Note that if $|t - t_0| \leq \delta$ then

$$|(\varphi(t), t) - (x_0, t_0)|^2 = (\varphi(t) - x_0)^2 + (t - t_0)^2 \leq \delta^2 + K^2\delta^2 < \delta_0^2,$$

hence $|F(\varphi(t), t)| \leq K$. It follows from this that

$$|T(\varphi)(t) - x_0| \leq K\delta$$

for all t satisfying $|t - t_0| < \delta$. The function $T(\varphi)$ is continuous, and is therefore a well-defined element of $C(I, J)$ for all $\varphi \in C(I, J)$.

We now show that T is a contraction mapping on $C(I, J)$. Let φ and ψ be elements of $C(I, J)$. The hypotheses of the theorem ensure that

$$|F(\varphi(t), t) - F(\psi(t), t)| \leq M|\varphi(t) - \psi(t)| \leq M\rho(\varphi, \psi)$$

for all $t \in I$, where $\rho(\varphi, \psi) = \sup_{t \in I} |\varphi(t) - \psi(t)|$. Therefore

$$\begin{aligned} |T(\varphi)(t) - T(\psi)(t)| &= \left| \int_{t_0}^{t} \left(F(\varphi(s), s) - F(\psi(s), s)\right) ds \right| \\ &\leq M|t - t_0|\rho(\varphi, \psi) \end{aligned}$$

for all t satisfying $|t - t_0| \leq \delta$. Therefore $\rho(T(\varphi), T(\psi)) \leq M\delta\rho(\varphi, \psi)$ for all $\varphi, \psi \in C(I, J)$. But δ has been chosen such that $M\delta < 1$. This shows that $T: C(I, J) \to C(I, J)$ is a contraction mapping on $C(I, J)$. It follows from the Contraction Mapping Theorem (Theorem 7.4) that there exists a unique element φ of $C(I, J)$ satisfying $T(\varphi) = \varphi$ This function φ is the required solution to the differential equation. ∎

A straightforward but somewhat technical least upper bound argument can be used to show that if $x = \psi(t)$ is any other continuous solution to the differential equation

$$\frac{dx}{dt} = F(x,t)$$

on the interval $[t_0 - \delta, t_0 + \delta]$ satisfying the initial condition $\psi(t_0) = x_0$, then $|\psi(t) - x_0| \leq K\delta$ for all t satisfying $|t - t_0| \leq \delta$. Thus such a solution to the differential equation must belong to the space $C(I, J)$ defined in the proof of Theorem 7.5. The uniqueness of the fixed point of the contraction mapping $T\colon C(I, J) \to C(I, J)$ then shows that $\psi = \varphi$, where $\varphi\colon [t_0 - \delta, t_0 + \delta] \to \mathbb{R}$ is the solution to the differential equation whose existence was proved in Theorem 7.5. This shows that the solution to the differential equation is in fact unique on the interval $[t_0 - \delta, t_0 + \delta]$.