

# Course 121: Examination questions from previous years

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The annual examination in course 121 in 2004 will be set on the work of Hilary and Trinity Terms, 2004. It will represent 70% of the credit for the course. Candidates will be asked to attempt 6 questions out of 9 questions on the paper. Below is a collection of questions taken from examination papers for short examinations set in previous years—these can be taken as examples of the sort of questions to be found on this year's examination paper.

1. (a) State and prove *Rolle's Theorem*. [You may assume without proof any standard property of continuous functions that you require, provided that any such result is clearly stated.]  
(b) Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function from  $\mathbb{R}$  to  $\mathbb{R}$  which is  $2k + 1$  times differentiable, for some non-negative integer  $k$ . Let  $a$  and  $b$  be real numbers satisfying  $a < b$ . Suppose that  $f^{(j)}(a) = 0$  and  $f^{(j)}(b) = 0$  for  $j = 0, 1, \dots, k$ . Prove that there exists some  $\xi \in \mathbb{R}$  satisfying  $a < \xi < b$  for which  $f^{(2k+1)}(\xi) = 0$ .  
(c) State the *Mean Value Theorem*, and show how it may be deduced as a corollary of *Rolle's Theorem*.
2. Let  $f: [a, b] \rightarrow \mathbb{R}$  be a bounded function defined on the interval  $[a, b]$ .  
(a) Define the concept of a *partition* of the interval  $[a, b]$ . Give the definition of the *lower sum*  $L(P, f)$  and the *upper sum*  $U(P, f)$  of  $f$  for the partition  $P$ .  
(b) Define the *lower Riemann integral*  $\mathcal{L} \int_a^b f(t) dt$  and the *upper Riemann integral*  $\mathcal{U} \int_a^b f(t) dt$  of  $f$  on the interval  $[a, b]$ . Define precisely what is meant by saying that the function  $f$  is *Riemann-integrable* on  $[a, b]$ , and define the *Riemann integral* of a Riemann-integrable function on  $[a, b]$ .

- (c) Explain why  $L(P, f) \leq U(Q, f)$  for all partitions  $P$  and  $Q$  of  $[a, b]$ , and show that

$$\mathcal{L} \int_a^b f(t) dt \leq \mathcal{U} \int_a^b f(t) dt.$$

[You may use, without proof, the fact that  $L(R, f) \geq L(P, f)$  and  $U(R, f) \leq U(P, f)$  for any refinement  $R$  of a partition  $P$  of the interval  $[a, b]$ .]

- (d) Let  $f: [a, b] \rightarrow \mathbb{R}$  be the constant function with value  $c$ . What are the values of the lower sum  $L(P, f)$  and the upper sum  $U(P, f)$  for any partition  $P$  of the interval  $[a, b]$ ?
- (e) Let  $f: [0, 1] \rightarrow \mathbb{R}$  be defined by

$$f(t) = \begin{cases} t & \text{if } t \text{ is rational;} \\ 0 & \text{if } t \text{ is irrational.} \end{cases}$$

Is the function  $f$  Riemann-integrable on  $[0, 1]$ ? [Justify your answer.]

3. Let  $f: [a, b] \rightarrow \mathbb{R}$  be a bounded function defined on the interval  $[a, b]$ .

- (a) Define the concept of a *partition* of the interval  $[a, b]$ . Give the definition of the *lower sum*  $L(P, f)$  and the *upper sum*  $U(P, f)$  of  $f$  for the partition  $P$ .
- (b) Define the *lower Riemann integral*  $\mathcal{L} \int_a^b f(t) dt$  and the *upper Riemann integral*  $\mathcal{U} \int_a^b f(t) dt$  of  $f$  on the interval  $[a, b]$ . Define precisely what is meant by saying that the function  $f$  is *Riemann-integrable* on  $[a, b]$ , and define the *Riemann integral* of a Riemann-integrable function on  $[a, b]$ .
- (c) Let  $f: [0, 1] \rightarrow \mathbb{R}$  be defined by  $f(t) = 1 - t^2$ . Calculate  $L(P_n, f)$  and  $U(P_n, f)$ , where  $P_n$  denotes the partition of  $[0, 1]$  into  $n$  subintervals of length  $1/n$  (i.e.,  $P = \{t_0, t_1, \dots, t_n\}$ , where  $t_i = i/n$  for  $i = 0, 1, \dots, n$ ). Hence show that

$$\mathcal{L} \int_0^1 f(t) dt = \mathcal{U} \int_0^1 f(t) dt = \frac{2}{3}.$$

[You may use, without proof, the following identities:

$$\sum_{i=1}^n i = \frac{1}{2}n(n+1), \quad \sum_{i=1}^n i^2 = \frac{1}{6}n(n+1)(2n+1).]$$

4. (a) Let  $f: [a, b] \rightarrow \mathbb{R}$  be a continuous function on the interval  $[a, b]$ , and let

$$F(x) = \int_a^x f(t) dt, \quad (a \leq x \leq b).$$

Prove that  $F'(x) = f(x)$  for all  $x$  satisfying  $a < x < b$ . (N.B., this is part of the *Fundamental Theorem of Calculus*.)

- (b) Use the rule for Integration by Parts to evaluate  $\int_0^s t^3 e^{-t} dt$  for any  $s > 0$ .

5. (a) Let  $f: [a, b] \rightarrow \mathbb{R}$  be a continuous function on the interval  $[a, b]$ , and let

$$F(x) = \int_a^x f(t) dt, \quad (a \leq x \leq b).$$

Prove that  $F'(x) = f(x)$  for all  $x$  satisfying  $a < x < b$ .

- (b) Find the derivative of the function  $g: \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$g(x) = \int_0^{x^4} t^2 e^{t^2} dt.$$

6. (a) Let  $f$  be a function that is  $k$  times differentiable and whose  $k$ th derivative is continuous on some open interval containing the real numbers  $a$  and  $a+h$ . Using the rule for integration by parts, show that

$$f(a+h) = f(a) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(a) + r_k(a, h),$$

where

$$r_k(a, h) = \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} f^{(k)}(a+th) dt.$$

- (b) Show that

$$\log(1+h) = \lim_{k \rightarrow +\infty} \sum_{n=1}^k \frac{(-1)^{(n-1)} h^n}{n}$$

for all  $h$  satisfying  $-1 < h < 1$ .

7. Let  $f_1, f_2, f_3, \dots$  be a sequence of continuous real-valued functions on an interval  $[a, b]$ , and let  $f$  be a real-valued function on  $[a, b]$ .

- (a) Define what is meant by saying that the functions  $f_n$  converge *uniformly* to  $f$  on the interval  $[a, b]$  as  $n \rightarrow +\infty$ .

- (b) Suppose that the functions  $f_n$  converge uniformly to  $f$  on  $[a, b]$  as  $n \rightarrow +\infty$ . Prove that

$$\lim_{n \rightarrow +\infty} \int_a^b f_n(t) dt = \int_a^b f(t) dt.$$

- (c) Give an example of a sequence  $f_1, f_2, f_3, \dots$  of continuous real-valued functions on an interval  $[a, b]$  and a continuous real-valued function  $f$  on  $[a, b]$  such that

$$\lim_{n \rightarrow +\infty} \int_a^b f_n(t) dt \neq \int_a^b f(t) dt,$$

even though  $\lim_{n \rightarrow +\infty} f_n(t) = f(t)$  for all  $t \in [a, b]$ .

8. (a) What is meant by saying that a infinite sequence  $z_1, z_2, z_3, \dots$  of complex numbers is *convergent*?
- (b) Suppose that  $z_n = x_n + iy_n$  for all natural numbers  $n$ , where  $x_n$  and  $y_n$  are real numbers and  $i = \sqrt{-1}$ . Suppose that the sequence  $x_1, x_2, x_3, \dots$  converges to  $\lambda$  and that the sequence  $y_1, y_2, y_3, \dots$  converges to  $\mu$ , where  $\lambda$  and  $\mu$  are real numbers. Prove that the sequence  $z_1, z_2, z_3, \dots$  converges to the complex number  $\lambda + i\mu$ .
- (c) Let  $(z_n)$  and  $(w_n)$  be convergent infinite sequences of complex numbers. Prove that the sequence  $(z_n + w_n)$  is convergent. [Any relevant results concerning infinite sequences of *real* numbers may be assumed without proof.]
- (d) What is meant by saying that an infinite series  $\sum_{n=1}^{+\infty} s_n$  is convergent?
- (e) Test the following infinite sequences for convergence:—

$$\sum_{n=1}^{+\infty} \frac{\sin n - 4}{\sqrt{2n - 1}}, \quad \sum_{n=1}^{+\infty} \frac{7 \sin n - 4}{\sqrt{2n^3 - n^2}}, \quad \sum_{n=1}^{+\infty} \frac{z^n}{(2n)!}, \quad \sum_{n=1}^{+\infty} \frac{n!}{n^{n+3}}.$$

9. (a) Define what is meant by saying that a sequence  $z_1, z_2, z_3, z_4, \dots$  of complex numbers is a *Cauchy sequence*.
- (b) Prove that every convergent sequence of complex numbers is a Cauchy sequence.
- (c) Prove the every Cauchy sequence of complex numbers is bounded.

- (d) Prove that every Cauchy sequence of complex numbers is convergent. [You may use, without proof, the Bolzano-Weierstrass Theorem, which states that every bounded sequence of complex numbers has a convergent subsequence.]

10. Test the following infinite series for convergence:—

$$(a) \sum_{n=1}^{+\infty} \frac{z^n}{n! \sqrt{n}} \quad (z \in \mathbb{C}), \quad (b) \sum_{n=1}^{+\infty} \frac{3 \sin n - 2}{n \sqrt{n}}, \quad (c) \sum_{n=2}^{+\infty} \frac{(-1)^n}{\log n},$$

$$(d) \sum_{n=1}^{+\infty} \frac{4 + \cos n^2}{n}, \quad (e) \sum_{n=1}^{+\infty} \frac{n!}{n^{n+2}}.$$

11. (a) Prove that the infinite series  $\sum_{n=1}^{+\infty} \frac{1}{n}$  is divergent
- (b) By using the same method as in (a), or otherwise, prove that the infinite series  $\sum_{n=2}^{+\infty} \frac{1}{n \log n}$  is divergent
12. Prove that the infinite sequence  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$  is convergent. [N.B., if you make use of the Alternating Series Test then you must prove that this test is a valid test for convergence.]
13. (a) State the *Alternating Series Test*, and prove that any infinite series satisfying the conditions of this test is convergent.
- (b) Does the infinite series

$$\sum_{n=1}^{+\infty} \frac{2 \cos n\pi + \sin \frac{1}{2}n\pi}{n^2}$$

satisfy the conditions of the Alternating Series Test? Is this infinite series convergent?

14. Let  $\sum_{n=0}^{+\infty} a_n z^n$  be a power series whose coefficients  $a_0, a_1, a_2, \dots$  are complex numbers.

- (a) Define the *radius of convergence* of this power series.

- (b) Prove that the power series  $\sum_{n=0}^{+\infty} a_n z^n$  converges if  $z < R_0$ , but diverges if  $z > R_0$ , where  $R_0$  is the radius of convergence of the power series.

15. Determine which of the following subsets of  $\mathbb{R}^2$  are open and which are closed:—

- (a)  $\{(x, y) \in \mathbb{R}^2 : (x - 2)^2 + y^2 < 9\}$ ,  
(b)  $\{(x, y) \in \mathbb{R}^2 : (x - 2)^2 + y^2 \geq 9\}$ ,  
(c)  $\{(x, y) \in \mathbb{R}^2 : (x - 2)^2 + y^2 < 9 \text{ and } x \leq 0\}$ ,  
(d)  $\{(x, y) \in \mathbb{R}^2 : (x - 2)^2 + y^2 \geq 9 \text{ or } x \leq 0\}$ ,  
[Briefly justify your answers.]

16. (a) Prove that a sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$  of points of  $\mathbb{R}^3$  converges to some point  $\mathbf{p}$  if and only if, given any open set  $U$  which contains  $\mathbf{p}$ , there exists some natural number  $N$  such that the point  $\mathbf{x}_j$  belongs to  $U$  for all  $j$  satisfying  $j \geq N$ .  
(b) Using (a), or otherwise, show that if  $F$  is a closed set in  $\mathbb{R}^2$ , and if

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \dots$$

is an infinite sequence of points of  $\mathbb{R}^2$  belonging to  $F$  which converges to some complex number  $\mathbf{p}$  then  $\mathbf{p} \in F$ .