HW9: Continuous systems

If you find misprints, have any questions, find some task difficult and want a hint, contact me by email vel145@gmail.com.
Each question is worth **2 points**.

Consider the system described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\left(\frac{\partial\phi}{\partial t}\right)^2 - c^2\left(\frac{\partial\phi}{\partial x}\right)^2\right) - V(\phi)\,, \tag{1}$$

where c is some constant. ϕ is a function of x and t.

1. Find its energy density and momentum density. Write integrals that define energy and momentum of the system.

2. Write down equations of motion.

3. Show that the system is relativistic invariant if c is interpreted as a speed of light. This means: consider a Lorentz boost transformation $(c\,t') = \gamma(c\,t) - \beta\,\gamma\,x$, $x' = \gamma\,x - \beta\,\gamma(c\,t)$ and show that a) the Lagrangian looks exactly the same in new (primed) coordinates as in the original coordinates and b) equations of motion look exactly the same.

4. For $V = \frac{c^2}{2}\frac{1}{(1+\phi^2)^2}$, find a solution of equations of motion if it is known that this solution does not depend on time and it has the following boundary condition: $\phi = \pm\infty$ and $\frac{\partial\phi}{\partial x} = 0$ at $x = \pm\infty$.

   Your final answer should be that this solution satisfies $\phi + \frac{1}{3}\phi^3 = x - x_0$. Do not solve this cubic equation explicitly (possible, but useless for our goals).

   Hint: when solution does not depend on time, it satisfies an ordinary 2nd order differential equation. This equation looks very similar to the one of Lagrangian mechanics for a system with one degree of freedom. Use your experience from the MA2341 then.

5. For the solution from the previous exercise, depict qualitatively the energy density as a function of x. Precise how energy density behaves near $x = x_0$ and $x = \pm\infty$ (i.e. provide the Taylor expansion of the energy density as a function of x at these points, up to the first order that depends on x.)