HW7: Special relativity

If you find misprints, have any questions, find some task difficult and want a hint, contact me by email vel145@gmail.com.
Each question except the last one is worth **1 point**.

Lorentz boost transformation

Below we will use $\beta = v/c$ and $\gamma(\beta) = \frac{1}{\sqrt{1-\beta^2}}$. The formula for relativistic addition of velocities reads

$$\beta = \frac{\beta' + \beta''}{1 + \beta'\beta''}\,. \tag{1}$$

It was asked to be derived during the tutorial 6.

1. From (1), prove that if $|\beta'| < 1$, $|\beta''| < 1$ then $|\beta| < 1$. Interpretation: if particle is moving slower than light in one reference frame then it moves slower than light in any other reference frame. How to interpret the case $\beta'' = 1$?

2. Introduce the rapidity variable by the rule $\beta = \tanh\theta$. Find the value of $\sinh\theta$ and $\cosh\theta$ in terms of β and γ. Rewrite the addition of velocities formula in terms of rapidities.

3. In special relativity, the momentum of the particle of mass m with velocity v is given by $p = \gamma\, m\, v$ and the energy is given by $E/c = \sqrt{p^2 + m^2\, c^2}$. These facts were derived in HW1 from the relativistic Lagrangian.

   Find the energy as a function of β. Find the transformation law of momentum and energy under Lorentz boost in x-direction. Show that $\{E/c, \mathbf{p}\}$ transforms as either 4-vector or 4-covector (which of two?).

   In particular, an analog of the interval invariance would be written as $(E/c)^2 - \mathbf{p}^2 = m^2\, c^2$ in any reference frame.

4. In 1+1 dimensional case, it is typical to use the rapidity θ parameterisation:

   $$E = A\, c^2 \cosh\theta\,, \qquad p = A\, c \sinh\theta\,. \tag{2}$$

   What is the meaning of A? How does θ change when Lorentz boost with rapidity θ' (i.e. with $\beta' = \tanh\theta'$) is applied?

5. Consider the total momentum and energy of a collection of N particles: $\mathbf{P} = \sum_{i=1}^{N} \mathbf{p_i}\,, \quad E = \sum_{i=1}^{N} E_i\,.$ Here i labels particles, not coordinates in space.

   Suppose that velocities of the particles depend on some parameter a in a way that the total momentum and energy do not depend on a in a given reference frame. Prove that the total momentum and energy do not depend on a in any other reference frame.

Covariance

6. Write explicitly how a covariant 2-tensor $A_{\mu\nu}$ transforms under Lorentz boost in the x-direction. Note that $\mu, \nu \in \overline{1,4}$. Lorentz boost in x-direction is as usual $(x^0)' = \gamma x^0 - \beta\,\gamma x^1$ and $(x^1)' = \gamma x^1 - \beta\,\gamma x^0$, $(x^2)' = x^2$, $(y^2)' = y^2$.

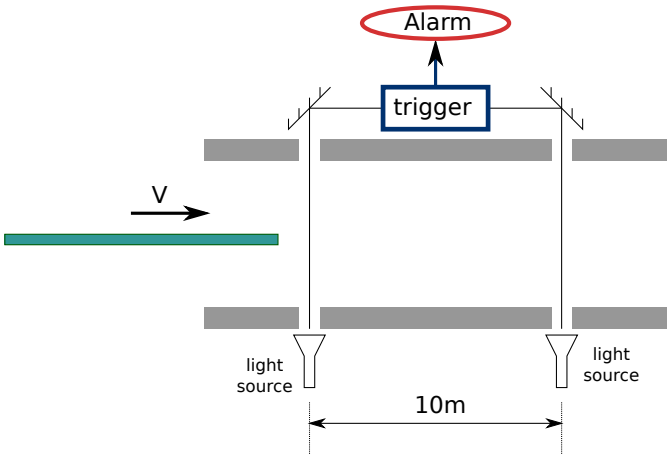

Figure 1: Movement detection paradox.

7. The Minkowski metric $\eta_{\mu\nu}$ is a covariant 2-tensor. Show that it is invariant under Lorentz boost in the x-direction.

8. If consider $A_{\mu\nu}$ as a 4×4 matrix, one can compute $\det A$. Find how $\det A$ transforms under an arbitrary Lorentz transformation.

Paradox

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9. [3 points] Resolve the paradox outlined below. Will the alarm sound or not?

   Disposition: We have a movement control system which consists of two light beams and a trigger, as shown on the figure below. The trigger is activated only if he stops receiving signals from both beams. If the trigger is activated, alarm sounds.

   Consider a rod moving through the system. The rod's length at rest is 11 meters. If the speed of rod is small then its length is not contracted significantly, so it will be able to block both lights, and the alarm will sound.

   The apparent paradox comes when the speed of the rod is comparable with the speed of light. You can check that at $v = 0.5c$ the length of the rod becomes approximately 9.5 meters, which is less than the distance between beams.

   Reference frame of the trigger: Length of the rod is shorter than the distance between the beams. So it will never block both of them, so alarm will not sound.

   Reference frame of the rod: The whole control system is moving with $v = -0.5c$. Distance between two beams is approximately 8.7 meters, while the length of the rod is 11 meters. Hence the rod will block both beams at some point and alarm will sound.

   Note: Triggering or not of the alarm is a universal output which cannot depend on the reference frame.