

① $V \otimes V$ is a vector space. We can define addition as follows

$$(*) \quad \forall \sum_{i,j} a_{ij} x^i \otimes x^j + \sum_{i,j} b_{ij} x^i \otimes x^j = \sum_{i,j} (a_{ij} + b_{ij}) x^i \otimes x^j$$

And multiplication by a number as:

$$(**) \quad \lambda \cdot \left(\sum_{i,j} a_{ij} x^i \otimes x^j \right) = \sum_{i,j} (\lambda a_{ij}) x^i \otimes x^j$$

The only think we have to verify that (*) and (**) can be well-defined on $S^2(V)$ and $\Lambda^2(V)$.

If $a_{ij} = a_{ji}$ then we can identify $\sum a_{ij} x^i \otimes x^j$ with elements of $S^2(V)$.

$a_{ij} + b_{ij}$ preserves the property of being symmetric.

λa_{ij} preserves the property of being symmetric.

Hence (*) and (**) are ~~possibly~~ restricted to $S^2(V)$, hence they define the vector space.

The same for $a_{ij} = -a_{ji}$.

② (a) $(5\theta_1 + 3\theta_2) \wedge (\theta_1 - 2\theta_3 + 1) = 5\theta_1 \wedge \theta_1 - 10\theta_1 \wedge \theta_3 + 5\theta_1 \wedge 1$
 $+ 3\theta_2 \wedge \theta_1 - 6\theta_2 \wedge \theta_3 + 3\theta_2 \wedge 1 = (*)$

$$\theta_1 \wedge \theta_1 = -\theta_1 \wedge \theta_1 \Rightarrow 2\theta_1 \wedge \theta_1 = 0 \Rightarrow \theta_1 \wedge \theta_1 = 0$$

$$\theta_i \wedge 1 = \theta_i \text{ (by definition).}$$

$$(*) = 5\theta_1 + 3\theta_2 - 3\theta_1 \wedge \theta_2 - 6\theta_2 \wedge \theta_3 - 10\theta_1 \wedge \theta_3$$

(b), (c), (d). Note: If ω is a homogeneous polynomial in θ 's of degree 1 then it is clear that $\omega \wedge \omega = -\omega \wedge \omega \Rightarrow \omega \wedge \omega = 0$

Hence (b): 0 (c): 0 (d): 0.

answers:

$$(e) \quad e^{\overbrace{\theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4}^x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots ; \quad \begin{cases} x^2 = 2\theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \theta_4 \\ x^3 = 0 \\ x^n = 0, n \geq 2 \end{cases} ; \quad \begin{cases} \omega = 1 + \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4 \\ \quad + \theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \theta_4 \end{cases}$$

[3]

The generic element in this algebra is:

$$\sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_k} c_{i_1, i_2, \dots, i_k} \theta^{i_1} \wedge \theta^{i_2} \wedge \dots \wedge \theta^{i_k}$$

When $k > n$, $\theta^{i_1} \wedge \dots \wedge \theta^{i_k} = 0$ because at least two of the θ 's are the same.

Hence, in truth, generic element is

$$(*) \quad \sum_{k=0}^n \underbrace{\sum_{\substack{1 \leq i_1 < i_2 < \dots < i_k \leq n}}}_{\text{finite number of terms.}} c_{i_1, i_2, \dots, i_k} \theta^{i_1} \wedge \theta^{i_2} \wedge \dots \wedge \theta^{i_k}$$

this is
an element in $\Lambda^k(W)$

this is
an element in $\Lambda(V)$

$$\dim \Lambda^k(W) = C_n^k \quad (\text{a way to choose } k \text{ integers from } n \text{ integers. This is the number of terms in the inner sum of (*)})$$

$$\dim \Lambda(V) = \sum_{k=0}^n C_n^k = 2^n$$

↑ this is known e.g. from:

$$2^n = (1+1)^n = \sum_{k=0}^n C_n^k \cdot 1^k \cdot 1^{n-k}$$

Another way to get this number:

$\prod_{i=1}^n (1 + \theta_i)$ ← this obviously generates all monomials in θ 's
 From each term in the product we choose either 1 or $\theta_i \Rightarrow$
 $= 2^n$ possibilities

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$$X = \alpha \wedge \beta = \alpha_i \theta^i \wedge \beta_j \theta^j = \sum_{i,j} (\alpha_i \beta_j) \theta^i \wedge \theta^j =$$

HW2 (3)
solutions

$$= \frac{1}{2} \sum_{i,j} (\alpha_i \beta_j) \theta^i \wedge \theta^j + \frac{1}{2} \sum_{j,i} (\alpha_j \beta_i) \theta^i \wedge \theta^j =$$

MA2392
2014

$$= \frac{1}{2} \sum_{i,j} (\alpha_i \beta_j - \alpha_j \beta_i) \theta^i \wedge \theta^j = \sum_{i,j} (\alpha_i \beta_j - \alpha_j \beta_i) \theta^i \wedge \theta^j$$

By definition: $X \wedge X = \sum_{i,j} X_{ij} \theta^i \wedge \theta^j$

hence:

$$\boxed{X_{ij} = \alpha_i \beta_j - \alpha_j \beta_i}$$

$$Y = \alpha \wedge \omega = \alpha_i \theta^i \wedge \frac{1}{2} \omega_{jk} \theta^j \wedge \theta^k = \frac{1}{2} \sum_{i,j,k} (\alpha_i \omega_{jk}) \theta^i \wedge \theta^j \wedge \theta^k =$$

$$= \text{(antisymmetrisation)} = \frac{1}{3!} \sum_{i,j,k} (\alpha_i \omega_{jk} + \alpha_j \omega_{ki} + \alpha_k \omega_{ij}) \theta^i \wedge \theta^j \wedge \theta^k =$$

$$= \sum_{i,j,k} (-1)^{i+j+k} \theta^i \wedge \theta^j \wedge \theta^k$$

$$\boxed{Y_{ijk} = \alpha_i \omega_{jk} + \alpha_j \omega_{ki} + \alpha_k \omega_{ij}}$$

$$Z = \omega \wedge \eta = \frac{1}{2} \omega_{ij} \theta^i \wedge \theta^j \wedge \frac{1}{2} \eta_{ke} \theta^k \wedge \theta^e =$$

$$= \frac{1}{4} \sum_{i,j,k,e} \omega_{ij} \eta_{ke} \theta^i \wedge \theta^j \wedge \theta^k \wedge \theta^e =$$

$$= \frac{1}{4} \frac{1}{6} \sum_{i,j,k,e} (\omega_{ij} \eta_{ke} - \omega_{ik} \eta_{je} + \omega_{je} \eta_{ik} - \omega_{kj} \eta_{ie} - \omega_{ej} \eta_{ki} + \omega_{ek} \eta_{ij}).$$

because
6 terms

$\underbrace{24}_{\text{exactly factors for}} \rightarrow \sum_{i,j,k,e} \sum_{i,j,k,e}$

4-continued

Conclusion:

HW2 (4)
Solutions

$$\begin{aligned} Z_{ijk\ell} = & \omega_{ij}\eta_{ke} - \omega_{ie}\eta_{je} + \omega_{ik}\eta_{je} - \omega_{kj}\eta_{il} - \\ & - \omega_{lj}\eta_{ki} + \omega_{ke}\eta_{ij} \end{aligned}$$

In general: if ω - form of degree k
 η - form of degree l

$$(\omega \wedge \eta)_{i_1 \dots i_{k+l}} = \frac{1}{k!l!} \sum_{\text{permutations}} \text{sign}(\sigma) \omega_{i_{\sigma(1)} \dots i_{\sigma(k)}} \eta_{i_{\sigma(k+1)} \dots i_{\sigma(k+l)}}$$

5 Let $\alpha = \alpha_i \theta^i$. Without loss of generality,
assume $\alpha_1 \neq 0$. Then we can use
instead of basis $\theta^1, \dots, \theta^n$ a basis

$$\alpha, \theta^2, \theta^3, \dots, \theta^n$$

$$\omega = \sum_{1 \leq i, j \leq n} w_{ij} \theta^i \theta^j + \sum_{1 \leq i \leq n} w_{ii} \alpha \wedge \theta^i$$

$$\alpha \wedge \omega = \underbrace{\sum_{1 \leq i, j \leq n} w_{ij} \alpha \wedge \theta^i \wedge \theta^j}_{\substack{\text{all terms here} \\ \text{are linearly independent, hence}}} + 0$$

each is equal 0 separately. $\Rightarrow \omega = \alpha \wedge (\underbrace{w_{ii} \theta^i}_{\text{define } \beta})$

Hence $\omega = \alpha \wedge \beta$.

6 We prove: If $\omega \neq \alpha \wedge \beta$ then $\omega \wedge \omega \neq 0$.

Answers:
yes, true

Since $\omega \neq \alpha \wedge \beta$, generically

we can present

$$\omega = \sum_{i=1}^K \alpha_i \wedge \beta_i \quad \text{for some finite } K.$$

It is always possible: Just take $\alpha_i = \theta^{i'}$
 $\beta_i = \omega \theta^{i'} \theta^{j'}$

(6-continued)

HW 2 (5)

solutions

It is always possible to find such α_i and β_i

that they all are linearly independent. Indeed,

$$\text{if } \alpha_1 = \sum_{i=2}^k c_i \alpha_i + \sum_{i=1}^k d_i \beta_i \text{ then}$$

$$\omega = \alpha_1 \wedge \beta_1 + \sum_{i=2}^k \alpha_i \wedge \beta_i = \sum_{i=2}^k (c_i \alpha_i + d_i \beta_i) \wedge \beta_1 + \sum_{i=2}^k \alpha_i \wedge \beta_i =$$

$$= \cancel{\sum_{i=2}^k c_i \alpha_i \wedge (\beta_1 + \beta_i)} + \cancel{\sum_{i=2}^k d_i \beta_i \wedge (\alpha_i - d_i \beta_i)} \wedge \beta_1 =$$

$$= \sum_{i=2}^k (d_i - d_i \beta_i) \wedge (\beta_i + c_i \beta_1)$$

I.e. we reduced k by to $k-1$.

We do not need to prove that there exists a minimal k such that all α_i and β_i are linearly independent.

Important that it exists. Clearly, $K \leq \lceil \frac{n}{2} \rceil$, and by condition of the theorem we want to prove, $K > 1$,

$$\omega = \sum_{i=1}^k \alpha_i \wedge \beta_i$$

$$\omega^K = K! \alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_K \wedge \beta_1 \wedge \dots \wedge \beta_K \neq 0$$

because of
linear independence
of α_i and β_i
(cf. exercise 5).

Since $K > 1$, we conclude
that $\omega \wedge \omega \neq 0$

So, we proved: if $\omega \neq \alpha \wedge \beta \Rightarrow \omega \wedge \omega \neq 0$

The inverse statement is

$$\boxed{\omega \wedge \omega = 0 \Rightarrow \omega = \alpha \wedge \beta} \quad \underline{\text{QED}}$$

6-continued]. Finding dimensions of subspace defined

Hwk2 ⑥

by $\omega \wedge \omega = 0$
Denote it as $\dim(n)$

solutions

• Cheating method :)

Assume that $\dim(n)$ is a polynomial

There is no justification of this assumption, it can be function like $\dim(n) = [\frac{n}{2}] \cdot n$ (for instance) which is even not analytic in n . But it is always good to be optimistic.

Since $\dim \Lambda^2(V) = \underbrace{\frac{n(n-1)}{2}}_{\text{dimension of space with coordinates } w_{ij}} \sim n^2$ we conclude

that $\dim(n)$ grows at most quadratically,
so!

$$\dim(n) = An^2 + Bn + C.$$

Now we know: $n=2: \omega \wedge \omega = 0 \text{ always} \Rightarrow \dim(2) = \dim \Lambda^2(\mathbb{R}^2) = \underline{1}$

$n=3: \omega \wedge \omega = 0 \text{ always} \Rightarrow \dim(3) = \dim \Lambda^2(\mathbb{R}^3) = \underline{3}$

$n=4: \omega \wedge \omega =$

$$= 2(\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23}) \theta^1 \theta^2 \theta^3 \theta^4 = 0$$

One condition

$$\Rightarrow \underline{\dim(4)} = \dim \Lambda^2(\mathbb{R}^4) - 1 = 5$$

$$\dim(2) = 1$$

$$\dim(3) = 3$$

$$\dim(4) = 5$$

$$\Rightarrow \boxed{\dim(n) = 2n-3} \leftarrow \text{answer.}$$

6-continued

Finding $\dim(n)$.

HW2

(7)

solutions

- ~~the honest method~~

$$\text{Since } \omega \wedge \omega \Rightarrow \omega = \alpha \wedge \beta$$

$$\begin{aligned} \alpha &= \alpha_i \theta^i \leftarrow n \text{ parameters} \\ \beta &= \beta_j \theta^j \leftarrow n \text{ parameters} \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} 2n \text{ parameters}$$

$$\text{But } \alpha \wedge \beta = \frac{(\alpha + c_1 \beta) \wedge (\beta + c_2 \alpha)}{1 - c_1 c_2} \quad -2 \text{ parameters}$$

$$\alpha \wedge \beta = (\lambda \alpha) \wedge (\frac{1}{\lambda} \beta) \quad -1 \text{ parameters.}$$

$$\dim(n) = \boxed{2n-3}$$

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$$n=2: \quad \omega = \omega_{12} \theta^1 \wedge \theta^2 \quad \omega_{12} = \sqrt{\det \omega} \quad \omega = \begin{pmatrix} 0 & \omega_{12} \\ -\omega_{12} & 0 \end{pmatrix}$$

$$n=4: \quad \underbrace{\omega = 2(\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23})}_{\text{Pf}(\omega) = \sqrt{\det \omega}} \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4,$$

$$\text{Pf}(\omega) = \sqrt{\det \omega} \quad \omega = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} & \omega_{14} \\ -\omega_{12} & 0 & \omega_{23} & \omega_{24} \\ -\omega_{13} & -\omega_{23} & 0 & \omega_{34} \\ -\omega_{14} & -\omega_{24} & -\omega_{34} & 0 \end{pmatrix}$$

Good thing to do this exercise
at least once in the life.

$$\text{Generic } n: \quad \omega_{i_1 \dots i_n} = \frac{1}{2^{n/2}} \sum_{j_1 \dots j_n} \epsilon_{i_1 \dots i_n} w_{i_1 j_1} w_{i_2 j_2} \dots \underbrace{w_{i_n j_n}}$$

This is so called Plucker identity.

$$\text{Pf}(n/2)! \cdot \text{Pf}(\omega) \geq$$

$\epsilon_{i_1 \dots i_n} \delta_{j_1} = \epsilon_{j_1 \dots j_n} \delta_{i_1} + \epsilon_{j_1 \dots j_n} \delta_{i_1} \dots + \epsilon_{i_1 \dots i_n} \delta_{j_n}$

The point is that only one of terms on the R.H.S is non-zero.

$$\text{Pf}(\omega) = \sqrt{\det \omega} \quad (\text{Proof is not included, you can find it online})$$

8 $dx^1 \wedge dx^2 \wedge dx^3 (x_i \otimes x_j \otimes x_k) = \epsilon_{ijk}$

$$\vec{u} \otimes \vec{v} \otimes \vec{w} = u^i v^j w^k dx_i \otimes dx_j \otimes dx_k$$

$$(dx^1 \wedge dx^2 \wedge dx^3) (\vec{u} \otimes \vec{v} \otimes \vec{w}) = \epsilon_{ijk} u^i v^j w^k$$

On the other hand, the volume of parallelogram
is computed from:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{u} (u_i, \epsilon_{ijk} v_j w_k) = \epsilon_{ijk} u^i v^j w^k.$$

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$$\omega_{i_1 \dots i_k}^{(x)} = \frac{\partial y^{j_1}}{\partial x^{i_1}} \dots \frac{\partial y^{j_k}}{\partial x^{i_k}} \omega_{j_1 \dots j_k}^{(y)}$$

A Explanation

$$\begin{aligned} \omega &= \frac{1}{k!} \underbrace{\omega_{j_1 \dots j_k}^{(y)}}_{\text{J.P. } k!} dy^{j_1} dy^{j_2} \dots dy^{j_k} = \\ &= \underbrace{\frac{1}{k!} \omega_{j_1 \dots j_k}^{(y)} \frac{\partial y^{j_1}}{\partial x^{i_1}} \dots \frac{\partial y^{j_k}}{\partial x^{i_k}}}_{\text{this should be}} dx^{i_1} \wedge \dots \wedge dx^{i_k} \end{aligned}$$

10 (a) Answers: for $\theta - x = r \sin \theta \cos \varphi$
 $y = r \sin \theta \sin \varphi$
 $z = r \cos \varphi$

$$dx \wedge dy \wedge dz = r^2 \sin \theta \, dr \wedge d\theta \wedge d\varphi$$

Derivation is up to you. I will only check signs!

For this we can consider small θ and small φ

~~then $\sin \theta \approx \theta$; $\cos \theta \approx 1$~~ (but after taking $d\theta$)
 ~~$dr \approx 0$~~ ~~Keep only linear terms~~

$$dx \approx dr \theta + r d\theta \approx$$

$$dy \approx r d\varphi + \varphi r d\theta$$

$$dz = dr - \varphi r d\varphi$$

10 - continued

HW2 ④

solutions

$$dx \wedge dy \wedge dz = (\cancel{\theta dr + r d\theta}) \wedge (\cancel{\theta r d\varphi + \varphi r d\theta}) \wedge (dr - \cancel{r d\varphi}) =$$

already
linear

$$= r d\theta \wedge (\theta r d\varphi) \wedge dr = r^2 \theta dr \wedge d\theta \wedge d\varphi.$$

this checks sign.

(b) $\underline{dx \wedge dy = \frac{1}{2} dz \wedge d\bar{z}}$

Derivation: $x = \frac{1}{2}(z + \bar{z}) \Rightarrow dx = \frac{1}{2}(dz + d\bar{z})$
 $y = \frac{1}{2i}(z - \bar{z}) \Rightarrow dy = -\frac{1}{2i}(dz - d\bar{z})$

$$dx \wedge dy = -\frac{1}{2i}(dz + d\bar{z}) \wedge (dz - d\bar{z}) = \frac{1}{2} dz \wedge d\bar{z}$$

II g_{ij} transforms under change of coordinates

as:

$$g''^{ij} = \frac{\partial y^i}{\partial x^j} \frac{\partial y^j}{\partial x^i} g^{ij}$$

$$\text{Hence } \det g''^{ij} = (\det J)^2 \det g^{ij}$$

$$g''^{ij} = \det J \sqrt{g^{ij}}$$

$$dx^1 \wedge \dots \wedge dx^n =$$

$$= \frac{\partial x^1}{\partial y^{i_1}} \dots \frac{\partial x^n}{\partial y^{i_n}} dy^{i_1} \wedge \dots \wedge dy^{i_n}$$

$$= \epsilon_{i_1 \dots i_n} \frac{\partial x^1}{\partial y^{i_1}} \dots \frac{\partial x^n}{\partial y^{i_n}} dy^{i_1} \wedge \dots \wedge dy^{i_n}$$

$$= \det(J^{-1}) dy^1 \wedge \dots \wedge dy^n$$

↓

$\int g''^{ij} dx^1 \wedge \dots \wedge dx^n$ is invariant.

!!

$$\int g^{ij} dy^1 \wedge \dots \wedge dy^n$$

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$$d(z^2 dx \wedge dy + e^x dx \wedge dz + (x-y)^3 dy \wedge dz) =$$

HW2 (16)
solutions

$$= (2z dz \wedge dx \wedge dy + \underbrace{e^x dx \wedge dx \wedge dz}_{0} + (3(x-y)^2 dx + 3(x-y)^2 dy) dy \wedge dz) \wedge dy \wedge dz / s$$

$$= dx \cdot (2z + 3(x-y)^2) dx \wedge dy \wedge dz.$$

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$$d\omega = dx^k \frac{\partial}{\partial x^k} \wedge \left(\frac{1}{2} \omega_{ij} dx^i \wedge dx^j \right) =$$

$$= \frac{1}{2} \partial_k \omega_{ij} dx^k \wedge dx^i \wedge dx^j = (\text{antisymmetrisation})$$

$$= \frac{1}{6} (\partial_k \omega_{ij} + \partial_i \omega_{jk} + \partial_j \omega_{ki}) \cancel{dx^k} dx^i \wedge dx^j \wedge dx^k$$

$$(d\omega)_{ijk} = \partial_i \omega_{jk} + \partial_j \omega_{ki} + \partial_k \omega_{ij}$$

(c.f. Maxwell
equations
 $\cancel{dF = 0}$)

\cancel{F} as ω .

14, 15 - text book material.