Results for Mandelstam variables

Donie O'Brien
Email: donie@maths.tcd.ie

The Mandelstam variables are defined as follows

$$s = (k + p)^2 = (k' + p')^2$$
$$t = (p' - p)^2 = (k - k')^2$$
$$u = (k' - p)^2 = (p' - k)^2$$

using $p \cdot p = p' \cdot p' = m^2$ and $k \cdot k = k' \cdot k' = M^2$ because of elastic scattering we can square above to get.

$$s = m^2 + M^2 + 2\,p \cdot k = m^2 + M^2 + 2\,p' \cdot k'$$
$$t = 2\,m^2 - 2\,p \cdot p' = 2\,M^2 - 2\,k \cdot k'$$
$$u = m^2 + M^2 - 2\,p \cdot k' = m^2 + M^2 - 2\,p' \cdot k$$

hence we see that

$$p \cdot k = p' \cdot k'$$
$$p \cdot k' = p' \cdot k$$

and we have the defining relation

$$s + t + u = \sum_{i=1}^{4} m_i^2$$

Which in our case of elastic scatering is

$$s + t + u = 2\,m^2 + 2\,M^2$$

Rearranging the t equation we get

$$p \cdot p' = \left(m^2 - \tfrac{t}{2} \right)$$
$$k \cdot k' = \left(M^2 - \tfrac{t}{2} \right)$$

Similarily for s and u

$$p' \cdot k = p \cdot k' = \tfrac{1}{2} \left(m^2 + M^2 - u \right)$$
$$p \cdot k = p' \cdot k' = -\tfrac{1}{2} \left(m^2 + M^2 - s \right)$$

For convenience lets define

$$R_\mu = \left(k_\mu + k'_\mu \right)$$
$$r_\nu = \left(p_\nu + p'_\nu \right)$$

Thus we have

$$R^2 = R^\mu R_\mu = k'^2 + 2\,k \cdot k' + k^2 = 2\,M^2 + 2\,M^2 - t = \left(4\,M^2 - t \right)$$
$$r^2 = r^\nu r_\nu = \cdots = \left(4\,m^2 - t \right)$$

Now

$$\begin{aligned} R \cdot r &= R^\mu r_\mu = \left(p^\mu + p'^\mu \right) \left(k_\mu + k'_\mu \right) = \left(p \cdot k + p' \cdot k + p \cdot k' + p' \cdot k' \right) \\ &= 2 \left[-\frac{1}{2} \left(m^2 + M^2 - s \right) \right] + 2 \left[\frac{1}{2} \left(m^2 + M^2 - u \right) \right] = 2 \left[\frac{1}{2} \left(s - u \right) \right] \\ &= (s - u) \end{aligned}$$

Similarily

$$R \cdot p = R \cdot p' = r \cdot k = r \cdot k' = \frac{1}{2} \left(s - u \right)$$