Assignment 8  MA 1124

5.2, 5.5, 5.4, Assignment.

a. Also try to use the L.U.B. axiom to prove [a, b] is compact.

Hint: [a, b] can be covered by one O_y for some c > a.
51. Prove: If \( a_n \to a \) and \( b_n \to b \) where \( b_n \neq 0 \) and \( b \neq 0 \), then the sequence \( \{a_n/b_n\} \) converges to \( a/b \).

52. Prove: If the sequence \( (a_n) \) converges to \( b \), then every subsequence \( (a_{n_k}) \) of \( (a_n) \) also converges to \( b \).

53. Prove: If the sequence \( (a_n) \) converges to \( b \), then either the range \( \{a_n\} \) of the sequence \( (a_n) \) is finite, or \( b \) is an accumulation point of the range \( \{a_n\} \).

54. Prove: If the sequence \( (a_n) \) of distinct elements is bounded and the range \( \{a_n\} \) of \( (a_n) \) has exactly one limit point \( b \), then the sequence \( (a_n) \) converges to \( b \).

(Remark: The sequence \( \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \) shows that the condition of boundedness cannot be removed from this theorem.)

CONTINUITY

55. Prove: A function \( f: \mathbb{R} \to \mathbb{R} \) is continuous at \( a \in \mathbb{R} \) if and only if for every sequence \( (a_n) \) converging to \( a \), the sequence \( (f(a_n)) \) converges to \( f(a) \).

56. Prove: Let the function \( f: \mathbb{R} \to \mathbb{R} \) be continuous at \( p \in \mathbb{R} \). Then there exists an open interval \( S \) containing \( p \) such that \( f \) is bounded on the open interval \( S \).

57. Give an example of a function \( f: \mathbb{R} \to \mathbb{R} \) which is continuous at every point in the open interval \( S = (0, 1) \) but which is not bounded on the open interval \( S \).

58. Prove: Let \( f: \mathbb{R} \to \mathbb{R} \) be continuous at every point in a closed interval \( A = [a, b] \). Then \( f \) is bounded on \( A \). (Remark: By the preceding problem, this theorem is not true if \( A \) is not closed.)

59. Prove: Let \( f: \mathbb{R} \to \mathbb{R} \) and \( g: \mathbb{R} \to \mathbb{R} \) be continuous. Then the sum \( (f+g): \mathbb{R} \to \mathbb{R} \) is continuous, where \( f+g \) is defined by \( (f+g)(x) = f(x) + g(x) \).

60. Prove: Let \( f: \mathbb{R} \to \mathbb{R} \) be continuous, and let \( k \) be any real number. Then the function \( (kf): \mathbb{R} \to \mathbb{R} \) is continuous, where \( kf \) is defined by \( (kf)(x) = kf(x) \).

61. Prove: Let \( f: \mathbb{R} \to \mathbb{R} \) and \( g: \mathbb{R} \to \mathbb{R} \) be continuous. Then \( \{x \in \mathbb{R} : f(x) = g(x)\} \) is a closed set.

62. Prove: The projection \( \pi_x: \mathbb{R}^2 \to \mathbb{R} \) is continuous where \( \pi_x \) is defined by \( \pi_x((a, b)) = a \).

63. Consider the functions \( f: \mathbb{R} \to \mathbb{R} \) and \( g: \mathbb{R} \to \mathbb{R} \) defined by

\[
f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}
\]

\[
g(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}
\]

Prove \( g \) is continuous at 0 but \( f \) is not continuous at 0.

64. Recall that every rational number \( q \in \mathbb{Q} \) can be written uniquely in the form \( q = a/b \) where \( a \in \mathbb{Z} \), \( b \in \mathbb{N} \), and \( a \) and \( b \) are relatively prime. Consider the function \( f: \mathbb{R} \to \mathbb{R} \) defined by

\[
f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/b & \text{if } x \text{ is rational and } x = a/b \text{ as above} \end{cases}
\]

Prove that \( f \) is continuous at every irrational point, but \( f \) is discontinuous at every rational point.

Answers to Supplementary Problems

57. Consider the function \( f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ 1/x & \text{if } x > 0 \end{cases} \)

The function \( f \) is continuous at every point in \( \mathbb{R} \) except at 0 as indicated in the adjacent graph of \( f \). Hence \( f \) is continuous at every point in the open interval \((0, 1)\). But \( f \) is not bounded on \((0, 1)\).

58. Hint: Use the result stated in Problem 56 and the Heine-Borel Theorem.