1. p233, 22,30 attached.

2. p64, 44,45,47,48 attached.

3. If \( A_\alpha \) are open, what about finite unions? countable unions? All unions? Same for intersections?

4. If \( A_\alpha \) are closed, same questions.

5. Prove that \( A \) is open iff \( \forall x \in A, x_n \to x_n \) implies the sequence \( x_n \) is ultimately in \( A \), i.e. \( \exists N \) such that \( n \geq N \to x_n \in A \).
Solution:
Suppose $p_1$ and $p_2$ belong to every interval. If $p_1 \neq p_2$, then $|p_1 - p_2| = \delta > 0$. Since
\[ \lim_{n \to \infty} (b_n - a_n) = 0, \]
there exists an interval $I_{n_0} = [a_{n_0}, b_{n_0}]$ such that the length of $I_{n_0}$ is less than
the distance $|p_1 - p_2| = \delta$ between $p_1$ and $p_2$. Accordingly, $p_1$ and $p_2$ cannot both belong to $I_{n_0}$, a
contradiction. Thus $p_1 = p_2$, i.e., only one point can belong to every interval.

Supplementary Problems

FIELD AXIOMS

20. Show that the Right Distributive Law [D4] is a consequence of the Left Distributive Law [D3] and
the Commutative Law [M4].

21. Show that the set $\mathbb{Q}$ of rational numbers under addition and multiplication is a field.

22. Show that the following set $A$ of real numbers under addition and multiplication is a field:
\[ A = \{a + b\sqrt{2} : a, b \text{ rational}\} \]

23. Show that the set $A = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ of even integers under addition and multiplication
satisfies all the axioms of a field except $[M_5]$, $[M_4]$ and $[M_4]$, that is, is a ring.

INEQUALITIES AND POSITIVE NUMBERS

24. Rewrite so that $x$ is alone between the inequality signs:
(i) $4 < -2x < 10$, (ii) $-1 < 2x - 3 < 5$, (iii) $-3 < 5 - 2x < 7$.

25. Prove: The product of any two negative numbers is positive.

26. Prove Theorem A.2(ii): If $a < b$, then $a + c < b + c$.

27. Prove Theorem A.2(iv): If $a < b$ and $c$ is positive, then $ac < bc$.

28. Prove Corollary A.3: The set $\mathbb{R}$ of real numbers is totally ordered by the relation $a \leq b$.

29. Prove: If $a < b$ and $c$ is positive, then:
(i) $\frac{a}{c} < \frac{b}{c}$, (ii) $\frac{c}{b} < \frac{c}{a}$.

30. Prove: $\sqrt[n]{ab} = (a+b)/2$. More generally, prove $\sqrt[n]{a_1a_2\cdots a_n} = (a_1 + a_2 + \cdots + a_n)/n$.

31. Prove: Let $a$ and $b$ be real numbers such that $a < b + \epsilon$ for every $\epsilon > 0$. Then $a = b$.

32. Determine all real values of $x$ such that:
(i) $x^3 + x^2 - 6x > 0$, (ii) $(x-1)(x+3)^2 \leq 0$.

ABSOLUTE VALUES

33. Evaluate:
(i) $|2| + |3 - 4|$, (ii) $|3 - 8| - |1 - 9|$, (iii) $|-4| - |2 - 7|$.

34. Rewrite, using the absolute value sign:
(i) $-3 < x < 9$, (ii) $2 \leq x \leq 8$, (iii) $-7 < x < -1$.

35. Prove:
(i) $|a| = |a|$, (ii) $a^2 = |a|^2$, (iii) $|a| = \sqrt{a^2}$, (iv) $|x| < a$ iff $-a < x < a$. 


We claim that \( f(p) = 0 \). If \( f(p) < 0 \), then, by the preceding problem, there is an open interval \((p - \delta, p + \delta)\) in which \( f \) is negative, i.e.,

\[ (p - \delta, p + \delta) \subseteq A \]

So \( p \) cannot be an upper bound for \( A \). On the other hand, if \( f(p) > 0 \), then there exists another interval \((p - \delta, p + \delta)\) in which \( f \) is positive; so

\[ (p - \delta, p + \delta) \cap A = \emptyset \]

which implies that \( p \) cannot be a least upper bound for \( A \). Thus \( f(p) \) can only be zero, i.e., \( f(p) = 0 \).

Remark. The theorem is also true and proved similarly in the case \( f(b) < 0 < f(a) \).

37. Prove Theorem (Weierstrass) 4.9: Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous on a closed interval \([a, b]\). Then the function assumes every value between \( f(a) \) and \( f(b) \).

Solution:
Suppose \( f(a) < f(b) \) and let \( y_0 \) be a real number such that \( f(a) < y_0 < f(b) \). We want to prove that there is a point \( p \) such that \( f(p) = y_0 \). Consider the function \( g(x) = f(x) - y_0 \) which is also continuous. Note that \( g(a) < 0 < g(b) \).

By the preceding problem, there exists a point \( p \) such that \( g(p) = f(p) - y_0 = 0 \). Hence \( f(p) = y_0 \).

The case when \( f(b) < f(a) \) is proved similarly.

Supplementary Problems

OPEN SETS, CLOSED SETS, ACCUMULATION POINTS
38. Prove: If \( A \) is a finite subset of \( \mathbb{R} \), then the derived set \( A' \) of \( A \) is empty, i.e. \( A' = \emptyset \).

39. Prove: Every finite subset of \( \mathbb{R} \) is closed.

40. Prove: If \( A \subset B \), then \( A' \subset B' \).

41. Prove: A subset \( B \) of \( \mathbb{R}^2 \) is closed if and only if \( d(p, B) = 0 \) implies \( p \in B \), where \( d(p, B) = \inf \{ d(p, q) : q \in B \} \).

42. Prove: \( A \cup A' \) is closed for any set \( A \).

43. Prove: \( A \cup A' \) is the smallest closed set containing \( A \), i.e. if \( F \) is closed and \( A \subset F \subset A \cup A' \) then \( F = A \cup A' \).

44. Prove: The set of interior points of any set \( A \), written \( \text{int}(A) \), is an open set.

45. Prove: The set of interior points of \( A \) is the largest open set contained in \( A \), i.e. if \( G \) is open and \( \text{int}(A) \subset G \subset A \), then \( \text{int}(A) = G \).

46. Prove: The only subsets of \( \mathbb{R} \) which are both open and closed are \( \emptyset \) and \( \mathbb{R} \).

SEQUENCES
47. Prove: If the sequence \( (a_n) \) converges to \( b \in \mathbb{R} \), then the sequence \( (|a_n - b|) \) converges to 0.

48. Prove: If the sequence \( (a_n) \) converges to 0, and the sequence \( (b_n) \) is bounded, then the sequence \( (a_n b_n) \) also converges to 0.

49. Prove: If \( a_n \to a \) and \( b_n \to b \), then the sequence \( (a_n + b_n) \) converges to \( a + b \).

50. Prove: If \( a_n \to a \) and \( b_n \to b \), then the sequence \( (a_n b_n) \) converges to \( ab \).