Assignment 5 MA1124 Due Wednesday24th.


2. On page 234, attached, 37,38,42
Solution:
Suppose \( p_1 \) and \( p_2 \) belong to every interval. If \( p_1 \neq p_2 \), then \( |p_1 - p_2| = \delta > 0 \). Since \( \lim \Delta_n = 0 \), there exists an interval \( I_{\epsilon} = [a_{\epsilon}, b_{\epsilon}] \) such that the length of \( I_{\epsilon} \) is less than the distance \( |p_1 - p_2| = \delta \) between \( p_1 \) and \( p_2 \). Accordingly, \( p_1 \) and \( p_2 \) cannot both belong to \( I_{\epsilon} \), a contradiction. Thus \( p_1 = p_2 \), i.e. only one point can belong to every interval.

## Supplementary Problems

### FIELD AXIOMS

20. Show that the Right Distributive Law \([D_3]\) is a consequence of the Left Distributive Law \([D_1]\) and the Commutative Law \([M_3]\).

21. Show that the set \( \mathbb{Q} \) of rational numbers under addition and multiplication is a field.

22. Show that the following set \( A \) of real numbers under addition and multiplication is a field:
\[
A = \{a + b\sqrt{2} : a, b \text{ rational}\}
\]

23. Show that the set \( A = \{\ldots, -4, -2, 0, 2, 4, \ldots\} \) of even integers under addition and multiplication satisfies all the axioms of a field except \([M_3],[M_4]\) and \([M_5]\), that is, is a ring.

### INEQUALITIES AND POSITIVE NUMBERS

24. Rewrite so that \( x \) is alone between the inequality signs:
   (i) \( 4 < -2x < 10 \), (ii) \( -1 < 2x - 3 < 5 \), (iii) \( -3 < 5 - 2x < 7 \).

25. Prove: The product of any two negative numbers is positive.

26. Prove Theorem A.3(ii): If \( a < b \), then \( a + c < b + c \).

27. Prove Theorem A.3(iv): If \( a < b \) and \( c \) is positive, then \( ac < bc \).

28. Prove Corollary A.3: The set \( \mathbb{R} \) of real numbers is totally ordered by the relation \( a \leq b \).

29. Prove: If \( a < b \) and \( c \) is positive, then:
   (i) \( \frac{a}{c} < \frac{b}{c} \), (ii) \( \frac{a}{b} < \frac{c}{a} \).

30. Prove: \( \sqrt{ab} = (a + b)/2 \). More generally, prove \( \sqrt[3]{a_1 a_2 \cdots a_n} = (a_1 + a_2 + \cdots + a_n)/n \).

31. Prove: Let \( a \) and \( b \) be real numbers such that \( a < b + \epsilon \) for every \( \epsilon > 0 \). Then \( a \leq b \).

32. Determine all real values of \( x \) such that: (i) \( x^2 + x^2 - 6x > 0 \), (ii) \( (x - 1)(x + 3)^2 \leq 0 \).

### ABSOLUTE VALUES

33. Evaluate: (i) \( | -2 | + |1 - 4| \), (ii) \( |3 - 8| - |1 - 9| \), (iii) \( | -4 | - |2 - 7| \).

34. Rewrite, using the absolute value sign: (i) \( -8 < x < 9 \), (ii) \( 2 \leq x \leq 8 \), (iii) \( -7 < x < -1 \).

35. Prove: (i) \( | -a | = |a| \), (ii) \( a^2 = |a|^2 \), (iii) \( |a| = \sqrt{a^2} \), (iv) \( |x| < a \iff -a < x < a \).
36. Prove Proposition A.4(i): \(|ab| = |a| |b|\).

37. Prove Proposition A.4(iv): \(|a| - |b| \leq |a - b|\).

**Least Upper Bound Axiom**

38. Prove: Let \(A\) be a set of real numbers bounded from below. Then \(A\) has a greatest lower bound, i.e. \(\inf(A)\) exists.

39. Prove: (i) Let \(x \in \mathbb{R}\) such that \(x^2 < 2\); then \(\exists n \in \mathbb{N}\) such that \((x + 1/n)^2 < 2\).
   
   (ii) Let \(x \in \mathbb{R}\) such that \(x^2 > 2\); then \(\exists n \in \mathbb{N}\) such that \((x - 1/n)^2 > 2\).

40. Prove: There exists a real number \(a \in \mathbb{R}\) such that \(a^2 = 2\).

41. Prove: Between any two positive real numbers lies a number of the form \(r^2\), where \(r\) is rational.

42. Prove: Between any two real numbers there is an irrational number.

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**Answers to Supplementary Problems**

24. (i) \(-5 < x < -2\) (ii) \(1 < x < 4\) (iii) \(-1 < x < 4\)

32. (i) \(-3 < x < 0\) or \(x > 2\), i.e. \(x \in (-3, 0) \cup (2, \infty)\) (ii) \(x = 1\)

33. (i) 5 (ii) -3 (iii) 1

34. (i) \(|x - 3| < 6\) (ii) \(|x - 5| = 3\) (iii) \(|x + 4| < 3\)