1 (a) Use True/False Tables to prove

(i) \( P \implies Q \equiv Q' \implies P' \)

The definition of \( P \implies Q \) is given by

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>( P \implies Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

So

<table>
<thead>
<tr>
<th>Q'</th>
<th>( P' )</th>
<th>( Q' \implies P' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Since the \( P \implies Q \) and \( Q' \implies P' \) columns contain the same truth values in all rows (for all possible truth values of the variables involved), it follows that \( P \implies Q \equiv Q' \implies P' \).

(ii) \( (P \lor Q)' \equiv P' \land Q' \)

\[
\begin{array}{c|c|c|c|c|c}
P & Q & P \lor Q & (P \lor Q)' & P' & Q' \\
---&---&---&---&---&---
T & T & T & F & F & F \\
T & F & T & F & F & T \\
F & T & T & F & F & F \\
F & F & F & T & T & T \\
\end{array}
\]

(iii) \( (P \implies Q) \lor (R \implies Q) \equiv (P \land R) \implies Q \)

\[
\begin{array}{c|c|c|c|c|c|c|c}
P & Q & R & P \implies Q & R \implies Q & (P \implies Q) \lor (R \implies Q) & P \land R & (P \land R) \implies Q \\
---&---&---&---&---&---&---&---
T & T & T & T & T & T & T & T \\
T & T & F & T & T & F & T & T \\
T & F & T & F & F & F & T & T \\
T & F & F & T & T & F & T & T \\
F & T & T & T & T & F & T & T \\
F & T & F & T & T & F & T & T \\
F & F & T & T & F & T & T & T \\
F & F & F & T & T & F & T & T \\
\end{array}
\]

(b) (i) \( \forall x \exists y \ (xy = 1)' \) (where ‘ denotes negation)

\[
\begin{align*}
\forall x \exists y \ (xy = 1)' &= \exists x \ [\exists y \ (xy = 1)']' \\
&= \exists x \ \forall y \ (xy = 1)' \\
&= \exists x \ \forall y \ xy \neq 1
\end{align*}
\]
(ii) $\exists x \exists y \ P'(x,y) \land \ \forall x \ \forall y \ Q(x,y)$

$$\exists x \exists y \ P'(x,y) \land \ \forall x \ \forall y \ Q(x,y) = \ [\exists x \exists y \ P'(x,y) \lor [\ \forall x \ \forall y \ Q(x,y)]$$

$$\ [\exists x \exists y \ P'(x,y) \lor [\ \forall x \ \forall y \ Q(x,y)] = [\ \forall x \ \forall y \ P(x,y) \lor [\exists x \exists y \ Q'(x,y)]$$

2 (a) Two sets $X$, $Y$ have the same cardinal number means that there exists a bijective mapping $f : X \to Y$. [Not same # of elements.]

To show this is an equivalence relation, suppose $X \sim Y$ so that there is a bijection $f : X \to Y$. Since $f$ is a bijection, $f^{-1} : Y \to X$ exists and is also a bijection. Thus $Y \sim X$.

For any $X$, there exists the identity mapping $id_X : X \to X$ by $id_X(x) = x$.

This is clearly bijective so $X \sim X$.

Suppose $X \sim Y$ and $Y \sim Z$ so that there are bijections $f : X \to Y$ and $g: Y \to Z$. But then we proved $g \circ f : X \to Z$ is bijective, so $X \sim Z$.

Hence $\sim$ is symmetric, reflexive, and transitive.

(b) If $n$ is the cardinal number of a set $X$, and $m$ is the cardinal number of a set $Y$, then $n \leq m$ means that there exists $f : X \to Y$ which is $1-1$ or injective. Or equivalently $X$ has the same cardinal number as a subset of $Y$. We can write $\#X \leq \#Y$.

Since $id_X : X \to X$ is injective $\#X \leq \#X$, so $\leq$ is reflexive.

If $\#X \leq \#Y$ and $\#Y \leq \#X$, then by the Cantor-Bendixson theorem $\#X = \#Y$, so $\leq$ is antisymmetric.

If $\#X \leq \#Y$ and $\#Y \leq \#Z$, there are injections $f : X \to Y$ and $g: Y \to Z$.

Then $g \circ f : X \to Z$ is injective, and we have $\leq$ is transitive.

Hence $\leq$ is a partial order.

(c) (i) $(A \cup B)^c = A^c \cap B^c$ De Morgan’s Laws (Note: Venn Diagrams give an illustration, not a proof.)

The only way to prove it is as follows. Let $x \in (A \cup B)^c$. Then:

$$\implies \ x \text{ is not in } A \cup B$$
$$\implies \ x \text{ is not in } A \text{ and } x \text{ is not in } B$$
$$\implies \ x \in A^c \text{ and } x \in B^c$$
$$\implies \ x \in A^c \cap B^c$$

Hence $(A \cup B)^c \subseteq A^c \cap B^c$. Now let $x \in A^c \cap B^c$. Then:

$$\implies \ x \text{ is not in } A \text{ and } x \text{ is not in } B$$
$$\implies \ x \text{ is not in } A \cup B$$
$$\implies \ x \in (A \cup B)^c$$

Hence $A^c \cap B^c \subseteq (A \cup B)^c$. By these two results $(A \cup B)^c = A^c \cap B^c$. 

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let $x \in A \cap (B \cup C)$. Then:

$\Rightarrow \quad x \in A$ and $x \in (B \cup C)$

$\Rightarrow \quad x \in A$ and $(x \in B$ or $x \in C)$

$\Rightarrow \quad (x \in A$ and $x \in B$) or $(x \in A$ and $x \in C)$

$\Rightarrow \quad x \in (A \cap B) \cup (A \cap C)$

Let $x \in (A \cap B) \cup (A \cap C)$. Then:

$\Rightarrow \quad x \in (A \cap B)$ or $x \in (A \cap C)$

$\Rightarrow \quad (x \in A$ and $x \in B$) or $(x \in A$ and $x \in C)$

$\Rightarrow \quad x \in A$ and $(x \in B$ or $x \in C)$

$\Rightarrow \quad x \in A \cap (B \cup C)$

Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(d) A function $f$ is uniformly continuous on a subset $A$ of $\mathbb{R}$ means that

$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall c \in A \quad \forall x \in A \quad \left( |x - c| < \delta \implies |f(x) - f(c)| < \epsilon \right)$.

(3) (a) Prove $\lim_{x \to a} f(x) = L \iff \forall_{a < x_n \neq a} f(x_n) \to L$.

Given $\lim_{x \to a} f(x) = L$ and $x_n \to a$ we want to show $f(x_n) \to L$ i.e.

$\forall \epsilon > 0 \quad \exists N \quad \forall_{n \geq N} \quad |f(x_n) - L| < \epsilon$.

Now $\lim_{x \to a} f(x) = L$ says that given $\epsilon > 0$, $\exists \delta$ such that

$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$.

And $x_n \to a$, $x_n \neq a$

$\implies$ given $\delta \exists N$ such that if $n \geq N$ then $0 < |x_n - a| < \delta$.

Putting these together, given $\epsilon > 0 \exists N$ such that if $n \geq N$ then $|f(x_n) - L| < \epsilon$.

Conversely: to show given $\forall x_n \to a, x_n \neq a, f(x_n) \to L$, then $\lim_{x \to a} f(x) = L$.

If you think about this for a while, there is no obvious way to go directly from knowing about all sequences to the limit statement. So we try to assume $\lim_{x \to a} f(x) \neq L$, and try to construct $x_n \to a$ such that $f(x_n) \not\to L$.

Now $\lim_{x \to a} f(x) \neq L$, means $\exists \epsilon$ such that $\forall \delta, \exists x$ with $0 < |x - a| < \delta$ but $|f(x) - L| \geq \epsilon$. For this $\epsilon$, take $\delta = 1/n$, then $\exists x_n$ such that $0 < |x_n - a| < \frac{\delta}{n}$.

But $|f(x_n) - L| \geq \epsilon$. Since $0 < |x_n - a| < 1/n$, we have $x_n \neq a$ and $x_n \to a$. But $|f(x_n) - L| \geq \epsilon \implies f(x_n) \not\to L$. This is ouer contradiction.
(b) Let \( \{a_n\} \) be a sequence. \( \liminf_{k \to \infty} \{a_n\} = \lim_{k \to \infty} \inf_{n \geq k} \{a_n\} \).

Note if \( G_k = \inf_{n \geq k} \{a_n\} \), then \( G_k \uparrow \), so if this sequence is bounded \( \lim G_k \) exists.

\( \limsup_{k \to \infty} \{a_n\} = \lim_{k \to \infty} \sup_{n \geq k} \{a_n\} \).

Note if \( L_k = \sup_{n \geq k} \{a_n\} \), then \( L_k \downarrow \), so if this sequence is bounded \( \lim L_k \) exists.

We have \( G_n \leq L_n \) \( \forall n, m \) so \( \liminf \{a_n\} \leq \limsup \{a_n\} \) and

\( \liminf \{a_n\} = \limsup \{a_n\} \iff \lim \{a_n\} \) exists.

If \( \lim \{a_n\} \) exists then \( \limsup \{a_n\} = \liminf \{a_n\} \).

(c) \( \bar{A} \), the closure of \( A = \{x : \exists x_n \in A \text{ with } x_n \to x\} \)

\( A^o \), the interior of \( A = \{x : \exists \epsilon \text{ with } N(x, \epsilon) \subset A\} \)

To show \( (\bar{A})^c = (A^o)^c \) let \( x \in (\bar{A})^c \). Then \( x \notin \bar{A} \)

\[ \because \not\exists x_n \in A \text{ such that } x_n \to x \]

\[ \because \exists N(x, \epsilon) \text{ such that } N(x, \epsilon) \cap A = \emptyset \]

(Note if \( \exists N(x, \epsilon) \cap A \neq \emptyset \) all \( n \), then we get a sequence \( x_n \to x \) with \( x_n \in A \).) But \( N(x, \epsilon) \cap A = \emptyset \implies N(x, \epsilon) \subset A^c \). Hence \( x \in (A^c)^c \).

Hence \( (\bar{A})^c \subset (A^c)^c \).

Conversely, let \( x \in (A^c)^c \). Then

\[ \implies \exists N(x, \epsilon) \subset A^c \]

\[ \implies N(x, \epsilon) \cap A = \emptyset \]

\[ \implies \not\exists x_n \in A \text{ such that } x_n \to x \]

\[ \implies x_n \notin A \]

\[ \implies x \in (\bar{A})^c \]

Hence \( (\bar{A})^c = (A^c)^c \).

(d) Prove that a function from \( \mathbb{R} \) to \( \mathbb{R} \) is continuous \( \iff \) the inverse image of every open set is open, i.e.

Assume \( f \) is continuous. Then \( \forall a \exists \epsilon > 0, \exists \delta > 0 \text{ such that } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon \). Let \( \mathcal{O} \) be open to show \( f^{-1}(\mathcal{O}) = \{x : f(x) \in \mathcal{O}\} \) is open.

(Note: \( f^{-1}(A) \) is always defined for a set \( A \). This is not the inverse function \( f^{-1}(y) \) which may or may not exist.) Let \( a \in f^{-1}(\mathcal{O}) \), then \( f(a) \in \mathcal{O} \). Since \( \mathcal{O} \) is open, \( \exists \epsilon > 0 \) such that \( N(f(a), \epsilon) \subset \mathcal{O} \).

But \( f \) is continuous at \( a \), so given this \( \epsilon \), \( \exists \delta > 0 \) such that \( |x - a| < \delta \implies |f(x) - f(a)| < \epsilon \), i.e. \( x \in N(a, \delta) \implies f(x) \in N(f(a), \epsilon) \). This last is contained in \( \mathcal{O} \), so \( N(a, \delta) \subset f^{-1}(\mathcal{O}) \). Hence \( a \) is an interior point of \( f^{-1}(\mathcal{O}) \). This is true for all \( a \), so \( f^{-1}(\mathcal{O}) \) is open.

4 (a) (i) Let \( A \subset \mathbb{R} \) be closed and bounded. Prove that every open cover of \( A \) has a finite subcover. This means that if \( A \subset \bigcup \mathcal{O}_\gamma, \mathcal{O}_\gamma \text{ open}, all \gamma \), then we can find a finite number of these, say \( \mathcal{O}_{\gamma_1}, \mathcal{O}_{\gamma_2}, \ldots, \mathcal{O}_{\gamma_n} \), such that \( A \subset \mathcal{O}_{\gamma_1} \cup \mathcal{O}_{\gamma_2} \cup \cdots \cup \mathcal{O}_{\gamma_n} \).
Case 1: A an interval, $A = [a, b]$ say. Consider $[a, \frac{a+b}{2}] \subset \bigcup \mathcal{O}_n$, and $[\frac{a+b}{2}, b] \subset \bigcup \mathcal{O}_n$. If both of these have a finite subcover, then so does $[a, b]$ and we are done. If one does not, call it $I_1$. Split $I_1$ into two halves. If both halves have a finite subcover, so does $I_1$. So at least one does not. Call it $I_2$. Keep going. We get a sequence of closed intervals $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$. Non of these has a finite cover. By the Nested Intervals Property $\bigcap I_n \neq \emptyset$. Let $x \in \bigcap I_n$. Then $x \in A$, so $\exists \mathcal{O}_{\gamma_0}$ such that $x \in \mathcal{O}_{\gamma_0}$ and $\mathcal{O}_{\gamma_0}$ is open, so $\exists N(x, \epsilon) \subset \mathcal{O}_{\gamma_0}$ some $\epsilon$, and $x \in I_n$, all $n$, then if the length of $I_n < \epsilon$, we have $I_n \subset N(x, \epsilon) \subset \mathcal{O}_{\gamma_0}$. Here is the picture.

But then $I_n$ has a finite subcover – just $\mathcal{O}_{\gamma_0}$ – and that intervals as in the last one so don’t get them confused. $A$ closed, bounded $\implies A \subset [a, b]$. So $[a, b]$ contains all the infinite number of points in our sequence. Consider $[a, \frac{a+b}{2}]$, $[\frac{a+b}{2}, b]$. One of these must contain an infinite number of terms of the sequence. Call it $I_1$. Split $I_1$ into two halves. Again, one of the halves must contain an infinite number of terms of the sequence. Keep going. We get $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$ and each $I_n$ contains an infinite number of terms of the sequence. These $I_n$ are closed intervals contradicts our choice of all the $I_n$’s. This shows that $A = [a, b]$ must have a finite subcover.

Case 2: $A$ closed bounded. Then $\exists [c, d]$ with $A \subset [c, d]$. Suppose $A \subset \bigcup \mathcal{O}_n$, then $[c, d] \subset A^c \cup (\bigcup \mathcal{O}_n)$ and $A^c$ is open. Then $[c, d]$ has a finite subcover, say $[c, d] \subset A^c \cup \mathcal{O}_{\gamma_1} \cup \mathcal{O}_{\gamma_2} \cup \cdots \cup \mathcal{O}_{\gamma_n}$. But then $A \subset \mathcal{O}_{\gamma_1} \cup \mathcal{O}_{\gamma_2} \cup \cdots \cup \mathcal{O}_{\gamma_n}$ since $A^c$ is not needed to cover $A$.

(ii) $A \subset \mathbb{R}$, $A$ closed bounded, then every sequence of elements in $A$ has a subsequence that converges to an element of $A$.

Note: this features exactly the same sort of splitting so by the Finite Nested Interval Property, $\bigcap I_n \neq \emptyset$. Let $x \in \bigcap I_n$ and for each $I_n$, choose $x_n \in I_n$ to be one of the terms of the sequence. Then $x_n \in A$, and since the length of $I_n \to 0$, $x_n \to x$ and $A$ is closed, so $x \in A$.

(b) A sequence $\{x_n\} \subset \mathbb{R}$ is a Cauchy sequence means $\forall \epsilon > 0 \exists N$ such that $n, m \geq N \implies |x_n - x_m| < \epsilon$. Prove that every Cauchy sequence in $\mathbb{R}$ converges to a point in $\mathbb{R}$.

* Every Cauchy sequence is bounded. Hence $\{x_n\} \subset [a, b]$ some $[a, b]$. But then $\{x_n\}$ has a convergent subsequence by (a) (ii). But if

** a Cauchy sequence has a subsequence that converges, then the whole sequence converges to that point.

We will prove *.

Let $\epsilon = 1$, $\exists N$ such that $n, m \geq N \implies |x_n - x_m| < 1$. In particular $n \geq N \implies |x_n - x_N| < 1$. So all $x_n$, $n \geq 1$ lie in the interval $(x_N - 1, x_N + 1)$. Hence $\{x_n\}$ are bounded above by $\max \{x_1, x_2, \ldots, x_N, x_N + 1\}$ and below by
\[ \min \{x_1, x_2, \ldots, x_N, x_N - 1 \}. \]

We will prove **.

Given \( \{x_n\} \) Cauchy sequence and \( x_{n_k} \to x \). Claim \( x_n \to x \).

Given \( \epsilon > 0 \), \( \exists n_{k_0} \) such that \( n_k \geq n_{k_0} \implies |x_{n_k} - x| < \epsilon/2 \) (since \( x_{n_k} \to x \)), \( \\exists N \) such that \( n, m \geq N \implies |x_n - x_m| < \epsilon/2 \) (since \( x_n \) is a Cauchy sequence).

Hence if \( n \geq N \), choose \( n_{k_1} \geq \max \{N, n_{k_0}\} \) then \( |x_n - x| \leq |x_n - x_{n_{k_1}}| + |x_{n_{k_1}} - x| < \epsilon \).