21) True \[ h(1) = c \int_1^b \delta(x) \, dx = c(b-a) \]

22) False \[ \int_{\aleph_1}^{\aleph_2} \text{A is defined} \]

\[ \int_{\aleph_1}^{\aleph_2} \text{would cross many times} \]

23) True if \[ \int_0^1 f(x) g(x) \, dx = 0 \]

then \[ f(x) - g(x) = 0 \] somewhere on \[ f(x) - g(x) > 0 \] somewhere \[ \Rightarrow \int_0^1 [f(x) - g(x)] \, dx = 0 \]

\[ \Rightarrow f(x) = g(x) \] somewhere and they cross

24) True if \[ A = \left| \int_0^1 f(x) - g(x) \, dx \right| \]

then \[ \int_0^1 f(x) - g(x) > 0 \] everywhere \[ \text{or } \int_0^1 f(x) - g(x) = 0 \] everywhere

\[ A = \text{the sum of } \int_0^1 f(x) - g(x) \, dx \]

12. \[ y = x^2 \]

Points of intersection

\[ x^2 = x + 2 \]
\[ x^2 - x - 2 = (x-2)(x+1) = 0 \]
\[ x = -1, 2 \]

Area \[ = \int_{-1}^{2} x^2 + 2 - x^2 \, dx \]

\[ = \frac{x^3}{3} + 2x - \frac{x^3}{3} \bigg|_{-1}^{2} \]

\[ = 6 - \frac{8}{3} - (-\frac{1}{3} + \frac{1}{3}) = \frac{7}{2} \]
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\[ \int_0^k x^2 \, dx = \int_0^k x^2 \, dx \]

\[ \frac{k^3}{3} = \frac{8}{3} - \frac{k^3}{3} \]

\[ 2k^2 = 8 \]

\[ k = 2 \sqrt{2} \]

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Diagram of a square with side \( 1 - y^2 \)

Area = \((1 - y^2)^2\)

Volume = \( \frac{1}{2} \int_{-1}^{1} (1 - y^2)^2 \, dy \)

\[ = \int_{-1}^{1} 1 - 2y^2 + y^4 \, dy = \left[ y - \frac{2}{3} y^3 + \frac{y^5}{5} \right]_{-1}^{1} \]

\[ = 1 - \frac{2}{3} + \frac{1}{5} - \left( -1 + \frac{1}{3} - \frac{1}{5} \right) \]

\[ = 2 - \frac{2}{3} + \frac{2}{5} = \frac{7}{3} + \frac{2}{5} = \frac{16}{15} \]
When the region is rotated about the y-axis, the cross section is a washer of inner radius $1-y^2$ and outer radius $2+y^2$.

So, the volume is:

$$V = \int_{-1}^{1} \pi (2+y^2)^2 - \pi (1-y^2)^2 \, dy$$

$$= \pi \int_{-1}^{1} 4 + 4y^2 + y^4 - (1 - 2y^2 + y^4) \, dy$$

$$= \pi \int_{-1}^{1} 3 + 6y^2 \, dy = \pi (3y^3 + 2y^3) \bigg|_{-1}^{1}$$

$$= \pi \left( 5 - (-3 - 2) \right) = 10 \pi$$
a) \[
r = \frac{1-x}{\sqrt{1-x^2}}
\]
\[
h = \sqrt{1-x^2}
\]
\[
A(x) = 2\pi (1-x) \sqrt{1-x^2}
\]
\[
V = \int_0^1 2\pi (1-x) \sqrt{1-x^2} \, dx
\]

b) \[
r = \frac{y+1}{\sqrt{1-y^2}}
\]
\[
h = \sqrt{1-y^2}
\]
\[
A(x) = 2\pi (y+1) \sqrt{1-y^2}
\]
\[
V = \int_0^1 2\pi (y+1) \sqrt{1-y^2} \, dy
\]
\[
\begin{align*}
    r &= 1 - y \\
    h &= x = \sqrt{y} \\
    A(y) &= 2\pi (1 - y) \sqrt{y} \\
    V &= \int_0^1 2\pi (1 - y) \sqrt{y} \, dy \\
    &= \frac{9\pi}{14}
\end{align*}
\]

\[\text{pp 375-376}\]

\[\text{A9) False}\]

\[f'(x) = -\frac{x}{(1-x^2)} \text{ is not continuous on } [-1,1]\]

\[\text{A10) True}\]

\[\text{MVT should be applied to the term:}\]
\[\left[ f(x_k) - f(x_{k-1}) \right]^2 \text{ to obtain the form of a Riemann sum.}\]

\[\text{A11) True}\]

\[\text{Each } h_k \text{ decreases exact an evenhth.}\]

\[\text{A12) False}\]

\[\text{This requirement applies to } f(x).\]
\[ L = \int_0^1 \left[ \left( 2(1+t) \right)^2 + \left( 3(1+t)^2 \right)^2 \right]^{1/2} \, dt \]

\[ = \int_0^1 (1+t)^2 \sqrt{4 + 9(1+t)^2} \, dt, \quad u = (1+t)^2 \]

\[ \frac{du}{dt} = 2(1+t) \]

\[ t=0 \Rightarrow u=1 \]

\[ t=1 \Rightarrow u=4 \]

\[ = \frac{1}{2} \int \left( u^2 + 4 \right)^{3/2} \frac{du}{u} \]

\[ = \frac{1}{2} \cdot \frac{2}{9} \left( 4 + 9u \right)^{3/2} \left. \right|_{u=1} \approx 7.6339 \]

\[ \overline{y} = \sqrt{x} \]

\[ A = \int_0^1 \sqrt{x} \, dx = \frac{4}{5} \]

\[ M = 8 \cdot A = 15 \cdot \frac{4}{5} = 12 \text{ unit of mass} \]

\[ \bar{x} = \frac{1}{A} \int_0^1 \sqrt{x} \, dx = \frac{5}{4} \cdot \frac{4}{9} = \frac{5}{9} \]

\[ \bar{y} = \frac{1}{A} \int_0^1 \frac{1}{2} \sqrt{x} \, dx = \frac{5}{4} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{5}{12} \]
Problem (363: 21-24). (Solid $S$ of volume $V$ is bounded by two parallel planes at $x = a$ and $x = b$ with cross sectional area $A(x).$)

If each cross section of $S$ perpendicular to the $x$ axis is a square then $S$ is a rectangular parallelepiped.

If each cross section of $S$ is a disk or washer, then $S$ is a volume of revolution.

If $x$ is in centimeters, then $A(x)$ must be a quadratic function of $x.$

The average value of $A(x)$ on the interval $[a, b]$ is $V/(b-a).$

Solution: False, $S$ could be a square based pyramid with axis of symmetry on the x-axis.

False. Consider a solid whose cross sectional area to x-axis is a circle, but the centre of each circle lies on the line $y = x$. This is also not a solid of revolution around $y = x$, otherwise the cross sectional area perpendicular to the x axis would be elliptic.

False. The units of $A(x)$ are dependent upon the units of the y axis, also see examples in the book where $A$ is a linear function etc.

True. Since $V = \int_a^b A(x)dx$, this is true by the MVT of integration.

Problem (34). Find the volume of the solid when the region enclosed by $y = \sqrt{x}, y = 0, x = 9$ is revolved about the line $y = 3$.

Solution: The region is shown below, and we’re to revolve about $y = 3$. This is best done using the “washer” method. The outer radius is $r_2 = 3$ and the inner is $r_1 = 3 - \sqrt{x}$:

$$V = \pi \int_0^9 (r_2(x))^2 - (r_1(x))^2 \, dx$$

$$= \pi \int_0^9 3 - (3 - \sqrt{x})^2 \, dx$$

$$= 135\pi/2$$
False. Section 5.2 outlines how to find the volume given the cross sectional area, cylindrical shells does not use cross sectional area.

True.

True. If $f = c$ is a constant then the fact that this is exact is equivalent to

$$V = c \sum_{k=1}^{n} 2\pi x_k^2 \Delta x_k = c \int_a^b 2\pi x dx$$

$$\sum_{k=1}^{n} \frac{(x_k - x_{k-1})}{2}(x_k - x_{k+1}) = \frac{(b^2 - a^2)}{2}$$

$$= \sum_{k=1}^{n} \frac{(x_k^2 - x_{k-1}^2)}{2}$$

The series on the left is telescoping, so only the first and last term contribute $= b^2 - a^2/2$. \qed