Problem (10 - p234). Find the point in the first quadrant of the curve $y = x^{-2}$ such that a rectangle with sides on the coordinate axes and a vertex $P$ has the smallest possible perimeter.

Solution: The perimeter function is:

$$f(x, y) = 2x + 2y$$

$$f(x) = 2x + \frac{2}{x^2}$$

$$\Rightarrow f'(x) = 2 - \frac{4}{x^3}$$

$$\Rightarrow f''(x) = \frac{12}{x^4}$$

Since $f''(x) > 0$ in the first quadrant, the solution to $f'(x) = 0$ will give us the minimum, which is $P = (2^{1/3}, 2^{-2/3}) \Box$

Problem (14 - p234). A wire of length 12cm can be bent into a circle, square, or cut into two pieces to make both. Find the length of wire used for the circle which a) minimises and b) maximises the enclosed area.

Solution: Let the square have length $a$ and the circle have radius $r$. Then the function we want to maximise is $f(a, r) = a^2 + \pi r^2$ subject to constraint $12 = 4a + 2\pi r$, so we substitute $a = 3 - \frac{r}{2}$ to get the function:

$$f(r) = (3 - \frac{r}{2})^2 + \pi r^2$$

$$\Rightarrow f'(r) = -\pi(3 - \frac{r}{2}) + 2\pi r$$

$$\Rightarrow f''(r) = \frac{\pi^2}{2} + 2\pi$$

Since $f'' > 0$, the solution to $f'(r) = 0$ will be a local minimum, and since $f(r)$ is quadratic this is also the global minimum. This is at $r_0 = \frac{6}{4\pi}$. To determine the global max we must look at the endpoints, $r = 0, a = 0$, which give $r_1 = 0, r_2 = 6/\pi$ when the wire is used entirely for the circle or the square. Now

$$f(r = 0) = 9$$

$$f(r = 6/\pi) = \frac{36}{\pi}$$

Since $\pi < 4$, we know $\frac{36}{\pi} > \frac{36}{4}$ and so the area is maximised when the wire is used entirely to form a circle. (In general this is the isoperimetric inequality). \Box

Problem (30 - p234). A closed cylindrical can is to have a surface area $S$. Show that the volume is maximised when the height is equal to the diameter of the base.

1
\[ a = 0 \Rightarrow r = \sqrt{\frac{S}{4\pi}} \]

\[ r = 0 \Rightarrow a = \sqrt{\frac{S}{6}} \]

\[ V(a = 0) = \frac{4}{3} \pi \left( \frac{S}{4\pi} \right)^{3/2} \]

\[ V(r = 0) = \left( \frac{S}{6} \right)^{3/2} \]

Numerically checking gives that the volume is maximised when \( a = 0 \) and the surface area is entirely used for the sphere. (This is the 3d version of the isoperimetric inequality). \( \square \)
(ii) 

a) \( \mathcal{Q}(x) = x \cdot p = x \cdot (1000 - x) \)

b) \( p(x) = \mathcal{Q}(x) - \mathcal{C}(x) = x \cdot (1000 - x) - (3000 + 20x) \\
= 980x - x^2 - 3000 \)

c) \( p'(x) = 980 - 2x = 0 \Rightarrow x = 490 \)

d) \( p(490) = 237100 \quad \Rightarrow p(500) = 237,100 \)

e) \( p(p) = 980(1000 - p) - (1000 - p)^2 - 3000 \)

\[ \frac{dp}{dp} = -980 + 2(1000 - p) = 0 \]

\[ \Rightarrow p = 510 \]

\[ \text{Q4b) } C(t, n) = 15 \cdot t + 2.5 \cdot n, \quad n: \text{total gallon of diesel fuel} \]

\[ t = \frac{x}{ne} \quad \text{and} \quad n = \frac{x}{10 - 0.09ne}, \quad x: \text{total distance} \]

\[ \Rightarrow C(x, ne) = \frac{15x}{ne} + 2.5 \cdot \frac{x}{10 - 0.09ne}, \quad \text{total cost} \]

Cost per mile: \( c(ne) = \frac{15}{ne} + \frac{2.5}{10 - 0.09ne} \)

\[ \frac{dc}{dne} = -\frac{15}{ne^2} + \frac{(10,07t) \cdot 2.5}{(10 - 0.09ne)^2} \]

\[ \Rightarrow -15 \cdot (10 - 0.07ne)^2 + 2.5 \cdot 0.07 \cdot ne^2 = 0 \]

\[ \Rightarrow ne = 56,1759 \text{ miles/hour} \]
\[ a) \quad T(x) = \frac{\left(\frac{x^2+1}{3}\right) + \frac{1-x}{5}}{9/5} \quad \Rightarrow x = \frac{3}{4} \text{ critical point} \]

\[ T(0) = 0.533 \quad T\left(\frac{3}{4}\right) = 0.466 \quad T(1) = 0.471 \]

\[ \Rightarrow x = \frac{3}{4} \text{ mi} \rightarrow \text{minimum time} \]

\[ b) \quad T(x) = \sqrt{\frac{x^2+1}{4}} + \frac{1-x}{5} \]

\[ T'(x) = \frac{1}{4} \sqrt{x^2+1} - \frac{1}{5} = 0 \quad \Rightarrow x = \frac{1}{3} \text{ mi} \rightarrow \text{end of range} \]

\[ T(0) = 0.450 \quad T(1) = 0.854 \]

\[ \Rightarrow x = 1 \text{ mi is minimum} \]

\[ 62) \quad t(x) = \frac{\sqrt{a^2 + (c-x)^2}}{n_e} + \frac{\sqrt{b^2 + x^2}}{n_e}, \quad n_e: \text{speed of light, constant in uni. medium} \]

\[ \frac{dt}{dx} = \frac{1}{n_e} \left[ -\frac{c-x}{\sqrt{a^2 + (c-x)^2}} + \frac{x}{\sqrt{b^2 + x^2}} \right] = 0 \]

\[ \Rightarrow (c-x)^2 \cdot (b^2 + x^2) = x^2 \cdot (a^2 + (c-x)^2) \]

\[ \frac{b^2}{x^2} + 1 = \frac{a^2}{(c-x)^2} + 1 \quad \Rightarrow \cot \theta_2 = \cot \theta_1 \]

\[ \text{for } 0 < \theta_1, \theta_2 < 90^\circ \quad \Rightarrow \theta_1 = \theta_2 \]
24

(c) \( f(x) = \frac{1}{x} - a \),

\[
\frac{d}{dx} f(x) = -\frac{1}{x^2}
\]

\[
x_{n+1} = x_n - \frac{f(x_n)}{\frac{df}{dx}(x_n)}
\]

\[
= x_n - \frac{1}{x_n} - a
\]

\[
= x_n + x_n - a x_n
\]

\[
= x_n (2 - a x_n)
\]

(d) to approximate \( \frac{1}{17} \),

let \( x_1 = \frac{1}{20} = 0.05 \),

\[
x_2 = 0.05 (2 - 17(0.05))
\]

\[
= 0.0575
\]

\[
x_3 = 0.0575 (2 - 17(0.0575))
\]

\[
= 0.8588 \text{ corrected to 4 d.p.}
\]

Note correct answer \( 0.0588235 \ldots \)
11 False  Rolle's theorem also need $f'(x)$ to exist on $(a, b)$.

Counter example $\exists$ no tangent.

12 True  $f'(c) = \frac{f(b) - f(a)}{b - a} = \text{Average rate of change}$

13 False  $f(x) = 2x  \quad g(x) = 3x$.

$f'(x) = 2  \quad g'(x) = 3$

$f' - g' = \text{constant}  \quad f' \neq g'$

14 True  We used the M.V. Th. to prove Theorem 3.1.2.

15 $f(x) = \tan x  \quad f'(x) = \sec^2 x = \frac{1}{\cos^2 x}$

$\leq 0 \text{ on } (0, \pi)$

But $f(x)$ is not defl at $\pi/2$.

16 $f(x) = x^{\frac{2}{3}}  \quad a = -1  \quad b = 8$.

$f(a) = \frac{1}{2}  \quad f(b) = 4$

$f'(a) - f'(c) = \frac{4 - 1}{8 - 1} = \frac{3}{7} \neq \frac{1}{2}$.

$f'(c) = \frac{2}{3} x^{-\frac{1}{3}} = \frac{1}{6} \Rightarrow x^{\frac{1}{3}} = \frac{6}{2} = 3$

$\Rightarrow x^{\frac{1}{3}} = 2  \Rightarrow x = 8$

But $f(x)$ is not defl at $x = 0$. 
17. (a) The x-intercepts of \( f(x) \) are found by setting \( f(x) = 0 \):

\[
\Rightarrow f(a) = f(b) = 0.
\]

By Rolle's Theorem \( \exists c \in (a, b) \) with \( f'(c) = 0 \).

(b) \( y = -x^2 + 4 \) \( y = 0 \) at \( x = \pm 2 \)

\[
\frac{dy}{dx} = -2x = 0 \quad \text{at} \quad x = 0.
\]

\( y = \sin x \) \( y = 0 \) at \( x = 0, \pi \)

\[
\frac{dy}{dx} = \cos x = 0 \quad \text{at} \quad x = \frac{\pi}{2}.
\]

22.

\( f(x) = 3x^4 + x^2 - 4x \)

\( f(0) = 0 \quad f(1) = 0 \)

By Rolle's Theorem \( \exists c \in (0, 1) \) with \( f'(c) = 0 \).

\( f(0) = 12x^3 + 2x - 4 \)

26. (a) \( f(x) - f(y) = f'(c) \times (x-y) \)

1. \( f(x) - f(y) = |f'(c)| |x-y| \)

2. \( M |x-y| \).
(b) \[ \frac{d}{dx} \tan x = \sec^2 x \quad \text{on } \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \]

\[ |\frac{d}{dx} \tan x| = |\sec^2 x| > 1 \quad \text{on } \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \]

\[ \therefore \quad |\tan x - \tan y| > |x - y| \quad \forall x, y \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \]

(c) \[ y \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \implies -y \]

and \[ \tan(-y) = \frac{\sin(-y)}{\cos(-y)} = -\tan y \]

\[ \therefore \quad |\tan x - \tan(-y)| = |x - (-y)| \]

\[ \therefore \quad \left| \tan x + \tan y \right| > |x + y| \]