Mathematics 11 23
Sample Test
January '12
Answer all questions

1. (i) Define what is meant by a function from a set $X$ to a set $Y$.
(ii) Define what it means for $f : X \rightarrow Y$ to be onto (surjective).
(iii) What does the vertical line test of a graph tell?
(iv) Define $\lim_{x \to a^-} f(x) = L_1$, $\lim_{x \to a^+} f(x) = L_2$
(v) If $\lim_{x \to a^-} f(x) = L_1$ and $\lim_{x \to a^+} f(x) = L_2$

prove $L_1 \neq L_2 \Rightarrow \lim_{x \to a} f(x)$ does not exist.
2. (i) Let \( f(x) = \begin{cases} -x & x \leq 0 \\ x^2 & x > 0 \end{cases} \)

Find \( f'(0) \) or show it does not exist.

(ii) Let \( f(x) = x \sin \frac{1}{x} \quad x \neq 0 \)

\[ f(x) = 0 \quad x = 0. \]

Find \( f'(0) \) or show it does not exist.

(iii) Is \( f(x) \) continuous at \( x = 0 \)?

\[ \lim_{x \to a} f(x) = L_1, \quad \text{and} \quad \lim_{x \to a} g(x) = L_2 \]

\[ \lim_{x \to a} f(x) g(x) = L_1 L_2. \]

(iv) Find the linear approximation and then the quadratic approximation to \( \sin(0.05) \).

3. (i) Find \( \frac{dy}{dx} \).

(a) \( y = xe^{\sin x} \)

(b) \( y = \ln(\sin(x^2 + 1)^2) \)

(c) \( x^2 y + y x^2 = 1 \)

(d) \( x = t^2 + 1, \ y = 2t + 1 \).
(ii) Let \( f(x) = \frac{2(x^2 - 9)}{x^2 - 4} \), find where
for increases, decreases, inflection, concave up, concave down, local extrema,
and points of reflection. Hence sketch this function.

(i) State and prove Rolle's Theorem

(ii) State the Mean Value Theorem and prove that if \( f(x) = 0 \),
all \( x \) in \([a, b]\), then \( f'(x) = 0 \) is constant on \([a, b]\).

(iii) A boat B is 2 miles from the shore. The person in the boat
wants to get to a point that is 1 mile in land from a point three miles down the shore. If the person can
row at 2 miles per hour,
and walk at 4 miles per hour.

what route should they take?

5 (i) Use Riemann sums to determine the formula for the arc length of \( y = f(x) \) from \( x = a \) to \( x = b \).

(ii) If \( f(x) = 1 \), is it rectifiable?

\[
\int_a^b f(x) \, dx
\]

What is \( \int_a^b f(x) \, dx \)? Explain your reasoning.

(iii) Integrate (a) \( \int x \ln x^2 \, dx \)

(b) \( \int x^2 \ln x \, dx \)

(c) \( \int \frac{x^2 + 1}{x^2 + 2} \, dx \)

(d) \( \int \frac{x + 2}{x^2 + 1} \, dx \)

(e) \( \int \frac{x + 2}{(x - 1)^2(x + 2)} \, dx \)

(f) \( \int \cos^2 x \sin^2 x \, dx \)
6 (i) Prove that the volume of a pyramid with square base is \( \frac{1}{3} \text{ Bh} \), where \( h \) is the height and \( B \) is the area of the base.

(ii) Find the volume of the solid formed by revolving the region bounded by \( y = x^2 + 1 \), \( y = 0 \), \( x = 0 \) and \( x = 1 \), about the y-axis. Do it first by disks and then by cylindrical shells.

(iii) Define hyperbolic sine and cosine. Why are they called hyperbolic?