MA2223: METRIC SPACES

Contents

1. Metric spaces

Definition 1.1. A *metric space* is a pair (X, d) consisting of a non-empty set X and a map $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$,

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0$ if and only if $x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ (the Triangle Inequality).

We will call the elements of X *points*. The mapping d is called a *metric* and we can think of $d(x, y)$ as the distance between two points x and y. Our goal is to develop a theory for metric spaces which we can apply in a variety of different situations. Our first examples of metric spaces are the Euclidean spaces $\mathbb{R}^n$.

1.1. **Euclidean spaces.** We denote by $\mathbb{R}^n$ the set of ordered n-tuples of real numbers,

$$\begin{aligned}\mathbb{R}^n &= \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\} \\ &= \mathbb{R} \times \overset{\leftarrow n \rightarrow}{\cdots} \times \mathbb{R}\end{aligned}$$

$\mathbb{R}^n$ is a vector space (over $\mathbb{R}$) with the following operations of addition and scalar multiplication: If $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are points in $\mathbb{R}^n$ then

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n)$$

If $\lambda \in \mathbb{R}$ is a scalar then

$$\lambda\,\mathbf{x} = (\lambda x_1, \ldots, \lambda x_n)$$

The dot product of $\mathbf{x}$ and $\mathbf{y}$ is

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n$$

The Euclidean norm of a point $\mathbf{x} = (x_1, \ldots, x_n)$ is

$$\begin{aligned}\|x\| &= \sqrt{\mathbf{x} \cdot \mathbf{x}} \\ &= \sqrt{x_1^2 + \cdots + x_n^2}\end{aligned}$$

Theorem 1.2. (Cauchy-Schwarz inequality) *Let* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, \ldots, y_n)$ *be points in* $\mathbb{R}^n$. *Then*

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|$$

Corollary 1.3. *Let* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, \ldots, y_n)$ *be points in* $\mathbb{R}^n$. *Then*

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$$

Corollary 1.4. *The mapping* $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *defined by*

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$

is a metric on $\mathbb{R}^n$.

The metric d defined in Corollary 1.4 is called the *Euclidean metric* on $\mathbb{R}^n$. We call $d(\mathbf{x}, \mathbf{y})$ the *Euclidean distance* between the points $\mathbf{x}$ and $\mathbf{y}$. The metric space $(\mathbb{R}^n, d)$ will be called *n-dimensional Euclidean space*. Unless otherwise stated it can be assumed that $\mathbb{R}^n$ denotes n-dimensional Euclidean space.

Note that by expanding out the Euclidean norm we get

$$\begin{aligned} d(\mathbf{x}, \mathbf{y}) &= \|\mathbf{x} - \mathbf{y}\| \\ &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} \end{aligned}$$

Example 1.5. (a) The Euclidean metric on $\mathbb{R}$. For real numbers $x, y \in \mathbb{R}$ the Euclidean distance is expressed in terms of absolute value

$$d(x, y) = |x - y|$$

(b) The Euclidean metric on $\mathbb{R}^2$. We can think of the elements of $\mathbb{R}^2$ as coordinates for points in the plane. The Euclidean distance between two points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ is

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

(c) The Euclidean metric on $\mathbb{R}^3$. We can think of the elements of $\mathbb{R}^3$ as coordinates for points in space. The Euclidean distance between two points $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ is

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

1.2. More examples of metric spaces.

Example 1.6. Let X be any non-empty set.

The *discrete metric* on X is defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

for all points x, y in X.

Example 1.7. Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ be points in $\mathbb{R}^2$.

(a) The *taxi-cab metric* on $\mathbb{R}^2$ is defined by

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|$$

(b) The *Irish rail metric* on $\mathbb{R}^2$ is defined by

$$d'(\mathbf{x}, \mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{y} \\ d(\mathbf{x}, \mathbf{0}) + d(\mathbf{0}, \mathbf{y}) & \text{if } \mathbf{x} \neq \mathbf{y} \end{cases}$$

where $\mathbf{0} = (0,0)$ is the origin in $\mathbb{R}^2$ and d is the Euclidean metric on $\mathbb{R}^2$.

Example 1.8. The *complex numbers.*

$(\mathbb{C}, d)$ is a metric space where d is defined by

$$d(z, w) = |z - w|, \quad \forall\, z, w \in \mathbb{C}$$

Example 1.9. A *function space.*

Let $C[0,1]$ be the set of all continuous functions $f : [0,1] \to \mathbb{R}$.

The following define two different metrics on $C[0,1]$,

(a) $d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$

(b) $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$

for all $f, g \in C[0,1]$.

Example 1.10. A *sequence space.*

Let c_0 be the set of all sequences $(x_k)_{k=1}^{\infty}$ of real numbers which converge to 0. Then (c_0, d) is a metric space where we define

$$d(\mathbf{x}, \mathbf{y}) = \sup_k |x_k - y_k|$$

for all points $\mathbf{x} = (x_k)_{k=1}^{\infty}$ and $\mathbf{y} = (y_k)_{k=1}^{\infty}$ in c_0.

Example 1.11. *Subspaces.*

If (X, d) is a metric space and A is a subset of X then (A, d_A) is a metric space where we define

$$d_A(x, y) = d(x, y) \quad \forall\, x, y \in A$$

(A, d_A) is called a *subspace* of (X, d) and d_A is called the *induced metric.*

1.3. **Open balls.**

Definition 1.12. Let (X, d) be a metric space. For each $x \in X$ and each positive real number $r > 0$ define

(i) the *open ball* with centre x and radius r,

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

(ii) the *sphere* with centre x and radius r,

$$S(x, r) = \{y \in X : d(x, y) = r\}$$

Example 1.13. 1-dimensional Euclidean space $\mathbb{R}$.

We can write the open ball with centre x and radius $r > 0$ in the following ways,

$$\begin{aligned} B(x, r) &= \{y \in \mathbb{R} : d(x, y) < r\} \\ &= \{y \in \mathbb{R} : |x - y| < r\} \\ &= (x - r,\ x + r) \end{aligned}$$

Example 1.14. 2-dimensional Euclidean space $\mathbb{R}^2$.

We can write the open ball with centre $\mathbf{x} = (x_1, x_2)$ and radius $r > 0$ in the following ways,

$$\begin{aligned} B(\mathbf{x}, r) &= \{\mathbf{y} \in \mathbb{R}^2 : d(\mathbf{x}, \mathbf{y}) < r\} \\ &= \{\mathbf{y} \in \mathbb{R}^2 : \|\mathbf{x} - \mathbf{y}\| < r\} \\ &= \{\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2 : \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} < r\} \end{aligned}$$

Example 1.15. 3-dimensional Euclidean space $\mathbb{R}^3$.

We can write the open ball with centre $\mathbf{x} = (x_1, x_2, x_3)$ and radius $r > 0$ in the following ways,

$$
\begin{aligned}
B(\mathbf{x}, r) &= \{\mathbf{y} \in \mathbb{R}^3 : d(\mathbf{x}, \mathbf{y}) < r\} \\
&= \{\mathbf{y} \in \mathbb{R}^3 : \|\mathbf{x} - \mathbf{y}\| < r\} \\
&= \{\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3 : \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} < r\}
\end{aligned}
$$

1.4. **Bounded sets.**

Definition 1.16. Let (X, d) be a metric space and let A be a subset of X. Define the *diameter* of A to be

$$diam(A) = \sup_{x,y \in A} d(x, y)$$

or if this supremum does not exist then $diam(A) = \infty$.

Theorem 1.17. *Let $B(x, r)$ be an open ball in a metric space (X, d). Then the diameter of $B(x, r)$ is $\leq 2r$.*

Example 1.18. The diameter of an open ball $B(\mathbf{x}, r)$ in Euclidean space $\mathbb{R}^n$ is $2r$.

Example 1.19. Let d be the discrete metric on a set X which contains at least two points. Then for each $x \in X$ the open ball $B(x, r)$ has diameter 0 if $r \leq 1$ and diameter 1 if $r > 1$.

Definition 1.20. Let (X, d) be a metric space and let A be a subset of X. Then A is called a *bounded set* if there exists an open ball $B(x, r)$ which contains A.

Theorem 1.21. *Let (X, d) be a metric space and let A be a subset of X. Then A is a bounded set if and only if A has finite diameter.*

1.5. **Open sets.**

Definition 1.22. Let (X, d) be a metric space and let A be a subset of X. A point $x \in A$ is called an *interior point* of A if there exists an open ball $B(x, r)$ with centre x which is contained in A.

A subset A of X is called an *open set* if every point in A is an interior point of A.

Theorem 1.23. *Let (X, d) be a metric space. Every open ball in (X, d) is an open set.*

Theorem 1.24. *Let (X, d) be a metric space. Then*

(i) $\emptyset$ and X are open sets,

(ii) the union of any collection of open sets is an open set,

(iii) the intersection of any finite collection of open sets is an open set.

Example 1.25. Part *(iii)* of Theorem 1.24 does not extend to infinite collections. For example, consider the 1-dimensional Euclidean space $\mathbb{R}$. For each n, the open interval $(-\frac{1}{n}, \frac{1}{n})$ is an open set. However,

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

is not an open set.

The *interior* of A, denoted $int(A)$ or A°, is defined as the set of all interior points of A. The collection of all open sets in a metric space (X, d) is called the *metric topology* on X.

Example 1.26. In $\mathbb{R}$ every open interval (a, b) is an open set. Intervals of the form $(a, b]$, $[a, b)$, $[a, b]$ are not open sets. A set consisting of a single point is not an open set. The interior of the closed interval $[a, b]$ is the open interval (a, b).

1.6. **Closed sets.**

Definition 1.27. Let (X, d) be a metric space. A *sequence* in X is a mapping $s : \mathbb{N} \to X$ and is usually written as (x_n) or $(x_n)_{n=1}^{\infty}$ where $x_n = s(n)$ for each $n \in \mathbb{N}$.

A sequence (x_n) is said to *converge* to a point $x \in X$ if given any positive real number $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) \leq \epsilon \quad \text{for all } n \geq N$$

The point x is called the *limit* of the sequence and we write $\lim_{n\to\infty} x_n = x$.

A sequence (x_n) is said to be *bounded* if the set $\{x_n : n \in \mathbb{N}\}$ is a bounded set in (X, d).

Theorem 1.28. *Every convergent sequence in a metric space is bounded and has a unique limit.*

Example 1.29. A sequence $(\mathbf{x}_j)$ in $\mathbb{R}^m$ converges to a point $\mathbf{x} \in \mathbb{R}^m$ if and only if each coordinate sequence converges in $\mathbb{R}$.

Definition 1.30. Let (X, d) be a metric space and let A be a subset of X. A point $x \in X$ is called a *limit point* of A if there exists a sequence (x_n) in $A \backslash \{x\}$ which converges to x.

A is called a *closed set* if it contains all of its limit points.

Theorem 1.31. *Let (X, d) be a metric space and let A be a subset of X. Then A is a closed set if and only if $X \backslash A$ is an open set.*

Theorem 1.32. *Let (X, d) be a metric space. Then*

(i) $\emptyset$ and X are closed sets,

(ii) the intersection of any collection of closed sets is a closed set,

(iii) the union of finitely many closed sets is a closed set.

Example 1.33. In $\mathbb{R}$ every closed interval $[a, b]$ is a closed set. The interval $(0, 1]$ is not closed since 0 is a limit point which is not contained in the set. Intervals of the form $(a, b]$, $[a, b)$, (a, b) are not closed sets. A set $\{x\}$ consisting of a single point is a closed set since it has no limit points.

The *closure* of A, denoted $\bar{A}$, is the union of A and the set of limit points of A.

Example 1.34. In $\mathbb{R}$, the closure of each of the intervals $(a, b]$, $[a, b)$ and (a, b) is $[a, b]$. The closure of $\mathbb{Q}$ is $\mathbb{R}$.

1.7. **Continuous mappings.**

Definition 1.35. Let (X, d) and (Y, d') be metric spaces. A mapping $T : X \to Y$ is a called *continuous* at a point $x_0 \in X$ if given any $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(x, x_0) < \delta \implies d'(T(x), T(x_0)) < \epsilon.$$

T is called continuous if it is continuous at every point of X.

Theorem 1.36. *Let (X, d) and (Y, d') be metric spaces. A mapping $T : X \to Y$ is a continuous mapping if and only if for every sequence (x_n) converging to a point x in (X, d), the sequence $(T(x_n))$ converges to $T(x)$ in (Y, d').*

Theorem 1.37. *Let (X, d) and (Y, d') be metric spaces. A mapping $T : X \to Y$ is continuous if and only if the preimage*

$$T^{-1}(U) = \{x \in X : T(x) \in U\}$$

is open in (X, d) for each U open in (Y, d').

Theorem 1.38. *The composition of two continuous mappings is a continuous mapping.*

Definition 1.39. Let (X, d) and (Y, d') be metric spaces. A mapping $T : X \to Y$ is called an *isometry* if

$$d'(T(x), T(y)) = d(x, y) \quad \text{for all } x, y \in X$$

(i.e. T preserves distances).

Proposition 1.40. *Let (X, d) and (Y, d') be metric spaces and let $T : X \to Y$ be an isometry. Then*

(i) T is one-to-one,

(ii) T *is continuous,*

(iii) $T^{-1} : T(X) \to X$ *is continuous.*

Definition 1.41. Let (X, d) be a metric space. A subset A of X is called *dense* in X if the closure of A is X. (i.e. $\bar{A} = X$).

Example 1.42. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Theorem 1.43. *Let* (X, d) *and* (Y, d') *be metric spaces and suppose* $S : X \to Y$ *and* $T : X \to Y$ *are continuous mappings. If*

$$S(x) = T(x) \quad \textit{for all } x \in A$$

where A *is a dense subset of* X *then*

$$S(x) = T(x) \quad \textit{for all } x \in X$$

1.8. **Complete metric spaces.**

Definition 1.44. Let (X, d) be a metric space. A sequence (x_n) in X is called a *Cauchy sequence* if given any positive real number $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_m, x_n) < \epsilon \quad \text{for all } m, n \geq N.$$

Theorem 1.45. *Every convergent sequence in a metric space (X, d) is a Cauchy sequence in (X, d).*

A Cauchy sequence is not necessarily a convergent sequence. Consider the subspace $\mathbb{R}\backslash\{0\}$ of $\mathbb{R}$. The sequence $(\frac{1}{n})_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}\backslash\{0\}$ but does not converge in $\mathbb{R}\backslash\{0\}$.

Definition 1.46. A metric space (X, d) is called *complete* if every Cauchy sequence in X is a convergent sequence in X.

Example 1.47. $\mathbb{R}^n$ is complete.

Theorem 1.48. *Let (X, d) be a metric space and let A be a subset of X.*

(i) If (A, d_A) is complete then A is a closed set.

(ii) If (X, d) is complete and A is a closed set then (A, d_A) is complete.

Example 1.49. The set ℓ^∞ of all bounded sequences of real numbers is a complete metric space with metric

$$d(\mathbf{x}, \mathbf{y}) = \sup_n |x_n - y_n|$$

The set c_0 of all sequences of real numbers which converge to 0 is a closed subspace of ℓ^∞ and hence by the above theorem is also complete.

Definition 1.50. Let (X, d) be a metric space. A metric space $(\tilde{X}, \tilde{d})$ is called a *completion* for (X, d) if

(i) there exists an isometry $i : X \to \tilde{X}$ such that $i(X)$ is dense in $\tilde{X}$,

(ii) $(\tilde{X}, \tilde{d})$ is a complete metric space.

Example 1.51. $\mathbb{R}$ is a completion of $\mathbb{Q}$. The required isometry is the inclusion map $i : \mathbb{Q} \hookrightarrow \mathbb{R}$, $x \mapsto x$.

Theorem 1.52. *Every metric space* (X, d) *has a completion* $(\tilde{X}, \tilde{d})$ *.*

Definition 1.53. Let (X, d) be a metric space. A mapping $T : X \to X$ is called a *contraction* if there exists a real number α with $\alpha < 1$ such that

$$d(T(x), T(y)) \leq \alpha\, d(x, y) \quad \text{for all } x, y \in X$$

Theorem 1.54 (Banach's Fixed Point Theorem)**.** *Let* (X, d) *be a complete metric space. If* $T : X \to X$ *is a contraction then* T *has a unique fixed point.*

(i.e. there exists exactly one element $x \in X$ *such that* $T(x) = x$*).*

Applications of Banach's fixed point theorem arise in differential equations. See for example *Picard's theorem* which provides conditions for existence and uniqueness of a solution to the initial value problem

$$\frac{df}{dx} = f(x, y(x)), \;\; y(x_0) = y_0$$