Representations of $\mathfrak{su}(2)$

The purpose of these notes is to construct the representations of $\mathfrak{su}(2)$ using the method of weight-vectors, based on the discussion of the representations of $\mathfrak{sl}(2, \mathbb{R})$ in the notes for course Ma424 Group Representations by Dr Timothy Murphy. This is interesting because it corresponds to the quantum mechanical description of angular momentum, which we very briefly discuss at the end.

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1 Lie and linear groups

We begin with some background material. Recall that a **Lie group** is a differential manifold with a group structure, such that the group operations of multiplication and inversion are differentiable, and that the **Lie algebra** of a Lie group is the tangent space to the group at the identity. The most commonly used Lie groups are matrix groups, and we will focus on these, disregarding the manifold structure.

We denote the set of all $n \times n$ matrices over the field k by $\mathrm{Mat}(n, k)$, where $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$. The space of invertible matrices over k is denoted by $\mathrm{GL}(n, k)$. Such matrices form a group under matrix multiplication. A **linear group** is a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$.

For $X \in \mathrm{Mat}(n, k)$ we define the **exponential map** $\exp : \mathrm{M}(n, k) \to \mathrm{GL}(n, k)$ by

$$\exp X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

Convergence follows using the matrix norm, by comparison with the scalar case. Note that if X has eigenvalues λ_i then $\exp X$ has eigenvalues e^{λ_i}. From this we have the useful property that the determinant of $\exp X$ is the exponential of the trace of X, as the determinant of a matrix is the product of the eigenvalues while the trace is the sum, so $\det \exp X = e^{\lambda_1} \dots e^{\lambda_n} = e^{\lambda_1 + \dots + \lambda_n}$.

The **commutator** of two matrices is defined by

$$[X, Y] = XY - YX$$

If $[X, Y] = 0$ then $\exp X \exp Y = \exp Y \exp X = \exp(X + Y)$. From this it follows that $\exp X$ is always invertible with inverse $\exp(-X)$.

The **Lie algebra of a linear group** linear group G is the space $\mathcal{L}G$ defined by

$$\mathcal{L}G = \{X \in \mathrm{Mat}(n, \mathbb{R}) : \exp(tX) \in G \, \forall t \in \mathbb{R}\}$$

It is a vector space, and the commutator gives a bilinear skew-symmetric form on $\mathcal{L}G$ satisfying Jacobi's identity: $[X, [Y, Z]]$ + cyclic permuations = 0. In this context, the commutator is called the **Lie product** on $\mathcal{L}G$. This definition of the Lie algebra of a linear group agrees with the general definition of a Lie algebra as a vector space over k equipped with a bilinear skew-symmetric form satisfying Jacobi's identity.

A **homomorphism of Lie algebras** $f : \mathcal{L} \to \mathcal{M}$ is a linear map which preserves the Lie product, $f([X, Y]) = [f(X), f(Y)]$. Given a homomorphism $F : G \to H$ of linear groups then there exists a unique homomorphism $f = \mathcal{L}F : \mathcal{L}G \to \mathcal{L}H$ of the corresponding Lie algebras, such that

$$\exp f(X) = F(\exp X)$$

for all $X \in \mathcal{L}G$. As a representation of a group G is simply a homorphism from the group to the space of invertible linear maps over some vector space V, this has implications for the representations of Lie algebras. First let us give an exact definition: a **representation of a Lie algebra** over k in a vector space V over k is defined by a bilinear map $\mathcal{L} \times V \to V$, denoted by $(X, v) \mapsto Xv$, satisfying $[X, Y]v = X(Yv) - Y(Xv)$.

Note that we always view $\mathcal{L}G$ as a real vector space yet are primarily interested in representations over $\mathbb{C}$. We thus introduce the **complexification of a Lie algebra** $\mathbb{C}\mathcal{L}$ as the complex Lie algebra derived from $\mathcal{L}$ by allowing multiplication by complex scalars. Then we have that every real representation α of the linear group G in V gives rise to a representation $\mathcal{L}\alpha$ of the corresponding Lie algebra $\mathcal{L}G$ in U, while every complex representation α of G in V gives rise to a representation $\mathcal{L}\alpha$ of the complexified Lie algebra $\mathbb{C}\mathcal{L}G$ in V, with each representation characterised by

$$\exp(\mathcal{L}\alpha X) = \alpha(\exp X)$$

for all $X \in \mathcal{L}G$.

A particular consequence of this definition is that the representations (over $\mathbb{C}$) of Lie algebras with isomorphic complexifications are in one-to-one correspondence.

The usual properties and operations for representations then hold for representations of Lie algebras. We also have that α is (semi)simple if and only if $\mathcal{L}\alpha$ is (semi)simple.

If G is simply connected then every representation α of $\mathcal{L}G$ can be lifted to a unique representation α' of G such that $\alpha = \mathcal{L}\alpha'$.

2 SU(2)

The group SU(2) is the group of all two-by-two unitary matrices with determinant equal to one:

$$\mathrm{SU}(2) = \{X \in \mathrm{Mat}(2, \mathbb{C}) : X^\dagger X = XX^\dagger = \mathbb{I}, \det X = 1\}$$

Say $X \in \mathfrak{su}(2)$, that is $\exp tX \in \mathrm{SU}(2) \forall t \in \mathbb{R}$, so

$$(\exp tX)^\dagger \exp tX = \exp t(X^\dagger + X) = \mathbb{I}$$

$$\Rightarrow X = -X^\dagger$$

and also

$$\det \exp tX = \exp \mathrm{tr}\, tX = \exp t \mathrm{tr}\, X = 1 \quad \forall\, t \in \mathbb{R}$$

$$\Rightarrow \mathrm{tr}\, X = 0$$

hence the Lie algebra $\mathfrak{su}(2)$ of SU(2) consists of all traceless two-by-two skew-hermitian matrices:

$$\mathfrak{su}(2) = \{X \in \mathrm{Mat}(2, \mathbb{C}) : X = -X^\dagger, \mathrm{tr}\, X = 0\}$$

A basis for this space is

$$U = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad V = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad W = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

(note that this is the usual basis for $\mathfrak{su}(2)$ rescaled by a factor of one-half). The Lie algebra structure is given by the commutators of the basis elements:

$$[U, V] = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = W$$

$$[V, W] = \frac{1}{4}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = U$$

$$[W, U] = \frac{1}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = V$$

hence

$$\mathfrak{su}(2) = \langle U, V, W : [U, V] = W, [W, U] = V, [V, W] = U \rangle$$

We will use an algebraic method to determine the representations of this Lie algebra. First, let

$$H = -2iW \quad E = U - iV \quad F = -U - iV$$

then

$$[H,E] = -2i\Big([W,U] - i[W,V]\Big) = -2i(V + iU) = 2E$$

$$[H,E] = -2i\Big(-[W,U] - i[W,V]\Big) = -2i(-V + iU) = -2F$$

$$[E,F] = -i[U,V] - i[V,U] = -2iW = H$$

Now let α be a simple representation of $\mathfrak{su}(2)$ in a finite dimensional vector space $\mathcal{V}$, and let us consider the eigenvalues and eigenvectors of H: $Hv = \lambda v$. We shall call λ a **weight** of H and v a **weight-vector**, with $S(\lambda) = \{v \in \mathcal{V} : Hv = \lambda v\}$ being the corresponding weight-space. Now,

$$HEv = (HE - EH + EH)v = (E + EH)v = (\lambda + 2)v \Rightarrow Ev \in S(\lambda + 2)$$

and

$$HFv = (HF - FH + FH)v = (-F + FH)v = (\lambda - 2)v \Rightarrow Fv \in S(\lambda - 2)$$

so we see that E is a **raising operator** while F is a **lowering operator**: applying E moves us "up" a weight-space, while applying F moves us "down" a weight-space. Applying first one then the other leaves us in the same space that we started in, $EFv, FEv \in S(\lambda)$. Now as our vector space $\mathcal{V}$ is finite-dimensional there can only be a finite number of weight-vectors, so if we start with some weight-vector v and apply E repeatedly we will eventually get zero, i.e. $E^r v \neq 0$ but $E^{r+1} v = 0$ and similarly for F.

Let us take a weight-vector v_μ with weight μ such that $Ee = 0$, so that μ is the maximal weight. Acting on e with F we obtain weight-vectors $v_{\mu-2} = Fv_\mu$, $v_{\mu-4} = F^2 v_\mu, \ldots$ with weights $\mu - 2, \mu - 4, \ldots$ and for some integer r we will have $F^r v_\mu \neq 0$, $F^{r+1} v_\mu = 0$. So in general we have

$$Hv_w = wv_w \qquad Fv_w = v_{w-2}$$

We now claim that

$$EFv_w = a(w)v_w \quad FEv_w = b(w)v_w$$

i.e. acting on v_w with first F then E or vice-versa gives us a scalar multiple of v_w. Note that we have

$$Hv_w = [E,F]v_w = a(w) - b(w) \Rightarrow a(w) - b(w) = w$$

We show the claim by induction on w. Starting at the maximal weight, we have

$$FEv_\mu = 0$$

hence the result holds with $b(\mu) = 0$ and $a(\mu) = \mu$. Now suppose it is true for $v_\mu, v_{\mu-2}, \ldots, v_{w+2}$. Then

$$FEv_w = FEFv_{w+2} = Fa(w+2)v_{w+2} = a(w+2)v_w$$

so the result holds with $b(w) = a(w+2)$, or

$$a(w) = a(w+2) + w$$

This gives us a recursive formula for $a(w)$. Note that we have $a(\mu) = \mu$, $a(\mu - 2) = a(\mu) + \mu - 2$, and for $w = \mu - 2k$,

$$a(\mu - 2k) = \mu + \mu - 2 + \cdots + \mu - 2k$$

$$\Rightarrow a(\mu - 2k) = \mu(k+1) - 2\sum_{i=0}^{k} i = \mu(k+1) + 2\frac{1}{2}k(k+1)$$

so

$$a(\mu - 2k) = (k+1)\,(\mu - k)$$

or using $k = \frac{1}{2}(\mu - w)$

$$a(w) = \frac{1}{4}(\mu - w + 2)(\mu + w) = \frac{1}{4}\Big(\mu(\mu+2) - w(w-2)\Big)$$

and as $b(w) = a(w+2)$,

$$b(w) = \frac{1}{4}\Big(\mu(\mu+2) - w(w+2)\Big)$$

Now applying F to $v_{\mu-2r}$ gives zero, hence $a(\mu - 2r) = 0$ meaning

$$(r+1)\Big(\mu - r\Big) = 0 \Rightarrow \mu = r$$

so we conclude that the weights run from r down to $-r$ and take only integral values. We also have now found that the space

$$\langle v_r, v_{r-2}, \dots, v_{-r}\rangle$$

is stable under $\mathfrak{su}(2)$ and so is the whole of $\mathcal{V}$ and carries a representation of degree $r+1$. It follows that there is one simple representation of $\mathfrak{su}(2)$ of each dimension.

The actions of F and E are found as follows:

$$Ev_w = EFv_{w+2} = a(w+2)v_{w+2} = \frac{1}{4}\Big(r(r+2) - w(w+2)\Big)v_{w+2}$$

$$Fv_w = FEv_{w-2} = a(w-2)v_{w-2} = \frac{1}{4}\Big(r(r+2) - w(w-2)\Big)v_{w-2}$$

from which we recover

$$Uv_w = \frac{1}{2}(E-F)v_w = \frac{1}{8}\Big[\Big(r(r+2) - w(w+2)\Big)v_{w+2} - \Big(r(r+2) - w(w-2)\Big)v_{w-2}\Big]$$

$$Vv_w = \frac{i}{2}(E+F)v_w = \frac{i}{8}\Big[\Big(r(r+2) - w(w+2)\Big)v_{w+2} + \Big(r(r+2) - w(w-2)\Big)v_{w-2}\Big]$$

and

$$Wv_w = \frac{i}{2}wv_w$$

Finally we can express the above formulae in terms of the weights $m = \frac{w}{2}$ of W and in terms of $j = \frac{r}{2}$:

$$Ev_m = \frac{1}{4}\Big(2j(2j+2) - 2m(2m+2)\Big)v_{m+1} = \Big(j(j+1) - m(m+1)\Big)v_{m+1}$$

$$Fv_m = \frac{1}{4}\Big(2j(2j+2) - 2m(2m-2)\Big)v_{m-1} = \Big(j(j+1) - m(m-1)\Big)v_{m-1}$$

$$Wv_m = imv_m$$

where we now index the vectors using m (note $v_m \equiv v_{2m} = v_w$), so that the representation is in the space

$$\mathcal{V} = \langle v_j, v_{j-1}, \dots, v_{-j}\rangle$$

and is of degree $2j+1$.

2.1 Alternative methods

Note that SU(2) is isomorphic to the three-sphere S^3 which is simply connected, and hence the representations of SU(2) and $\mathfrak{su}(2)$ are in fact in one-to-one correspondence. The representations of SU(2) are the representations in the space V_j of homogeneous polynomials in $z, w \in \mathbb{C}$ of degree $2j$ induced by the natural action of SU(2) on $(z\ w)^t \in \mathbb{C}^2$. This result can be established by restricting to the subgroup of diagonal matrices in SU(2) which is isomorphic to U(1), and showing that V_j splits as a sum of simple U(1) modules but not as a sum of simple SU(2) modules.

As a second alternative, we have that the Lie algebras of SU(2) and SL(2, $\mathbb{R}$) have the same complexification and so the same representations. The group SL(2, $\mathbb{R}$) is the group of two-by-two real matrices with unit determinant and its Lie algebra consists of traceless two-by-two matrices. As a basis we have

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and these satisfy $[H, E] = 2E$, $[H, F] = -2F$, $[E, F] = H$ and so can be used to construct the representations of $\mathfrak{sl}(2, \mathbb{R})$ and hence $\mathfrak{su}(2)$ exactly as we did using $\mathfrak{su}(2)$. (To see that $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{R})$ have the same complexification we note that the complexification of each is the set of all traceless complex two-by-two matrices.)

3 In quantum mechanics

In quantum mechanics we are concerned with the generators of infinitesimal rotations, J_x, J_y, J_z, which satisfy

$$[J_l, J_m] = i\varepsilon_{lmn} J_n$$

(with $\hbar = 1$). An important difference between these generators and the basis for $\mathfrak{su}(2)$ used above is that the J_i are hermitian rather than skew-hermitian, hence

$$J_x \equiv -iV \quad J_y \equiv -iU \quad J_z \equiv -iW$$

so that we have the raising and lowering operators

$$J_+ = J_x + iJ_y = E \quad J_- = J_x - iJ_y = F$$

while also

$$J_z = \frac{H}{2}$$

The total angular momentum operator $J^2 = J_x^2 + J_y^2 + J_z^2$ has eigenvalues $j(j+1)$ while J_z has eigenvalues m. A procedure somewhat similar to the one above using the raising and lowering operators tells us that m runs from j to $-j$ and can be integral or half-integral, and so the space of a given angular momentum j is $2j+1$-dimensional. Our basis elements are written using Dirac notation as $|jm\rangle \equiv v_m$, and are normalised so that

$$J_+|jm\rangle = \sqrt{j(j+1) - m(m+1)}|j, m+1\rangle$$

$$J_-|jm\rangle = \sqrt{j(j+1) - m(m-1)}|j, m-1\rangle$$