Noether's theorem in course 241

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I'm impressed that such things can be understood in such a general way - Albert Einstein

1 Introduction

This as close as I can get to explaining Noether's theorem as it occurs in second year Mechanics.

2 Noether's theorem

Consider a Lagrangian $L(q_i, \dot{q}_i, t)$, with equations of motion $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$. Let $q_i(t) \mapsto q_i'(t) = q_i(t) + \varepsilon \delta q_i(t)$ be a (continuous) transformation of the generalised coordinates q_i that leaves the equations of motion unchanged. The condition that the equations of motions are unchanged is equivalent to requring that the action $S = \int L\, dt$ be invariant, or more generally be changed by no more than an additive constant term (as the equations of motion are derived from $\delta S = 0$ such a term will vanish).

This means we can allow the Lagrangian to vary by no more than an overall total time derivative, $L \mapsto L' = L + \alpha \frac{d}{dt} J$. This is because the overall time derivative will integrate out immediately in the action, leaving just an additive constant, and so does not affect the equations of motion:

$$S = \int L'\, dt = \int \left(L + \alpha \frac{d}{dt} J \right) dt = \int L\, dt + \alpha J(t_2) - \alpha J(t_1) \Rightarrow \delta S = \delta \int L\, dt$$

We could then formally state the theorem as follows:

Theorem (Noether) *Let $q_i(t) \mapsto q_i'(t) = q_i(t) + \varepsilon_\alpha \delta q_i(t)$ be an infinitesimal transformation of the generalised coordinates, parametrised by the (infinitesimal) quantities ε_α such that under this transformation $L \mapsto L' = L + \varepsilon_\alpha \frac{d}{dt} J$, then the quantities j_α given by*

$$j_\alpha \varepsilon_\alpha = \frac{\partial L}{\partial \dot{q}_i} \delta q_i \varepsilon_\alpha - J \varepsilon_\alpha$$

are conserved.

Proof. Consider the variation in the Lagrangian caused by the change in the coordinates and their velocities:

$$\begin{aligned}
\delta L &= \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \\
&= \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i \\
&= \frac{\partial L}{\partial q_i} \delta q_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \\
&= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right)
\end{aligned}$$

where we have used the equations of motion to eliminate two of the terms. Now for each α, this variation multiplied by ε_α must be equal to the corresponding change $\varepsilon_\alpha \frac{d}{dt}J$ in the Lagrangian, so

$$\varepsilon_\alpha \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) = \varepsilon_\alpha \frac{d}{dt}J$$

and this implies

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i - J\right) = 0$$

hence

$$j_\alpha = \frac{\partial L}{\partial \dot{q}_i}\delta q_i - J$$

is conserved, or taking into account that the index α refers to numerous transformations we should write

$$j_\alpha \varepsilon_\alpha = \frac{\partial L}{\partial \dot{q}_i}\delta q_i \varepsilon_\alpha - J\varepsilon_\alpha$$

□

3 Notes

The important thing to note is that there is a separate conserved quantity for each ε_α - where α is used to index the different transformations. Note also that this formulation of the theorem does not really take into account transforming time, though we can sort of handle this - see the energy example below. Note also what the theorem essentially means is that for every continuous symmetry there corresponds a conserved quantity, which is a really cool result.

4 Examples

4.1 Energy

Let $t \mapsto t+\varepsilon$ be an infinitesimal time translation. Under this, $q_i(t) \mapsto q_i'(t) \approx q_i(t)+\varepsilon\dot{q}_i(t)$ (Taylor expansion), and $L \mapsto L' \approx L + \varepsilon\frac{d}{dt}L$. Hence in this case we have $J = L$, and the conserved quantity (only one) is

$$j = \frac{\partial L}{\partial \dot{q}_i}\delta q_i - L \equiv E$$

4.2 Momentum

Let $q_i(t) \mapsto q'(t) = q_i(t) + \varepsilon_i$ be an infinitesimal spatial translation, and let L be invariant, i.e. $\delta L = 0$ or $J = 0$, and so we have the conserved quantities

$$j_i = \frac{\partial L}{\partial \dot{q}_i} \equiv p_i$$

This requires that the coordinates only appear in the Lagrangian as differences of coordinates $(q_i - q_j)$, so that it does not change under spatial translation. For instance momentum is not conserved for the harmonic oscillator, $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$.

4.3 Angular momentum

Let $q_i(t) \mapsto q_i'(t) = q_i(t) + \varepsilon_{ij} q_j$ (summing over j) where $\varepsilon_{ij} = -\varepsilon_{ij}$ (the infinitesimal generators of the rotation group $SO(n)$ are $n \times n$ skew-symmetric matrices. This can be seen by observing that a rotation $\vec{x} \mapsto A\vec{x}$ preserves norms, so $|\vec{x}|^2 = \vec{x}^t \vec{x} \mapsto (A\vec{x})^t A\vec{x} = \vec{x}^t A^t A \vec{x} = \vec{x}^t \vec{x}$ so $A^t A = I$ - i.e. rotation matrices are orthogonal. Now infinitesimally we let $A = I + \varepsilon$, then we have $(I+\varepsilon)^t (I+\varepsilon) = I + \varepsilon^t + \varepsilon$ to first order, hence we must have $\varepsilon^t = -\varepsilon$). The conserved quantities are then given by

$$j_{ij}\varepsilon_{ij} = \frac{\partial L}{\partial \dot{q}_i} \varepsilon_{ij} q_j = \varepsilon_{ij} p_i q_j$$

There is a sum over i and j here. As ε_{ij} is skew-symmetric so too is j_{ij} (because it has the same index), note that every j_{ij} term then occurs twice (that is, $j_{ij}\varepsilon_{ij} = j_{ji}\varepsilon_{ji}$) so we put a factor of half in front on the left-hand side, at the same time rewriting the right-hand side so that we have

$$\frac{1}{2} j_{ij}\varepsilon_{ij} = \frac{1}{2} \varepsilon_{ij}(p_i q_j - p_j q_i)$$

(this should all make sense if you think for a moment about the summations and the skew-symmetry). We can then read off the conserved quantities to be:

$$j_{ij} = p_i q_j - p_j q_i$$

Note that j_{ij} is an $n \times n$ skew-symmetric matrix, so the entries on the diagonal are all zero, reducing the number of free entries in the matrix by n, and also the entries above the diagonal are equal to minus those below the diagonal, further dividing the number of free entries by two, hence the number of conserved quantities in this case is $\frac{1}{2}n(n-1)$.

In three dimensions, $n = 3$ giving three conserved quantities,

$$j_{12} = p_1 q_2 - p_2 q_1 = -M_3 \quad j_{23} = p_2 q_3 - p_3 q_2 = -M_1 \quad j_{31} = p_3 q_1 - p_1 q_3 = -M_2$$

corresponding to the different components of angular momentum (M_1 being angular momentum about the x-axis and so on).

References

The version of the theorem presented here is a sort of horrible mismash of what Frolov covered in lectures in 2007-08 and the version found in Peskin and Schroeder's quantum field theory book (altered to apply to the discrete rather than field case). Goldstein has a long derivation of a version of the theorem, but for fields. The Wikipedia article does a number of different versions, but uses a different approach for the case covered here. Emmy Noether's original paper is available online at `http://arxiv.org/PS_cache/physics/pdf/0503/0503066v1.pdf`.