Formulating Mechanics Problems: Supplementary Notes for 141 Mechanics

Brian Durcan

Contents

1 Introduction

Most problems in 141 Mechanics are solved by either

1. integrating an equation and applying some initial conditions

2. calculating a conserved quantity in two situations and using the equality of these two expressions to find some unknown.

or by some combination of both. These notes will not tell you how to do 1 or 2. The goal here is to explain how to read a mechanics problem, extract the information from it and express it as something soluble.

The only way to adequately prepare for the 141 exam is to study your lecture notes and practice doing **ALL** of the problem sets. There are many exceptional questions which the material below wont be of much help with.

The lists of forces and equations given below are **INCOMPLETE**. The lists are intended to encourage you to collect all the useful expressions you have come across. Add to them.

2 List all the information given

2.1 Read the question

Read through the question. Keep in mind that if something is stated in the question, it is because you *need* this information to solve it. There is no unnecessary description. The following words all have very specific meanings:

massless Gravity does not act on this body.

light The body has such a small mass that we can consider it to be **massless**.

smooth There is no friction on this surface.

rough There is friction on this surface.

inelastic If referring to a string, the length of that string will not change. If referring to a collision, energy is not conserved during the collision.

rigid The shape of a rigid body cannot be changed. A rigid rod has constant length.

rolling without slipping A ball or wheel which is rolling without slipping has its rotation and linear motion in synchronisation. That is, if it rotates through 360 degrees, then it travels a distance equal to its circumference. $V = \omega R$.

Look through the assignments and try to **ADD TO THIS LIST**.

As you read through the question, highlight terms like these which you understand the significance of. If you come across some information, and you don't know why you are told it, highlight it differently.

Pay particular attention to quantities which you are told are **constant**. Differentiation plays an important part in solving mechanics problems and if something is constant this can simplify things greatly.

2.2 Draw a picture

Draw the system which is described in the question. Draw it big. If there is already a picture in the question draw it again, but bigger. If you see someone else drawing the picture, draw yours bigger than theirs. You need to be able to fit as much information as possible into it. It helps to use chalk or pencil, so that you can rearrange things as you go.

The picture should be built up in stages:

1. Draw it as it would look, without any extra information added in. In a problem where things change dramatically with time (e.g. a collision) draw each stage of the system separately.

2. Next, you add information to the picture. Look through the question at the phrases you have highlighted. If a phrase has a simple way of being represented (e.g. '..a particle of mass m..' - put m beside the particle). Tick any phrases which you manage to put in the picture.

3. The most important step is to add forces to the diagram. Go through your **List of Forces** (see below) and decide which apply and which don't.

4. Any highlighted phrases which didn't go into the picture in some simple way should be written next to the it.

3 Choose Coordinates

Next you must answer the question: What is the best way to describe the positions of the objects in the system? Look for straight lines and circles in your picture. Whenever you find one ask whether or not this line / curve will be important in determining how things move.

e.g.1 A body sitting on an inclined surface.
$\rightarrow$ The inclined surface provides a natural direction which will be significant when the dynamics are examined.

e.g.2 A bowl on the inner surface of which a small particle is free to move
$\rightarrow$ The curve of the bowl will be important when looking at how the particle moves.

e.g.3 A pendulum
$\rightarrow$ A pendulum sweeps out a circle when it moves.

e.g.4 A charged particle in a constant electric field
$\rightarrow$ The direction of the electric field is a natural direction to consider when looking at how the charged particle moves. Similarly, for a body falling under gravity the direction of the gravitational field is usually the most significant.

3.1 Cartesian Coordinates

If you have found that there is some straight line in the system which will be relevant to the motion then set up a Cartesian coordinate system which respects that.

1. In your picture, add in an x-axis along this direction and a y-axis perpendicular to it.

2. Resolve all the forces along these directions. If there are any other vectors in your picture (pictures representing collisions should contain velocity vectors, for example) resolve these along those directions too.

3. Where did you put the origin of the x- and y-axes? See **section 5.10** below to see if you need to add a pseudoforce.

3.2 Polar Coordinates

If you found that there is some circle related to the motion, you should set up a polar coordinate system.

1. Fix the origin at the centre of the circle.

2. Resolve forces normal to the circle and tangent to it.

3. Where did you fix the origin? See **section 5.10** below to see if you need to add a pseudoforce.

4. Pay special attention to the equations of **section 6.4** on circular motion.

Extra note: If you continue to study mechanics next year, you will learn another method for describing the evolution of physical systems which can be expressed without coordinates. It is generally favourable throughout mathematics to express important equations without relying on particular coordinate systems or other arbitrary choices. In geometry (and in physics) it is very common that one coordinate system (such as Cartesian axes or polar coordinates) are insufficient to describe the whole space under consideration.

4 Write Equations

You are now prepared to express the physical problem as a mathematical one.

1. Look through your **List of Equations** and write as many as possible with the information from your force diagram.

2. Collect your equations neatly and see which functions and numbers are unknown.

3. Refer back to the question to see which of these unknowns you are required to find.

If all goes well, you will see how to proceed and if there are any pieces of information written next to the picture you will see how they can now be applied.

If you still don't know what to do, don't worry. Writing down these equations is an important part of the problem and you'll receive a good portion of the marks without having solved anything.

5 List of Forces

For each force you need to know:

1. in what situations does it apply?

2. what is its magnitude?

3. what is its direction?

4. if it is a conservative force, what is the associated potential energy?

5.1 Weight

The force of weight acts on a body **when you are told that it has mass**. For problems related to planets, stars, satellites etc. you will use the more general expression for gravity (see below).

The force is given by

$$F_W = -mg \tag{1}$$

where m is the mass and $g = \dfrac{GM_E}{R_E^2} \simeq 9.8\ \mathrm{ms}^{-2}$.

This is a conservative force, and the corresponding potential energy is

$$V_W = mgh \tag{2}$$

where h is the height above some fixed point (chosen by you).

Equation (1), with g as given above, is an approximation to the force of gravity. The approximation is valid near to the earth's surface.

5.2 Normal Reaction

Normal reactions are the forces which prevent objects from passing through each other. A normal reaction will act on a body **when it is in contact with another surface**.

The force is perpendicular (normal) to the surface of contact, and is directed from the surface towards the body. The strength of the force is often[1] equal to whatever force is pushing the body against the surface i.e. It is exactly as strong as it needs to be to prevent the body moving through the surface.

Because of Newton's third law, the surface experiences a force which is equal in magnitude and opposite in direction to that acting on the body. You will not need to consider this unless you are dealing with a problem in which the surface can move.

Reaction forces come about due to electromagnetic forces between atoms.

5.3 Tension

Tension should be included **when there is an inelastic string / rod connecting two objects**. Tension is effectively an attractive force between the objects which are connected.

As with the normal reaction, the strength of the tension force is determined by other forces in the system. Often[2] it is exactly as strong as it needs to be to prevent the length of the string / rod from changing.

Tension is caused by electromagnetic forces between atoms.

5.4 Friction

Friction acts **when a body is in contact with a rough surface**.

The force is parallel to the surface, and is directed in such a way that it opposes motion.

The friction force has a maximum magnitude:

$$F_{\text{max}} = \mu R \tag{3}$$

[1] There are exceptional cases which you should study e.g. A particle rolling down the side of a ball. There is some acceleration in the direction normal to the surface, so the forces in this direction cannot be cancelling each other.

[2] Again, if there is some acceleration in that direction, the forces will not cancel

where μ is the coefficient of friction of the surface (a measure of how rough the surface is) and R is the normal reaction on the body due to its contact with the surface.

When a body is stationary on a rough surface and you press against it with a force P parallel to the surface, one of two things can happen:

1. If $P < F_{\max}$ a friction force will act on the body, directed opposite to P, with strength equal to that of P. There will be zero net force and the body will not move. i.e. You did not push hard enough to overcome friction.

2. If $P > F_{\max}$ a friction force will act on the body, directed opposite to P, with strength $F_{\max}$. There will be a net force on the body (in the direction of P) and it will accelerate.

Often, the focus of a problem involving friction will be the situation in which $P = F_{\max}$. This happens when you are asked 'What is the minimum force required to move the body?' or 'What is the maximum force which can be applied to the body without it moving?'

Friction is a **non-conservative** force. Energy leaves the system which you are considering (a macroscopic body and a rough surface). This means you will have to modify some of the formulae associated to energy when solving the problem. Energy is never completely lost however - in this case it is converted into heat and sound.

Friction is caused by electromagnetic forces between atoms.

5.5 Restoring Force / Spring Force

This force applies **when a body is connected to a spring or to an elastic string**. If the spring has spring constant k and x represents the displacement from the *equilibrium position*, then the force is

$$F_S = -kx \tag{4}$$

The equilibrium position is the place where the restoring force cancels all other forces acting on the body, and the body can remain at rest. If there are no other forces then this is the place where the restoring force is zero. This corresponds to the *natural length* of the spring - it is neither stretched nor compressed.

This is a conservative force and the potential is

$$V_S = -\frac{1}{2}kx^2 \tag{5}$$

The restoring force is caused by electromagnetic forces between atoms.

Note: This type of force/potential and the harmonic oscillator solution are extremely important in physics. In *all* of physics. The reason is that potential energies in general physical systems can have very complicated forms, but they have local minima. These minima correspond to equilibrium states of the system. If we want to consider how a system behaves when it is not far from an equilibrium state, we can do a Taylor expansion of the potential about that position. The first non-zero term in the expansion will be of the form (5). So *ALL* physical systems, when they are slightly perturbed from a stable equilibrium state, will behave like a harmonic oscillator. The difficulties arise when we want to consider larger perturbations.

5.6 Gravity

The force of gravity acting on a mass M_1 due to a mass M_2 separated by a distance r is

$$F_G = \frac{GM_1M_2}{r^2} \tag{6}$$

Newton's constant G is one of the fundamental constants of nature, determining many of the features of the universe. It also appears in Einstein's general relatvity (of which Newton's theory of gravity is a very useful approximation).

The force of gravity in this form is used **when considering the motion of planets, stars, satellites etc.**

Gravity is an attractive force, so the direction is towards the body which is causing the force.

It is a conservative force, the potential is given by

$$V_G = -\frac{GM_1M_2}{r} \tag{7}$$

5.7 Coulomb Force

This is included next because it is so similar in form to gravity. It applies **when we are considering multiple charged objects**. For charged bodies with charges Q_1 and Q_2 separated by r we have:

$$F_C = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r^2} \tag{8}$$

$$V_C = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r} \tag{9}$$

The difference between this electric force and the gravitational one is that electric charge can be a positive number or a negative one whereas *gravitational charge* (mass) is always positive. For this reason the Coulomb interaction can be attractive or repulsive whereas gravity is always attractive. It is attractive when Q_1 and Q_2 have different signs and is repulsive when they have the same sign.

5.8 Force due to an Electric Field

This force applies **when you are told that there is a charged particle in an electric field** $\vec{E}$. The definition of electric field is in terms of force per unit charge. To recover a force from a force per unit charge, we simply multiple by the charge.

A particle with charge q in an electric field $\vec{E}$ is given by:

$$\vec{F}_E = q\vec{E} \tag{10}$$

5.9 Force due to a Magnetic Field

This force applies **when you are told that there is a charged particle in an magnetic field** $\vec{B}$.

A particle with charge q and velocity $\vec{v}$ in an electric field $\vec{B}$ is given by:

$$\vec{F}_B = q\vec{v} \times \vec{B} \tag{11}$$

5.10 Pseudoforce / Fictitious Force

This applies **when you are working in an accelerating coordinate system**. To decide if your frame is inertial (not accelerating) or non-inertial (accelerating) look at the picture of the system. Find some object / point which will not move e.g. In a problem with some objects resting on a table, you take the table to be this fixed point of reference.

If the origin of the coordinate system is moving with acceleration $\vec{a}_o$ relative to some fixed point, then a body with mass m will experience a pseudoforce

$$\vec{F}_P = -m\vec{a}_o \tag{12}$$

Once you include this force, you can apply the usual equations as if you were working in a non-accerlating reference frame. It can be thought of as a correction which compensates for the acceleration of the coordinate system.

If your reference frame is **rotating** with respect to some non-accelerating coordinate system, then you have to compensate for this in all expressions involving derivatives. See your notes of the Coriolis force for more information.

The pseudoforce is caused by the perspective of the observer.

6 List of Equations

Throughout the following we use $\vec{r}$ to describe the position of a particle, $\dot{\vec{r}}$ its velocity and $\ddot{\vec{r}}$ its acceleration. We use m for the mass of the particle.

6.1 Definitions

Momentum of a particle:

$$\vec{p} = m\dot{\vec{r}} \tag{13}$$

Kinetic energy of a particle:

$$E = \frac{1}{2}m\dot{\vec{r}} \cdot \dot{\vec{r}} \tag{14}$$

Total energy of a particle:

$$E = \frac{1}{2}m\dot{\vec{r}} \cdot \dot{\vec{r}} + V \tag{15}$$

The exact form of the potential energy V depends on the system in question.
Work done by a force $\vec{F}$ acting on a particle as it moves from a to b:

$$W = \int_a^b \vec{F} \cdot d\vec{r} \tag{16}$$

Work, if $\vec{F}$ does not depend on $\vec{r}$:

$$W = \vec{F} \cdot \int_a^b d\vec{r} = \vec{F} \cdot (\vec{r}_b - \vec{r}_a) \tag{17}$$

Work, if $\vec{F}$ is a conservative force with potential V:

$$W = -\left(V(\vec{r}_b) - V(\vec{r}_a)\right) \tag{18}$$

People (by which I mean me) often get confused by the signs when using formulae for work. It is most easily understood by looking at the final equation. If a particle loses potential energy then the work is positive. A body falling under gravity is having positive work done on it by the force.

Center of Mass:
If we have a collection of particles with masses $\{m_a | a = 1, \ldots, N\}$ and positions $\{\vec{r}_a | a = 1, \ldots, N\}$, the center of mass of that collection is located at

$$\vec{R}_{CM} = \sum_{a=1}^{N} \frac{m_a \vec{r}_a}{M} \tag{19}$$

where $M = \sum_{a=1}^{N} m_a$.

6.2 Laws of Motion

Newton's Second Law of Motion:

$$\vec{F}_{tot} = \frac{d\vec{p}}{dt} \tag{20}$$

The rate of change of the momentum of a body is equal to the net force acting on it. Newton's First Law of Motion can be derived from the second.

Very often, the mass of the body in question will be constant (problems with changing mass should be studied carefully, as they require a good understanding of the ideas surrounding momentum), in which case we have **Newton's Second Law of Motion (constant mass):**

$$\vec{F}_{tot} = m\ddot{\vec{r}} \tag{21}$$

Newton's Third Law of Motion:

$$\vec{F}_{AB} = -\vec{F}_{BA} \tag{22}$$

The force on body A due to body B is equal and opposite to the force on body B due to body A. This is often applied when dealing with normal reactions (if you want to know the force on the surface due to the body).

6.3 Conservation Laws

Conservation of energy applies when the system you are considering is **closed** - that is, nothing is leaving or entering the system:

$$\frac{dE}{dt} = 0 \tag{23}$$

When non-conservative forces are acting (friction, air resistance...) energy is not conserved.

Conservation of momentum applies to a single particle when no forces are acting on that particle - see equation (20). Conservation of momentum applies to a collection of particles when there are no external forces acting.

$$\frac{d}{dt}P_{tot} = \frac{d}{dt}\sum_{a=1}^{N} p_a = 0 \tag{24}$$

A force is **external** if it is not caused by any of the particles in the system you are considering.

6.4 Circular Motion

The unit vectors associated with the Cartesian coordinates x and y are $\hat{\imath}$ and $\hat{\jmath}$. That is, when you consider a general point (x, y) in this coordinate system, and advance one of the coordinates slightly:

$$(x, y) \to (x + c, y) \text{ for } c > 0$$

the direction in which you are moving is $\hat{\imath}$.

$$(x, y) \to (x, y + c) \text{ for } c > 0$$

corresponds to motion in the $\hat{\jmath}$ direction.

Now consider polar coordinates (r, θ) and ask what do we mean by the unit vectors $\hat{r}$ and $\hat{\theta}$. Using the same idea as above, $\hat{r}$ is the direction you move when r is increasing and θ is kept fixed. $\hat{\theta}$ is the direction you move when θ is increased and r is kept fixed. This leads to the following vectors:

$$\hat{r} = cos\theta\hat{i} + sin\theta\hat{j} \quad (25)$$
$$\hat{\theta} = -sin\theta\hat{i} + cos\theta\hat{j} \quad (26)$$

Draw these and make sure you understand why they take this form. From these expressions we can derive the following:

$$\dot{\hat{r}} = \dot{\theta}\hat{\theta} \quad (27)$$
$$\dot{\hat{\theta}} = -\dot{\theta}\hat{r} \quad (28)$$

A general vector in the plane is written in polar coordinates as $\vec{r} = r\hat{r}$. (Change back to $\hat{i}$ and $\hat{j}$ to check that this makes sense). We can differentiate it to get the following:

$$\vec{r} = r\hat{r} \quad (29)$$
$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \quad (30)$$
$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \quad (31)$$

We have assumed nothing about the motion of the particle to derive these equations - we haven't yet said that we are considering circular motion. These equations are valid for a particle moving in any way in a plane.

To consider circular motion we must impose the following equation:

$$\dot{r} = 0 \quad (32)$$

i.e. the radius is not changing. The equations above reduce to:

$$\vec{r} = r\hat{r} \quad (33)$$
$$\dot{\vec{r}} = r\dot{\theta}\hat{\theta} \quad (34)$$
$$\ddot{\vec{r}} = -r\dot{\theta}^2\hat{r} + r\ddot{\theta}\hat{\theta} \quad (35)$$

Now note that the vector $\hat{r}$ points from the centre of the circle outwards (it is normal to the circle). The vector $\hat{\theta}$ is tangent to the circle. This gives us the following formulae for circular motion (It helps if you understand the derivation above, but even if it is not yet clear you will still need the formulae below to solve mechanics problems):

Velocity and Acceleration Normal and Tangent to the Circle (ANY Circular Motion)
We put $\omega = \dot{\theta}$, $\dot{\vec{r}} = \vec{v} = v_n\hat{r} + v_t\hat{\theta}$ and $\ddot{\vec{r}} = \vec{a} = a_n\hat{r} + a_t\hat{\theta}$.

$$v_n = 0 \quad (36)$$
$$v_t = r\omega \quad (37)$$
$$a_n = -r\omega^2 \quad (38)$$
$$a_t = r\dot{\omega} \quad (39)$$

Velocity and Acceleration Normal and Tangent to the Circle (CONSTANT Circular Motion)
The only difference for constant circular motion is that $\dot{\omega} = 0$.

$$v_n = 0 \quad (40)$$
$$v_t = r\omega \quad (41)$$
$$a_n = -r\omega^2 \quad (42)$$
$$a_t = 0 \quad (43)$$

7 Important things these notes don't contain

1. Pictures! The easiest way to remember your list of forces is to draw pictures of situations where they apply.

2. The equations of **angular momentum**. You will study these in Trinity Term.

3. A list of exceptional problems that require modifications to the usual approach. You need to study the past problem sheets to find these.

4. Numerous things I've forgotten to mention (please email me with any of these)