Instructions to Candidates:

Credit will be given for the best 3 questions answered.

Materials Permitted for this Examination:

Formulae and Tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.
1. The population in country $A$ in year one is 1000 while in country $B$ it is 2000. Each year 25% of the population of country $A$ moves to country $B$ while 20% of country $B$’s population moves to country $A$.

(a) Find the population of countries $A$ and $B$

i. in year two.

ii. in year zero

(b) Find the population of countries $A$ and $B$ far in the future.

2. (a) Given the graph, $y = f(x)$, of a differentiable and invertible function $f$, derive two general formulae for the length of a segment of the graph, given either as integral over $x$ or $y$.

(b) Apply your result to the function $f(x) = \cosh(x) = (e^x + e^{-x})/2$ and calculate the length of the curve:

i. as an integral over $x$ between $x = 0$ and $x = \ln(2)$;

ii. as an integral over $y$ between $y = 1$ and $y = 5/4$.

Hint: you will need the identity $\cosh^2(x) - \sinh^2(x) = 1$ and the derivatives $\cosh' = \sinh$ and $\sinh' = \cosh$. 
3. Find the center of gravity of the homogeneous semicircular region bounded by \( y = \sqrt{1 - x^2} \) for \( x \in [-1, 1] \) and \( y = 0 \).

4. The trace \( \text{tr}(A) \) of an \( n \times n \) matrix \( A = [a_{ij}]_{n \times n} \) is the sum of the entries on the main diagonal, i.e., \( a_{11} + a_{12} + \ldots + a_{nn} \). It is known that

\[
\det(e^A) = e^{\text{tr}(A)}.
\]

Calculate \( e^A \) by eigenvector methods, and verify the formula, when

\[
A = \begin{bmatrix}
5 & -4 \\
2 & -1
\end{bmatrix}.
\]