Numerical Analysis of *Re*-Uniform Convergence for Boundary Layer Equations for a Flat Plate*

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Abstract. *In this article we consider grid approximations of a boundary value problem for boundary layer equations for a flat plate outside of a neighbourhood of its leading edge. The perturbation parameter* $\varepsilon = Re^{-1}$ *multiplying the highest derivative can take arbitrary values from the half-interval* $(0, 1]$*; here Re is the Reynolds number. We consider the case when the solution of this problem is self-similar. For this Prandtl problem by using piecewise uniform meshes, which are refined in the neighbourhood of a parabolic boundary layer, we construct a finite difference scheme that converges* ε*–uniformly. We present the technique for experimental substantiation of* ε*–uniform convergence of both the grid solution itself and its normalized difference derivatives, which are considered outside of a neighbourhood of the leading edge of the plate. We study also the applicability of fitted operator methods for the numerical approximation of the Prandtl problem. It is shown that the use of meshes condensing in the parabolic boundary layer region is necessary for achieving* ε*–uniform convergence.*

1 Introduction

Mathematical modelling of laminar flows of incompressible fluid for large Reynolds numbers Re often leads to a study of boundary value problems for boundary layer equations. These quasilinear equations are singularly perturbed, with the perturbation parameter ε defined by $\varepsilon = Re^{-1}$. The presence of parabolic boundary layers, i.e., layers described by parabolic equations, is typical for such problems [1, 2].

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The difficulties arising in the numerical solution even of linear singularly perturbed boundary value problems are well known. So, the application of numerical methods that were developed for regular boundary value problems (see, for example, [3, 4]) to the above problems yield error bounds which depend on the perturbation parameter ε. For small values of ε, the errors of such numerical methods may be comparable to, or even much larger than the solution of the boundary value problem. This behaviour of the approximate solutions creates the problem of the development of numerical methods with errors that are independent of the perturbation parameter ε, i.e., ε–uniformly convergent methods. The presence of a nonlinearity in the equations makes it considerably more difficult to construct ε–uniformly convergent numerical methods. For example, even in the case of ordinary differential quasilinear equations, fitted operator methods that converge ε–uniformly (see, e.g., [5, 6]) do not exist. Note that in the case of linear singularly perturbed problems with a parabolic boundary layer there are also no ε–uniformly convergent fitted schemes (see, for example, [7–9]). Thus, the development of special ε–uniform numerical methods for resolving boundary layer equations is of considerable interest.

At present, special finite difference schemes convergent ε–uniformly in the maximum norm are developed and investigated for wide classes of linear singularly perturbed boundary value problems, in particular, for problems with parabolic boundary layers (see, for example, [8–12]). It often occurs that the proved (i.e., theoretical) orders of ε-uniform convergence are quite low and would seem to imply that the constructed schemes will yield errors too large for these schemes to be of practical value. However, numerical (experimental) investigations of such schemes show that the actual orders of ε-uniform convergence are close to those typical for regular boundary value problems (see, e.g., [13, 14]). Thus, the experimental technique for *a posteriori* estimation of the parameters in ε–uniform error bounds seems to be crucial in the case of problems with rather complicated behaviour of the solution.

It is of interest to apply the existing technique to the construction of ε–uniformly convergent schemes for boundary layer equations in that part of the neighbourhood of the boundary where the boundary layer is parabolic. Note that, because of the nonlinearity of the boundary layer equations, the existing technique for justifying convergence of schemes and *a priori* estimates of the exact solutions do not allow us theoretically to prove ε–uniform convergence of the grid solutions in the L_∞–norm. In this connection, for the boundary layer equations we are forced to use only the alternative *a posteriori* method to study convergence, in particular, ε–uniform convergence of the grid solutions.

In the present paper we consider grid approximations of a boundary value problem for boundary layer equations for a flat plate outside of a neighbourhood of its leading edge. The boundary layer in the considered domain is parabolic. We consider the case when the solution of this Prandtl problem is self-similar. We construct a finite difference scheme, which is a natural development of monotone ε–uniformly convergent schemes for linear boundary value problems with a parabolic layer. For this we use classical grid approximations on piecewise uniform meshes, which are refined in the neighbourhood of the boundary layer. As is shown, the use of this condensing-mesh technique is necessary to achieve ε–uniform convergence. To justify convergence of the difference scheme constructed in this paper, we apply the experimental technique for *a posteriori* estimation of an error in the approximate solution.

We sketch an idea of experimentally studying ε–uniform convergence of numerical approximations for the Prandtl problem. Note that in the case of flow past a flat semi–infinite plate the Prandtl problem has a self-similar solution which can be expressed in terms of a solution of a quasilinear ordinary differential third-order equation, the so-called Blasius equation, defined on a semiaxis. To evaluate errors in the grid solutions of the boundary layer equations, as an approximation to the exact solution of the self-similar problem we use a linear interpolant of the grid solution to the corresponding Blasius equation. We present the technique how to study the behaviour of errors of the special difference scheme for the boundary layer equations in accordance with both the parameter ε and the number of mesh points. This method is used to justify ε–uniform convergence of both the grid solution itself and its normalized derivatives, which are considered outside of a neighbourhood of the leading edge of the plate.

We emphasize the growing interest in the performance of strong numerical investigations of a boundary layer; see, for example, [15]. Note that, in the neighbourhood of the parabolic boundary layer, the solutions of boundary layer equations for large Re numbers are close to the solution of the Navier-Stokes equations. The results obtained here and the research techniques can be used in the analysis of numerical methods for solving Navier-Stokes equations at high Reynolds numbers.

2 Problem formulation

In this section we give the problem formulation for boundary layer equations in the case of a bounded domain. Let a flat semi–infinite plate be represented by the semiaxis $P = \{(x,y):\ x \ge 0,\ y = 0\}$. The problem is considered to be symmetric with respect to the plane $y = 0$; we discuss the steady flow of an incompressible fluid on both sides of P, which is laminar and parallel to the plate (no separation occurs on the plate). We shall consider the solution of the problem on the bounded set

$$\overline{G}, \quad \text{where} \quad G = \{(x,y):\ x \in (d_1, d_2],\ y \in (0, d_0)\}\,, \quad d_1 > 0. \tag{2.1}$$

Let $G^0 = \{(x,y):\ x \in [d_1, d_2],\ y \in (0, d_0]\}$; note that $\overline{G}^0 = \overline{G}$. Assume $S = \overline{G} \setminus G$, $S = \cup S_j$, $j = 0,1,2$, where $S_0 = \{(x,y):\ x \in [d_1, d_2],\ y = 0\}$, $S_1 = \{(x,y):\ x = d_1,\ y \in (0, d_0]\}$, $S_2 = \{(x,y):\ x \in (d_1, d_2],\ y = d_0\}$,

$\overline{S}_0 = S_0$; $S^0 = \overline{G} \setminus G^0 = S_0$. On the set $\overline{G}$, it is required to find the function $U(x,y) = (u(x,y), v(x,y))$ as the solution of the following Prandtl problem:

$$\begin{aligned} L^1\left(U(x,y)\right) &\equiv \varepsilon \frac{\partial^2}{\partial y^2} u(x,y) - u(x,y) \frac{\partial}{\partial x} u(x,y) - \\ & \qquad - v(x,y) \frac{\partial}{\partial y} u(x,y) = 0, \quad (x,y) \in G, \qquad (2.2a) \\ L^2 U(x,y) &\equiv \frac{\partial}{\partial x} u(x,y) + \frac{\partial}{\partial y} v(x,y) = 0, \quad (x,y) \in G^0, \qquad (2.2b) \\ u(x,y) &= \varphi(x,y),\ \ (x,y) \in S, \quad v(x,y) = \psi(x,y),\ \ (x,y) \in S^0. \qquad (2.2c) \end{aligned}$$

Here $\varepsilon = Re^{-1}$; the parameter ε takes arbitrary values from the half-interval (0,1].

The solution of problem (2.2), (2.1) exists in that case, when the functions $\varphi(x,y)$, $\psi(x,y)$ are sufficiently smooth and also satisfy the compatibility conditions [2], respectively, on the sets $S^* = \overline{S}_1 \cap \left\{ S_0 \cup \overline{S}_2 \right\}$ (i.e., the set of the corner points adjoining to the side S_1) and $S^{0*} = \overline{S}_1 \cap S^0$.

We now wish to define the functions $\varphi(x,y)$ and $\psi(x,y)$ more exactly.

In the quarter plane

$$\overline{\Omega}, \quad \text{where} \quad \Omega = \left\{ (x,y) : \ x,\ y > 0 \right\}, \qquad (2.3)$$

let us consider the Prandtl problem whose solution is self-similar [1]:

$$\begin{aligned} L^1\left(U(x,y)\right) &= 0, & (x,y) &\in \Omega, \\ L^2 U(x,y) &= 0, & (x,y) &\in \overline{\Omega} \setminus P, \\ u(x,y) &= u_\infty, & x &= 0, \quad y \ge 0, \\ U(x,y) &= (0,0), & (x,y) &\in P. \end{aligned} \qquad (2.4)$$

The solution of problem (2.4), (2.3) can be written in terms of some function $f(\eta)$ and its derivative

$$\begin{aligned} u(x,y) &= u_\infty f'(\eta), \\ v(x,y) &= \varepsilon^{1/2} \left(2^{-1} u_\infty x^{-1} \right)^{1/2} \left(\eta f'(\eta) - f(\eta) \right), \end{aligned} \qquad (2.5)$$

where

$$\eta = \varepsilon^{-1/2} \left(2^{-1} u_\infty x^{-1} \right)^{1/2} y.$$

The function $f(\eta)$ is the solution of the Blasius problem

$$\begin{aligned} L(f(\eta)) \equiv f'''(\eta) + f(\eta) f''(\eta) = 0, \quad \eta \in (0,\infty), \\ f(0) = f'(0) = 0, \quad \lim_{\eta \to \infty} f'(\eta) = 1. \end{aligned} \qquad (2.6)$$

The functions $\varphi(x,y)$, $\psi(x,y)$ are defined by

$$\begin{aligned} \varphi(x,y) &= u_{(2.5)}(x,y), \qquad (x,y) \in S, \\ \psi(x,y) &= v_{(2.5)}(x,y), \qquad (x,y) \in S^0; \end{aligned} \tag{2.7}$$

note that $\varphi(x,y)=0$, $\psi(x,y)=0$, $(x,y) \in S^0$.

In the case of problem (2.2), (2.7), (2.1), as ε tends to zero, a parabolic boundary layer appears in a neighbourhood of the set S^0.

To solve problem (2.2), (2.7), (2.1) numerically, we will construct a finite difference scheme which converges ε–uniformly.

3 Difference scheme for problem (2.2), (2.7), (2.1)

Assume that we know the "coefficients" multiplying the derivatives $(\partial/\partial x)u(x,y)$ and $(\partial/\partial y)u(x,y)$ in the operator $L^1_{(2.2)}$; let these be some functions $u_0(x,y)$ and $v_0(x,y)$. In this case the transport equation takes the form

$$Lu(x,y) \equiv \left\{ \varepsilon \frac{\partial^2}{\partial y^2} - u_0(x,y)\frac{\partial}{\partial x} - v_0(x,y)\frac{\partial}{\partial y} \right\} u(x,y) = 0, \quad (x,y) \in G. \tag{3.1}$$

The function $u_0(x,y)$ outside of an $m\varepsilon$–neighbourhood of the set S^0 satisfies the condition [1] [1]

$$u_0(x,y) \geq m_0, \qquad \text{for} \quad (x,y) \in \overline{G} \quad \text{and} \quad r\left((x,y), S^0\right) \geq m\varepsilon, \tag{3.2a}$$

and also

$$u_0(x,y) > 0, \qquad \text{for} \quad (x,y) \in \overline{G} \quad \text{and} \quad y > 0, \tag{3.2b}$$

where $r\left((x,y), S^0\right)$ is the distance from the point (x,y) to the set S^0. By virtue of condition (3.2b) the operator $L_{(3.1)}$ is monotone [4].

For the function $v_0(x,y)$ the following estimate [1] is valid:

$$0 \leq v_0(x,y) \leq M\varepsilon^{1/2}, \quad (x,y) \in \overline{G}. \tag{3.2c}$$

This means that the product $\varepsilon^{-1/2}v_0(x,y)$ (i.e., the normalized component) is of order $O(1)$, that is, bounded ε–uniformly. Thus, in virtue of bounds (3.2) the singular part of the solution of (3.1) behaves similarly to the singular part of the singularly perturbed heat equation

$$Lu(x,y) \equiv \left\{ \varepsilon \frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial x} \right\} u(x,y) = 0. \tag{3.3}$$

[1] Here and below we denote by M (or m) sufficiently large (small) positive constants which do not depend on the value of the parameter ε and on the discretization parameters.

In the case of a boundary value problem for the singularly perturbed equation (3.3), special finite difference schemes on piecewise uniform meshes that converge ε–uniformly were developed and studied (see, e.g., [8, 9]). We shall use such meshes in the construction of ε–uniformly convergent schemes for problem (2.2), (2.7), (2.1).

To solve the boundary value problem (2.2), (2.7), (2.1) numerically, we use a classical finite difference scheme. At first we introduce the rectangular grid on the set $\overline{G}$:

$$\overline{G}_h = \overline{\omega}_1 \times \overline{\omega}_2, \tag{3.4}$$

where $\overline{\omega}_1$ and $\overline{\omega}_2$ are meshes on the segments $[d_1, d_2]$ and $[0, d_0]$, respectively; $\overline{\omega}_1 = \{x^i,\ i = 0, ..., N_1, x^0 = d_1, x^{N_1} = d_2\}$, $\overline{\omega}_2 = \{y^j,\ j = 0, ..., N_2, y^0 = 0, y^{N_2} = d_0\}$; $N_1 + 1$ and $N_2 + 1$ are the number of nodes in the meshes $\overline{\omega}_1$ and $\overline{\omega}_2$. Define $h_1^i = x^{i+1} - x^i$, $x^i, x^{i+1} \in \overline{\omega}_1$, $h_2^j = y^{j+1} - y^j$, $y^j, y^{j+1} \in \overline{\omega}_2$, $h_1 = \max_i h_1^i$, $h_2 = \max_j h_2^j$, $h = \max\,[\, h_1, h_2\,]$. We assume that $h \le MN^{-1}$, where $N = \min\,[\, N_1, N_2\,]$.

We approximate the boundary value problem by the finite difference scheme

$$\begin{aligned} \Lambda^1 \Big(U^h(x,y)\Big) \ &\equiv\ \varepsilon\delta_{\overline{y}\widehat{y}}\, u^h(x,y) - u^h(x,y)\delta_{\overline{x}}\, u^h(x,y) - \\ &\qquad -v^h(x,y)\delta_{\overline{y}}\, u^h(x,y) = 0, \quad (x,y) \in G_h, \end{aligned} \tag{3.5a}$$

$$\begin{aligned} \Lambda_1^2 U^h(x,y) \ &\equiv\ \delta_{\overline{x}}\, u^h(x,y) + \delta_{\overline{y}}\, v^h(x,y) = 0, \quad (x,y) \in G_h^0, \quad x > d_1, \\ \Lambda_2^2 U^h(x,y) \ &\equiv\ \delta_x\, u^h(x,y) + \delta_{\overline{y}}\, v^h(x,y) = 0, \quad (x,y) \in S_{1h}; \end{aligned} \tag{3.5b}$$

$$u^h(x,y) \ =\ \varphi(x,y), \qquad (x,y) \in S_h, \tag{3.5c}$$

$$v^h(x,y) \ =\ \psi(x,y), \qquad (x,y) \in S_h^0. \tag{3.5d}$$

Here $\delta_{\overline{y}\widehat{y}}\, z(x,y)$ and $\delta_x\, z(x,y), ..., \delta_{\overline{y}}\, z(x,y)$ are the second and first (forward and backward) difference derivatives (the bar denotes the backward difference):

$$\begin{aligned} &\delta_{\overline{y}\widehat{y}}\, z(x,y) = 2\left(h_2^{j-1} + h_2^j\right)^{-1} \left(\delta_y\, z(x,y) - \delta_{\overline{y}}\, z(x,y)\right), \\ &\delta_x\, z(x,y) = \left(h_1^i\right)^{-1} \Big(z(x^{i+1}, y) - z(x,y)\Big), \ \ldots\ , \\ &\delta_{\overline{y}}\, z(x,y) = \left(h_2^{j-1}\right)^{-1} \Big(z(x,y) - z(x, y^{j-1})\Big), \qquad (x,y) = (x^i, y^j). \end{aligned}$$

The difference scheme (3.5), (3.4) approximates problem (2.2), (2.1).

In that case when the “coefficients” multiplying the differences $\delta_{\overline{x}}$ and $\delta_{\overline{y}}$ in the operator Λ^1 are known (let these be the functions $u_0^h(x,y)$ and $v_0^h(x,y)$), and they satisfy the condition

$$u_0^h(x,y),\ v_0^h(x,y) \ge 0, \quad (x,y) \in \overline{G}_h,$$

the operator Λ^1 is monotone [4].

Let us introduce a piecewise uniform mesh, which is refined in a neighbourhood of the set S^0. On the set $\overline{G}$, we consider the grid

$$\overline{G}_h^* = \overline{\omega}_1 \times \overline{\omega}_2^*, \tag{3.6}$$

where $\overline{\omega}_1$ is a uniform mesh on $[d_1, d_2]$, $\overline{\omega}_2^* = \overline{\omega}_2^*(\sigma)$ is a special piecewise uniform mesh depending on the parameter σ and on the value N_2. The mesh $\overline{\omega}_2^*$ is constructed as follows. We divide the segment $[0, d_0]$ in two parts $[0, \sigma]$ and $[\sigma, d_0]$. The step-size of the mesh $\overline{\omega}_2^*$ is constant on the segments $[0, \sigma]$ and $[\sigma, d_0]$, and equal to $h_2^{(1)} = 2\sigma N_2^{-1}$ and $h_2^{(2)} = 2(d_0 - \sigma)N_2^{-1}$, respectively. The value of σ is defined by the relation

$$\sigma = \min\left[\, 2^{-1} d_0,\ m\varepsilon \ln N_2 \,\right],$$

where m is an arbitrary positive number.

In the case of the boundary value problem (2.2), (2.7), (2.1), it is required to study whether the solutions of the finite difference scheme (3.5), (3.6) converge to the exact solution. We mention certain difficulties that arise in the analysis of convergence.

Note that the difference scheme (3.5), (3.6), as well as the boundary value problem (2.2), (2.1), is nonlinear. To find an approximate solution of this scheme, we must construct a proper iterative method. It is of interest to investigate the influence of the parameter ε upon the number of iterations in the iterative numerical method, required for its convergence.

In the case of ε–uniformly convergent difference schemes for linear singular perturbation problems, techniques are well developed to determine numerically the parameters involved in the error bounds (orders of convergence and error constants for fixed values of ε and ε–uniformly), see, e.g., [14]. In these techniques, ε–uniform convergence is ascertained due to theoretical investigations. Formally the above mentioned techniques are inapplicable to problem (2.2), (2.7), (2.1) because the ε–uniform convergence of the finite difference scheme (3.5), (3.6) is not proved. Nevertheless, the results of such investigations of error bounds seem to be interesting from a practical viewpoint.

The pointwise comparison of the exact solutions of problem (2.2), (2.7), (2.1) with the solutions of the finite difference scheme (3.5), (3.6) could give us the most complete knowledge about the behaviour of the error bounds. To find the exact solutions of Prandtl's problem, we shall use the Blasius solution of problem (2.6). Note that the numerical solution of the Blasius problem yields its own additional errors. As for scheme (3.5), (3.6), it is of great interest to study errors for computation of which we use the "exact" solutions of the Prandtl problem obtained on the basis of the discrete solutions of Blasius' problem.

4 Iterative difference scheme for the Prandtl problem

Note that (2.2a) is a parabolic equation in which the variable x plays the role of time. The problem (3.5), (3.4) is solved on levels with respect to the variable $x^i \in \overline{\omega}_1$. To find the discrete solution at the level $x^{i_0} > d_1$, we use an iterative method.

In order to define the iterative difference scheme we must specify the boundary function $\varphi(x,y)$, $(x,y) \in S_h$ $(\psi(x,y)=0,\ (x,y) \in S_h^0)$. The function $\varphi(x,y)$ has no analytical representation. Instead of the function $\varphi(x,y)$, we use a function $\varphi^h(x,y)$ which can be found by using the grid solution of the Blasius problem.

Let us describe an iterative process used in the computation of the discrete solution at the level x^{i_0+1} for $x^{i_0} > d_1$. Assume that the solution of the grid problem (or its approximation) is known for $x = x^{i_0}$. The function $U^h(x,y)$ for $x = x^{i_0+1}$, $y \in \overline{\omega}_2$ is the solution of the nonlinear system of algebraic equations. To compute a new iteration for the component $u^h_{k+1}(x,y)$, $x = x^{i_0+1}$, we use (3.5a) in which we replace the coefficients multiplying the discrete derivatives $\delta_{\overline{x}} u^h_{k+1}$ and $\delta_{\overline{y}} u^h_{k+1}$ by the known components u^h_k and v^h_k from the previous iteration. The component $v^h_{k+1}(x,y)$, $x = x^{i_0+1}$ is computed from (3.5b) by using the known component u^h_{k+1}. We continue these iterations until the difference between the functions $u^h_k(x,y)$, $\varepsilon^{-1/2} v^h_k(x,y)$ for $x = x^{i_0+1}$, $y \in \overline{\omega}_2$ at the neighboring iterations becomes less than some prescribed sufficiently small value $\delta > 0$, which defines the desirable accuracy of the iterative solution. As an initial guess, namely, for the function $U^h_0(x,y)$, $x = x^{i_0+1}$, we use the known solution at the level $x = x^{i_0}$.

For $x = x^{i_0} = x^0 = d^1$, to compute the grid solution at $x = x^{i_0+1}$ we use the above-described iteration process in which we choose, as an initial guess $U^h_0(x,y)$, $x = x^{i_0+1}$, the function $U^h_0(x,y) = \Big(u^h_0(x,y) = \varphi^h(x,y),\ v^h_0(x,y) = \psi(x,y) = 0\Big)$, $x = x^1$, $y \in \overline{\omega}_2$.

The function $u^h(x,y)$ at the level $x = x^0 = d^1$ is known according to the problem formulation; the function $v^h(x,y)$ is computed from (3.5b).

Thus, we come to the following difference scheme

$$\Lambda^1\left(u^h_k(x,y);\ u^h_{k-1}(x,y),\ v^h_{k-1}(x,y)\right) \equiv \varepsilon\delta_{\overline{y}\widehat{y}} u^h_k(x,y) - u^h_{k-1}(x,y)\delta_{\overline{x}} u^h_k(x,y) -$$
$$-v^h_{k-1}(x,y)\delta_{\overline{y}} u^h_k(x,y) = 0, \quad y \in \omega_2,$$

$$\Lambda^2_1\Big(v^h_k(x,y); u^h_k(x,y), u^h_{K(x^{i-1})}(x^{i-1},y)\Big) \equiv (x^i - x^{i-1})^{-1}\Big[u^h_k(x,y) - u^h_{K(x^{i-1})}(x^{i-1},y)\Big] +$$
$$+\delta_{\overline{y}} v^h_k(x,y) = 0, \quad y \in \overline{\omega}_2, \quad y \neq 0,$$

$$u^h_k(x,y) = \varphi^h(x,y), \quad y = 0,\ d_0; \quad v^h_k(x,y) = 0, \quad y = 0;$$

$$
\begin{aligned}
&u_0^h(x,y) = \begin{cases} u^h_{K(x^{i-1})}(x^{i-1},y), & x^i \ge x^2, \\ \varphi^h(x=d_1,y), & x^i = x^1, \quad y \in \omega_2; \end{cases} \\
&v_0^h(x,y) = \begin{cases} v^h_{K(x^{i-1})}(x^{i-1},y), & x^i \ge x^2, \\ 0, \quad x^i = x^1, & y \in \overline{\omega}_2, \quad y \ne 0; \end{cases}
\end{aligned}
\tag{4.1}
$$

$$
\max_{y\in\overline{\omega}_2} \left| u^h_K(x,y) - u^h_{K-1}(x,y) \right|, \ \varepsilon^{-1/2} \max_{y\in\overline{\omega}_2} \left| v^h_K(x,y) - v^h_{K-1}(x,y) \right| \le \delta,
$$

$$
\max_{k<K} \left[\max_{y\in\overline{\omega}_2} \left| u^h_k(x,y) - u^h_{k-1}(x,y) \right|, \ \varepsilon^{-1/2} \max_{y\in\overline{\omega}_2} \left| v^h_k(x,y) - v^h_{k-1}(x,y) \right| \ \right] > \delta,
$$

$$
\text{for} \quad x = x^i, \quad i = 1, \dots, N_1, \quad k = 1, \dots, K, \quad K = K(x^i);
$$

$$
\begin{aligned}
\Lambda_2^2 \left(v^h(x,y);\, u^h_{K(x^1)}(x^1,y) \right) \ &\equiv \ (x^1 - x^0)^{-1} \left[u^h_{K(x^1)}(x,y) - \varphi^h(x,y) \right] + \\
&+ \delta_{\overline{y}}\, v^h(x,y) = 0, \quad y \in \overline{\omega}_2, \quad y \ne 0,
\end{aligned}
$$

$$
\text{for} \quad x = x^0 = d_1\,.
$$

The difference scheme (4.1), (3.6) permits us to compute the function $U^h(x,y) = \big(u^h(x,y), v^h(x,y)\big)$, $(x,y) \in \overline{G}_h$, namely, the components $u^h_{K(x^i)}(x,y)$, $v^h_{K(x^i)}(x,y)$ for $x^i \ge x^1$, $y \in \overline{\omega}_2$ and the function $v^h(x,y)$ for $x^i = x^0 = d_1$, $y \in \overline{\omega}_2$. The function $U^h(x,y)$, $(x,y) \in \overline{G}_h$ which satisfies (4.1) is called the solution of the iterative difference scheme (4.1), (3.6).

5 Approximation of the self-similar solution to the Prandtl problem by using the Blasius equation

In the case of scheme (4.1), (3.6), to analyze the approximation error for the solutions of problem (2.2), (2.7), (2.1) and their derivatives, we use the self–similar solution (2.5) defined by the solution of the Blasius' problem (2.6).

For the boundary value problem (2.6) we must construct a finite difference scheme that allows us to approximate both the Blasius solution and its derivatives on the semi-axis $\eta \ge 0$. It is required to find "constructive" difference schemes, i.e., difference schemes on meshes with a finite number of nodes.

We approximate problem (2.6) by the following differential problem on a finite interval. Let $f_\star(\eta)$, $\eta \in [0, L]$, where the length L of the interval is sufficiently large, be the solution of the boundary value problem

$$
\begin{aligned}
&L\left(f_\star(\eta)\right) \equiv f_\star'''(\eta) + f_\star(\eta) f_\star''(\eta) = 0, \quad \eta \in (0, L), \\
&f_\star(0) = f_\star'(0) = 0, \qquad f_\star'(L) = 1.
\end{aligned}
\tag{5.1a}
$$

We complete a definition of the function $f_\star(\eta)$ on the infinite interval (L,∞) by setting

$$f_\star(\eta) = f_\star(L) + (\eta - L), \quad \text{for all} \quad \eta > L. \tag{5.1b}$$

The continuous problem (5.1) is approximated by a discrete problem. For this we introduce a uniform mesh on the interval $[0, L]$ as follows:

$$\overline{\omega}_0 = \left\{\eta^i = ih,\ i = 0, 1, \ldots, N;\ \ \eta^0 = 0,\ \eta^N = L\right\} \tag{5.2}$$

with step-size $h = LN^{-1}$, where $N+1$ is the number of nodes in the mesh $\overline{\omega}_0$. Assume $L = \ln N$. On the mesh $\overline{\omega}_0$, we approximate problem (5.1a) by the grid problem

$$\begin{aligned} &\Lambda\left(f^h(\eta)\right) \equiv \delta_{\eta\overline{\eta}\,\overline{\eta}}\, f^h(\eta) + f^h(\eta)\delta_{\eta\overline{\eta}}\, f^h(\eta) = 0, \quad \eta \in \overline{\omega}_0, \quad \eta \neq \eta^0, \eta^1, \eta^N, \\ &f^h(0) = \delta_\eta\, f^h(0) = 0, \qquad \delta_{\overline{\eta}}\, f^h(L) = 1. \end{aligned} \tag{5.3a}$$

Here $\delta_{\eta\overline{\eta}}\, z(\eta)$ and $\delta_{\eta\overline{\eta}\,\overline{\eta}}\, z(\eta)$ are the second (central) and third difference derivatives:

$$\delta_{\eta\overline{\eta}}\, z(\eta) = h^{-1}(\delta_\eta\, z(\eta) - \delta_{\overline{\eta}}\, z(\eta)), \quad \delta_{\eta\overline{\eta}\,\overline{\eta}}\, z(\eta) = h^{-1}\left(\delta_{\eta\overline{\eta}}\, z(\eta) - \delta_{\eta\overline{\eta}}\, z(\eta^{i-1})\right), \quad \eta = \eta^i.$$

The function $f^h(\eta)$ on the interval (L,∞) is defined by

$$f^h(\eta) = f^h(L) + (\eta - L), \quad \eta \in (L, \infty). \tag{5.3b}$$

The equations (5.3) allows us to find the function $f^h(\eta)$ for $\eta \in \overline{\omega}_0$ and $\eta \in (L,\infty)$. To determine the components of the solution and their derivatives for the Prandtl problem, we need derivatives of the function $f^h(\eta)$. Let $\delta^k_\eta\, f^h(\eta) = \delta_\eta\left(\delta^{k-1}_\eta\, f^h(\eta)\right)$, $\eta \in \overline{\omega}_0$, $\eta \leq \eta^{N-k}$, $k \geq 1$, be the k-th difference derivatives of $f^h(\eta)$ on $\overline{\omega}_0$. Assume $\delta^k_\eta\, f^h(\eta) = 1$ for $k = 1$, $\eta = \eta^N$ and $\delta^k_\eta\, f^h(\eta) = 0$ for $k \geq 2$, $\eta \in \overline{\omega}_0$, $\eta^{N-k+1} \leq \eta \leq \eta^N$. By $\overline{f}^{\,h(k)}(\eta)$, $\eta \in [0, L]$, we denote the linear interpolant constructed from the values of the functions $\delta^k_\eta\, f^h(\eta)$, $\eta \in \overline{\omega}_0$, $k \geq 0$; $\delta^0_\eta\, f^h(\eta) = f^h(\eta)$. The function $\overline{f}^{\,h(k)}(\eta)$ is extended to the interval (L,∞) by the definitions: $\overline{f}^{\,h(k)}(\eta) = f^h(\eta)$ for $k = 0$, $\overline{f}^{\,h(k)}(\eta) = 1$ for $k = 1$, $\overline{f}^{\,h(k)}(\eta) = 0$ for $k \geq 2$, $\eta \in (L,\infty)$. We shall call the function $\overline{f}^{\,h}(\eta) = \overline{f}^{\,h(k=0)}(\eta)$, $\eta \in [0,\infty)$, defined in such a way, the solution of problem (5.3), (5.2), and the functions $\overline{f}^{\,h(k)}(\eta)$, $k \geq 1$, the derivatives (of order k) from the solution of problem (5.3), (5.2).

The problem (5.3), (5.2) is nonlinear. Let us give an iterative difference scheme that allows one to find the approximate solution of problem (5.3), (5.2).

On the mesh $\overline{\omega}_{0(5.2)}$, we find the function $f_R^h(\eta)$ by solving successively the problems

$$\Lambda\left(f_r^h(\eta), f_{r-1}^h(\eta)\right) \equiv \delta_{\eta\overline{\eta}\,\overline{\eta}}\, f_r^h(\eta) + f_{r-1}^h(\eta)\delta_{\eta\overline{\eta}}\, f_r^h(\eta) = 0, \quad \eta \in \overline{\omega}_0, \quad \eta \neq \eta^0, \eta^1, \eta^N, \tag{5.4a}$$

$$f_r^h(0) = \delta_\eta\, f_r^h(0) = 0, \qquad \delta_{\overline{\eta}}\, f_r^h(L) = 1, \qquad r = 1, \dots, R,$$

where $f_0^h(\eta) = \eta$, $\eta \in \overline{\omega}_0$, R is a sufficiently large given number. For $\eta \in (L, \infty)$ we define the function $f_R^h(\eta)$ by setting

$$f_R^h(\eta) = f_R^h(L) + (\eta - L), \quad \eta \in (L, \infty). \tag{5.4b}$$

The problem (5.4a), (5.2) is linear with respect to the function $f_r^h(\eta)$, $\eta \in \overline{\omega}$.

From the values of the function $f_R^h(\eta)$, similarly to the function $\overline{f}^{h(k)}(\eta)$, $\eta \in [0, \infty)$, we construct the function $\overline{f}_*^{h(k)}(\eta) = \overline{f}_R^{h(k)}(\eta)$, $\eta \in [0, \infty)$, $k \geq 0$. We shall call the function $\overline{f}_*^h(\eta) = \overline{f}_*^{h(k)}(\eta)$, $\eta \in [0, \infty)$ for $k = 0$ the solution of problem (5.4), (5.2), and the functions $\overline{f}_*^{h(k)}(\eta)$, $k \geq 1$ the derivatives of the problem solution.

Note that the derivatives of the function $\overline{f}_*^{h(k)}(\eta)$ have a discontinuity of the first kind at $\eta = \eta^{N-k}$, $k \geq 2$.

For

$$L = L(N) = M_1 \ln N, \qquad R = R(N) = M_2 \ln N, \tag{5.5}$$

where M_1, M_2 are sufficiently large numbers, the solution of problem (5.4), (5.2), (5.5) together with its derivatives up to order K (where K is fixed) converges, as $N \to \infty$, to the solution of problem (2.6) with the corresponding derivatives.

The theoretical and numerical analyses result in the estimates

$$\begin{gathered} \left|f'(\eta) - \overline{f}_*^{h(1)}(\eta)\right|, \quad \left|\eta f'(\eta) - f(\eta) - \left(\eta \overline{f}_*^{h(1)}(\eta) - \overline{f}_*^h(\eta)\right)\right|, \\ \left|\eta^k\left(f''(\eta) - \overline{f}_*^{h(2)}(\eta)\right)\right| \leq M N^{-\nu}, \quad \eta \in [0, \infty), \quad k = 0, 1, 2, \end{gathered} \tag{5.6}$$

where ν is some number $(0 < \nu < 1)$. It follows from estimates (5.6) that the difference scheme (5.4), (5.2), (5.5) in the case of Prandtl's problem (2.2), (2.7), (2.1) allows us to find the normalized components with the normalized (i.e., ε–uniformly bounded) derivatives, namely, $u(x, y)$, $\varepsilon^{-1/2}v(x, y)$, $(\partial/\partial x)u(x, y)$, $\varepsilon^{1/2}(\partial/\partial y)u(x, y)$, $\varepsilon^{-1/2}(\partial/\partial x)v(x, y)$, $(\partial/\partial y)v(x, y)$, $(x, y) \in \overline{G}_{(2.1)}$, with guaranteed (controlled) ε–uniform accuracy.

Thus, in the case of Prandtl's problem (2.2), (2.7), (2.1) the components of its solution with partial derivatives with respect to x and y, which are defined by using the solutions of difference scheme (5.4), (5.2), (5.5) for Blasius' problem (2.6), permit us to form the boundary conditions (with controlled ε–uniform accuracy) in the grid boundary value problem (4.1), (3.6). Moreover, the solutions of scheme (5.4), (5.2), (5.5) allow us to analyze ε–uniform convergence of special finite difference schemes, in particular, the schemes (4.1), (3.6) and (3.5), (3.6).

By numerical experiments carried out according to the above techniques, in [16] we show ε–uniform convergence of the finite difference schemes (4.1), (3.6) and (3.5), (3.6); also therein we find the parameters of ε–uniform error bounds for the numerical approximations to the solutions and derivatives for the Prandtl problem (2.2), (2.7), (2.1).

6 On fitted operator schemes for the Prandtl problem

As was shown in [8, 17] (see also [7, 9]) for a singularly perturbed parabolic equation with parabolic boundary layers, there do not exist fitted operator schemes on uniform meshes that converge ε–uniformly. Note that the coefficients of the terms with the first-order derivatives in time and the second-order derivatives in space did not vanish in the equation that was considered in [8, 17]. As for the case of the Prandtl problem, the coefficient multiplying the first derivative with respect to the variable x, which plays the role of the time variable, vanishes on the domain boundary for $y = 0$.

Unlike the problem that was studied in [8], where the boundary conditions did not obey any restriction, besides the requirement of sufficient smoothness, the problem (2.2), (2.7), (2.1) is essentially simpler. Its solution is defined only by the one parameter u_∞. In [18] an ε–uniform fitted operator method was constructed for a linear parabolic equation with a discontinuous initial condition in the presence of a parabolic (transient) layer. Such fitted operator schemes have been successfully constructed because all the variety of singular components of the solution (their main parts) is defined, up to some multiplier, only by one function. In view of the comparatively simple (depending on the one parameter u_∞ only) representation of the solution for the Prandtl problem, it is not obvious that for this problem there are no fitted schemes which converge ε–uniformly. So it is of interest to establish whether such fitted schemes on uniform meshes do exist for the Prandtl problem.

We shall try to construct a fitted operator scheme starting from (3.5a) under the assumption that the function $v^h(x,y)$ is known, and also $v^h(x,y) = v(x,y)$. Let us consider the fitted operator scheme in such a form:

$$\begin{aligned} \Lambda^{1*}\left(u^h(x,y)\right) &\equiv \varepsilon\gamma_{(2)}\delta_{y\overline{y}}u^h(x,y) - u^h(x,y)\delta_{\overline{x}}u^h(x,y) - \\ &\quad -\gamma_{(1)}v(x,y)\delta_{\overline{y}}u^h(x,y) = 0, \qquad (x,y) \in G_h, \qquad (6.1a) \\ u^h(x,y) &= \varphi(x,y), \qquad (x,y) \in S_h, \end{aligned}$$

where

$$\overline{G}_h \tag{6.2}$$

is a grid that is uniform with respect to both the variables, with steps h_1 and h_2 in x and y respectively; the parameters

$$\gamma_{(i)} = \gamma_{(i)}(x, y; \varepsilon, h_1, h_2), \quad i = 1,\ 2 \tag{6.1b}$$

are fitting coefficients.

The derivatives of the function $u(x, y)$ can be represented as follows:

$$\begin{aligned}
&\frac{\partial}{\partial x} u(x, y) = -2^{-1} u_\infty x^{-1} f''(\eta)\eta, \\
&\frac{\partial^2}{\partial x^2} u(x, y) = 4^{-1} u_\infty x^{-2} \left[f'''(\eta)\eta^2 + 3 f''(\eta)\eta \right], \\
&\frac{\partial^{k_2}}{\partial y^{k_2}} u(x, y) = 2^{-k_2/2} u_\infty^{1+k_2/2} \varepsilon^{-k_2/2} x^{-k_2/2} f^{(k_2+1)}(\eta),
\end{aligned} \tag{6.3}$$

$$k_2 \le 4, \qquad \eta = \eta_{(2.5)}(x, y; \varepsilon),$$

and for the function $v(x, y)$ we have the representation (2.5). Taking into account the last representations in (6.3) and also the estimates for the derivatives of the function $f(\eta)$, we find

$$\begin{aligned}
&\left| u(x, y) \left(\frac{\partial}{\partial x} - \delta_{\overline{x}} \right) u(x, y) \right| \le M h_1, \quad (x, y) \in G_h; \\
&\varepsilon \left(\delta_{y\overline{y}} - \frac{\partial^2}{\partial y^2} \right) u(x, y) \ge m h_2^2 \left(\varepsilon^{1/2} + h_2 \right)^{-2}, \\
&m\eta^2 h_2 \left(\varepsilon^{1/2} + h_2 \right)^{-1} \le -v(x, y) \left(\delta_{\overline{y}} - \frac{\partial}{\partial y} \right) u(x, y) \le M\eta^2 h_2 \left(\varepsilon^{1/2} + h_2 \right)^{-1},
\end{aligned} \tag{6.4}$$

$$(x, y) \in G_h, \qquad \eta \le M_0, \qquad \eta = \eta(x, y; \varepsilon).$$

From estimates (6.4) it follows that under the condition

$$\gamma_{(1)} = \gamma_{(2)} = 1 \tag{6.5}$$

the error in approximating the solution of the boundary value problem is of order 1, for the terms of the equation which contain the y–derivatives, when $\eta \le M_0$ and the step-size h_2 is of order $\varepsilon^{1/2}$. The error for the term involving the derivatives in x is ε–uniformly small for small values of h_1 on the whole domain $\overline{G}$.

Note that under condition (6.5) and for values of η, somewhat less than M_0, namely, for $\eta \le m_0$, the main term of the truncation error is generated by errors caused by the numerical approximation of the second derivatives

$$48^{-1}u_\infty^3\varepsilon^{-1}h_2^2x^{-2}\min_{\eta_1} f^{(5)}(\eta_1) \le \varepsilon\left(\delta_{y\,\overline{y}} - \frac{\partial^2}{\partial y^2}\right)u(x,y) \le$$

$$\le 48^{-1}u_\infty^3\varepsilon^{-1}h_2^2x^{-2}\max_{\eta_2} f^{(5)}(\eta_2), \quad \eta(x,y) \le m_0, \tag{6.6}$$

where $\eta_1,\ \eta_2 \in \left[\eta(x,y^{j-1}),\ \eta(x,y^{j+1})\right], \quad (x,y^j) \in G_h$.

Taking into account estimates (6.4), (6.6), we establish, similarly to considerations in [8], that there are no fitted operator schemes (6.1), (6.2) which converge ε–uniformly in the case of the Prandtl problem (2.2), (2.7), (2.1).

Theorem 1. *In the class of finite difference schemes (6.1), (6.2) there do not exist schemes, whose solutions converge as $N \to \infty$ to the solution of the boundary value problem (2.2), (2.7), (2.1) ε–uniformly.*

Thus, from here it follows that to construct ε–uniformly convergent schemes in the case of the Prandtl problem (2.2), (2.7), (2.1), the use of meshes condensing in the neighbourhood of the parabolic boundary layer is necessary.

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