QUIVER REPRESENTATIONS IN ABELIAN CATEGORIES

SERGEY MOZGOVOY

Abstract. We introduce the notion of (twisted) quiver representations in abelian categories and study the category of such representations. We construct standard resolutions and coresolutions of quiver representations and study basic homological properties of the category of representations. These results are applied in the case of parabolic vector bundles and framed coherent sheaves.

1. Introduction

Let $\Phi\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Its mapping cylinder $\mathcal{C}_\Phi$ is the category with objects consisting of triples (A, B, s), where

$$A \in \mathrm{Ob}(\mathcal{A}), \qquad B \in \mathrm{Ob}(\mathcal{B}), \qquad s \in \mathrm{Hom}_{\mathcal{B}}(B, \Phi A).$$

This category is abelian if Φ is left exact. For example, let $\mathcal{A} = \mathrm{Coh}\, Z$ be the category of coherent sheaves on an algebraic variety Z over a field $\Bbbk$, $\mathcal{B} = \mathrm{Vect}$ be the category of vector spaces over $\Bbbk$ and $\Phi = \Gamma(Z, -)$ be the section functor which is left exact. An object of the mapping cylinder is a triple (E, V, ϕ), where

$$E \in \mathrm{Coh}\, Z, \qquad V \in \mathrm{Vect}, \qquad s\colon V \to \Gamma(Z, E).$$

The category $\mathrm{Coh}_{\mathrm{f}}\, Z = \mathcal{C}_\Phi$ is the category of framed coherent sheaves over Z [15]. One can show [13] that if $\Phi\colon \mathcal{A} \to \mathrm{Vect}$ is exact and $\mathcal{A}$ is hereditary, then $\mathcal{C}_\Phi$ is also hereditary.

The goal of this note is to develop some basic homological algebra of mapping cylinders and, more generally, of categories of (twisted) quiver representations in abelian categories. Let $Q = (Q_0, Q_1, s, t)$ be a quiver and Φ be a Q^{op}-diagram of abelian categories (see §3) consisting of abelian categories Φ_i for all $i \in Q_0$ and left exact functors $a^* = \Phi_a\colon \Phi_j \to \Phi_i$ for all arrows $a\colon i \to j$ in Q. Define the category $\mathrm{Rep}(\Phi)$ of (twisted) Q-representations to have objects X consisting of the data: $X_i \in \mathrm{Ob}(\Phi_i)$ for all $i \in Q_0$ and $X_a\colon X_i \to a^* X_j$ for all arrows $a\colon i \to j$ in Q. We will see in Theorem 2.8 that $\mathrm{Rep}(\Phi)$ is an abelian category.

A different way of thinking about $\mathrm{Rep}(\Phi)$ is to consider Φ as a functor $\mathcal{P}^{\mathrm{op}} \to \mathrm{Cat}$, where $\mathcal{P}$ is the path category of Q (see §3) and Cat is the category of small categories. It induces a fibered category $\mathcal{F} \to \mathcal{P}$. Then the category $\mathrm{Rep}(\Phi)$ can be identified with the category of sections $\mathbf{\Gamma}(\mathcal{F}/\mathcal{P})$ (see Remark 2.7).

Apart from the usual Q-representations in the category of vector spaces, an important example of Q-representations was studied by Gothen and King [5], where all categories Φ_i are equal to the category $\mathrm{Sh}(Z, \mathcal{O}_Z)$ of $\mathcal{O}_Z$-modules over a ringed space $(Z, \mathcal{O}_Z)$ and the functors Φ_a are tensor products with locally free sheaves. This covers a lot of interesting situations, like Higgs bundles, Bradlow pairs and chains of vector bundles (see e.g. [3]), although not the category of framed objects introduced above or the category of parabolic bundles. The case of a one loop quiver and an auto-equivalence was studied in [14].

In this paper we describe homological properties of the category $\mathrm{Rep}(\Phi)$ in the spirit of [5]. In Theorems 3.3 and 3.4 we construct standard resolutions and coresolutions in $\mathrm{Rep}(\Phi)$ which are analogues of standard resolutions for usual quiver representations. In Theorem 4.1 we prove that $\mathrm{Rep}(\Phi)$ has enough injective objects. The main results of the paper are

Theorem 1.1. *Let Φ be a Q^{op}-diagram of Grothendieck categories such that the functor $\Phi_a\colon \Phi_j \to \Phi_i$ has an exact left adjoint functor $\Psi_a\colon \Phi_i \to \Phi_j$ for every arrow $a\colon i \to j$ in Q. Then, for any $X, Y \in \mathrm{Rep}(\Phi)$, there is a long exact sequence*

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathrm{Rep}(\Phi)}(X,Y) \to \bigoplus_i \mathrm{Hom}_{\Phi_i}(X_i,Y_i) \to \bigoplus_{a:i\to j} \mathrm{Hom}_{\Phi_j}(\Psi_a X_i, Y_j) \\ \to \mathrm{Ext}^1_{\mathrm{Rep}(\Phi)}(X,Y) \to \bigoplus_i \mathrm{Ext}^1_{\Phi_i}(X_i,Y_i) \to \bigoplus_{a:i\to j} \mathrm{Ext}^1_{\Phi_j}(\Psi_a X_i, Y_j) \to \dots \end{aligned}$$

Theorem 1.2. *Let Q be an acyclic quiver and Φ be a Q^{op}-diagram of Grothendieck categories such that the functor $\Phi_a\colon \Phi_j \to \Phi_i$ is exact for every arrow $a\colon i \to j$ in Q. Then, for any $X, Y \in \mathrm{Rep}(\Phi)$, there is a long exact sequence*

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathrm{Rep}(\Phi)}(X,Y) \to \bigoplus_i \mathrm{Hom}_{\Phi_i}(X_i,Y_i) \to \bigoplus_{a:i\to j} \mathrm{Hom}_{\Phi_i}(X_i, \Phi_a Y_j) \\ \to \mathrm{Ext}^1_{\mathrm{Rep}(\Phi)}(X,Y) \to \bigoplus_i \mathrm{Ext}^1_{\Phi_i}(X_i,Y_i) \to \bigoplus_{a:i\to j} \mathrm{Ext}^1_{\Phi_i}(X_i, \Phi_a Y_j) \to \dots \end{aligned}$$

Note that $\mathrm{Ext}^k_{\Phi_i}(X_i, \Phi_a Y_j) \not\simeq \mathrm{Ext}^k_{\Phi_j}(\Psi_a X_i, Y_j)$ in general, although these groups are isomorphic if both Φ_a and Ψ_a are exact (see Remark 4.6). Also note that even if we start with a quiver diagram of small abelian categories (like the category $\mathrm{Coh}\, Z$ for an algebraic variety Z) which can not be Grothendieck as they don't have small colimits, we can consider their Ind-categories which are Grothendieck (in the case of $\mathrm{Coh}\, Z$ we obtain the category $\mathrm{Qcoh}\, Z$ of quasi-coherent sheaves) and extend our diagram to a diagram of Grothendieck categories (see §4). Then we can apply the above results.

The paper is organized as follows. In §2 we introduce diagrams of categories and define their categories of representations. We prove that the category of representations is abelian under appropriate assumptions and we show that the forgetful functor sending a representation to one of its components has left and right adjoint functors. In §3 we restrict our attention to quiver diagrams and we construct standard resolutions and coresolutions of representations. In §4 we prove our main results about long exact sequences for quiver representations. In §5 we discuss several applications of these results, in particular, in the case of parabolic vector bundles and framed coherent sheaves.

2. Diagram representations

Let Cat be the category of small categories. Given a small category $\mathcal{I}$, define an $\mathcal{I}$-diagram (or just a diagram) of categories to be a functor $\Psi\colon \mathcal{I} \to \mathrm{Cat}$. For every object $i \in \mathrm{Ob}(\mathcal{I})$ let $\Psi_i = \Psi(i)$ and for every morphism $a \in \mathcal{I}(i,j) = \mathrm{Hom}_{\mathcal{I}}(i,j)$ let $a_* = \Psi_a = \Psi(a)\colon \Psi_i \to \Psi_j$. We will assume that the categories Ψ_i are abelian and the functors Ψ_a are additive unless otherwise stated. We will say that a diagram Ψ is exact (resp. left exact, right exact) if the functors $\Psi_a\colon \Psi_i \to \Psi_j$ are exact (resp. left exact, right exact) for all $a \in \mathcal{I}(i,j)$.

Definition 2.1. Given a diagram $\Psi\colon \mathcal{I} \to \mathrm{Cat}$, define the category $\overline{\mathrm{Rep}}(\Psi)$ of representations (also called a projective 2-limit of Ψ in a different context [2]) to have objects X consisting of the data:

(1) An object $X_i \in \Psi_i$ for all $i \in \mathrm{Ob}(\mathcal{I})$.
(2) A morphism $X_a\colon a_*(X_i) \to X_j$ in Ψ_j for every morphism $a\colon i \to j$ in $\mathcal{I}$.

subject to the condition:

(3) Given morphisms $a\colon i \to j$, $b\colon j \to k$ in $\mathcal{I}$, we have $X_{ba} = X_b \circ b_*(X_a)$.

A morphism $f: X \to Y$ between two objects in $\overline{\mathrm{Rep}}(\Psi)$ is a tuple $f = (f_i)_{i \in \mathrm{Ob}(\mathcal{I})}$ of morphisms $f_i: X_i \to Y_i$ in Ψ_i such that for every $a \in \mathcal{I}(i,j)$, the following diagram commutes

$$\begin{tikzcd} a_*(X_i) \arrow[r, "X_a"] \arrow[d, "a_*(f_i)"'] & X_j \arrow[d, "f_j"] \\ a_*(Y_i) \arrow[r, "Y_a"] & Y_j \end{tikzcd}$$

Example 2.2. Let $\mathcal{I}$ be the category with $\mathrm{Ob}(\mathcal{I}) = \{1,2\}$ and

$$\mathcal{I}(1,1) = \{\mathrm{Id}_1\}, \quad \mathcal{I}(2,2) = \{\mathrm{Id}_2\}, \quad \mathcal{I}(1,2) = \{a\}, \quad \mathcal{I}(2,1) = \varnothing.$$

Given an algebraic variety Z, let

$$\Psi_1 = \mathrm{Vect}, \qquad \Psi_2 = \mathrm{Coh}\, Z, \qquad \Psi_a: \mathrm{Vect} \to \mathrm{Coh}\, Z,\ V \mapsto V \otimes \mathcal{O}_Z.$$

Then the category $\overline{\mathrm{Rep}}(\Psi)$ consists of triples (V, F, s) with $V \in \mathrm{Vect}$, $F \in \mathrm{Coh}\, Z$ and $s: V \otimes \mathcal{O}_Z \to F$. Note that Ψ_a has a right adjoint functor $\Phi_a: \mathrm{Coh}\, Z \to \mathrm{Vect}$, $F \mapsto \Gamma(Z, F)$. We can consider s as a morphism $V \to \Phi_a(F) = \Gamma(Z, F)$ in Vect.

Definition 2.3. Given a diagram $\Phi: \mathcal{I}^{\mathrm{op}} \to \mathrm{Cat}$, define the category $\mathrm{Rep}(\Phi)$ to have objects X consisting of the data:

(1) An object $X_i \in \Phi_i$ for all $i \in \mathrm{Ob}(\mathcal{I})$.
(2) A morphism $X_a: X_i \to a^* X_j$ in Φ_i for every morphism $a: i \to j$ in $\mathcal{I}$, where $a^* = \Phi_a$.

subject to the condition:

(3) Given morphisms $a: i \to j$, $b: j \to k$ in $\mathcal{I}$, we have $X_{ba} = a^*(X_b) \circ X_a$.

Definition 2.4. We say that a diagram $\Psi: \mathcal{I} \to \mathrm{Cat}$ is left adjoint to a diagram $\Phi: \mathcal{I}^{\mathrm{op}} \to \mathrm{Cat}$ (or that Φ is right adjoint to Ψ) if $\Phi_i = \Psi_i$ for all $i \in \mathrm{Ob}(\mathcal{I})$ and $\Psi_a: \Psi_i \to \Psi_j$ is left adjoint to $\Phi_a: \Phi_j \to \Phi_i$ for all $a \in \mathcal{I}(i,j)$.

Remark 2.5. Let $\Psi: \mathcal{I} \to \mathrm{Cat}$ have a right adjoint diagram $\Phi: \mathcal{I}^{\mathrm{op}} \to \mathrm{Cat}$. Then $\overline{\mathrm{Rep}}(\Psi)$ is equivalent to $\mathrm{Rep}(\Phi)$. Moreover, the diagram Ψ is right exact, the diagram Φ is left exact, $\Psi_a: \Psi_i \to \Psi_j$ preserves coproducts and $\Phi_a: \Phi_j \to \Phi_a$ preserves products for all $a \in \mathcal{I}(i,j)$.

Remark 2.6. Given a diagram $\Psi: \mathcal{I} \to \mathrm{Cat}$, define the opposite diagram $\Psi^{\mathrm{op}}: \mathcal{I} \to \mathrm{Cat}$ with Ψ_i^{op} being the opposite category of Ψ_i and with the opposite functors $\Psi_a^{\mathrm{op}}: \Psi_i^{\mathrm{op}} \to \Psi_j^{\mathrm{op}}$ for $a \in \mathcal{I}(i,j)$. There is a canonical equivalence of categories

$$\mathrm{Rep}(\Psi^{\mathrm{op}}) \simeq \overline{\mathrm{Rep}}(\Psi)^{\mathrm{op}}. \tag{1}$$

This duality implies that the statements about the category $\mathrm{Rep}(\Psi^{\mathrm{op}})$ can be translated to the statements about the category $\overline{\mathrm{Rep}}(\Psi)$ and vice versa.

Remark 2.7. Given a diagram $\Phi: \mathcal{I}^{\mathrm{op}} \to \mathrm{Cat}$, the category $\mathrm{Rep}(\Phi)$ can be interpreted as the category of sections of the associated fibered category $p: \mathcal{F} \to \mathcal{I}$. More precisely, the category $\mathcal{F} = \int \Phi$, called the Grothendieck construction for Φ [6, §VI.8], has objects

$$\mathrm{Ob}(\mathcal{F}) = \coprod_{i \in \mathrm{Ob}(\mathcal{I})} \mathrm{Ob}(\Phi_i)$$

and, for $X_i \in \mathrm{Ob}(\Phi_i)$ and $X_j \in \mathrm{Ob}(\Phi_j)$, morphisms

$$\mathrm{Hom}_{\mathcal{F}}(X_i, X_j) = \coprod_{a \in \mathrm{Hom}_{\mathcal{I}}(i,j)} \mathrm{Hom}_{\Phi_i}(X_i, a^* X_j).$$

The functor $p: \mathcal{F} \to \mathcal{I}$ is defined by

$$\mathrm{Ob}(\Phi_i) \ni X_i \mapsto i \in \mathrm{Ob}(\mathcal{I}), \qquad \mathrm{Hom}_{\Phi_i}(X_i, a^* X_j) \ni X_a \mapsto a \in \mathrm{Hom}_{\mathcal{I}}(i,j).$$

Define the category of sections $\mathbf{\Gamma}(\mathcal{F}/\mathcal{I}) = \mathbf{Hom}_{\mathcal{I}}(\mathcal{I},\mathcal{F})$ whose objects are functors $X{:}\,\mathcal{I} \to \mathcal{F}$ such that $pX = \mathrm{Id}_{\mathcal{I}}$ and morphisms from $X{:}\,\mathcal{I} \to \mathcal{F}$ to $Y{:}\,\mathcal{I} \to \mathcal{F}$ are morphisms of functors $f{:}\,X \to Y$ such that $pf{:}\,pX \to pY$ is an identity morphism of functors. The last condition means that the component $f_i \in \mathrm{Hom}_{\mathcal{F}}(X_i, Y_i)$ of f is contained in $\mathrm{Hom}_{\Phi_i}(X_i, Y_i)$ for every $i \in \mathrm{Ob}(\mathcal{I})$. One can show that the category $\mathbf{\Gamma}(\mathcal{F}/\mathcal{I})$ is equivalent to the category of representations $\mathrm{Rep}(\Phi)$.

Theorem 2.8. *Let $\Phi{:}\,\mathcal{I}^{\mathrm{op}} \to \mathrm{Cat}$ be a left exact diagram. Then the category of representations $\mathrm{Rep}(\Phi)$ is abelian. A sequence of representations $X \to Y \to Z$ is exact in $\mathrm{Rep}(\Phi)$ if and only if the sequence $X_i \to Y_i \to Z_i$ is exact in Φ_i for all $i \in \mathrm{Ob}(\mathcal{I})$.*

Proof. We just have to show that there exist kernels and cokernels in $\mathcal{R} = \mathrm{Rep}(\Phi)$ which are defined componentwise. Let $f{:}\,X \to Y$ be a morphism in $\mathcal{R}$. We construct the kernel of f as follows. For every $i \in \mathrm{Ob}(\mathcal{I})$, let $h_i{:}\,K_i \to X_i$ be the kernel of $f_i{:}\,X_i \to Y_i$. For every arrow $a{:}\,i \to j$, consider the diagram

$$\begin{tikzcd} K_i \arrow[r, "h_i"] \arrow[d, dashed, "K_a"] & X_i \arrow[r, "f_i"] \arrow[d, "X_a"] & Y_i \arrow[d, "Y_a"] \\ a^*K_j \arrow[r, "a^*(h_j)"] & a^*X_j \arrow[r, "a^*(f_j)"] & a^*Y_j \end{tikzcd}$$

Morphism $a^*(h_j)$ is the kernel of $a^*(f_j)$ as a^* is left exact by our assumption. The composition $a^*(f_j)X_a h_i$ is zero, hence there exists a unique dashed arrow K_a making the left square commutative. In this way we obtain an object $K \in \mathcal{R}$ and a morphism $h{:}\,K \to X$ such that $fh = 0$. Let us show that h is the kernel of f. Let $g{:}\,Z \to X$ be a morphism in $\mathcal{R}$ such that $fg = 0$. Then every $g_i{:}\,Z_i \to X_i$ can be uniquely factored as

$$\begin{tikzcd} Z_i \arrow[r, "s_i"] \arrow[rr, bend left, "g_i"] & K_i \arrow[r, "h_i"] & X_i. \end{tikzcd}$$

To see that $s = (s_i)_{i \in \mathrm{Ob}(\mathcal{I})}$ defines a morphism $s{:}\,Z \to K$ such that $g = hs$, we have to verify commutativity of the left square in the diagram

$$\begin{tikzcd} Z_i \arrow[r, "s_i"] \arrow[d, "Z_a"] & K_i \arrow[r, "h_i"] \arrow[d, "K_a"] & X_i \arrow[d, "X_a"] \\ a^*Z_j \arrow[r, "a^*(s_j)"] & a^*K_j \arrow[r, "a^*(h_j)"] & a^*X_j \end{tikzcd}$$

This commutativity follows from the fact that

$$a^*(h_j)K_a s_i = X_a h_i s_i = a^*(h_j)a^*(s_j)Z_a$$

and that $a^*(h_j)$ is a monomorphism (as a^* is left exact). This proves that $h{:}\,K \to X$ is the kernel of $f{:}\,X \to Y$.

Construction of the cokernel is similar, with the only difference that one does not require any exactness properties of a^*. $\square$

Applying duality (1) we obtain

Corollary 2.9. *Let $\Psi{:}\,\mathcal{I} \to \mathrm{Cat}$ be a right exact diagram. Then the category of representations $\overline{\mathrm{Rep}}(\Psi)$ is abelian.*

Remark 2.10. In order to provide a motivation of the construction in the next theorem, let us consider the case of quiver representations in vector spaces. Let $A = \Bbbk Q$ be the path algebra of a quiver Q over a field $\Bbbk$ [1]. One can identify Q-representations over $\Bbbk$ with A-modules. For any vertex $i \in Q_0$, there is an idempotent $e_i \in A$ corresponding to the trivial path at i. The projective A-module Ae_i has a basis consisting of paths that start at i. Given a vector space V and a Q-representation X, the Q-representation $Ae_i \otimes V$ satisfies

$$\mathrm{Hom}_A(Ae_i \otimes V, X) \simeq \mathrm{Hom}_{\Bbbk}(V, X_i),$$

where $X_i = e_i X \in \mathrm{Vect}\,\Bbbk$. This implies that the forgetful functor $\mathrm{Mod}\,A \to \mathrm{Vect}\,\Bbbk$, $X \mapsto X_i$ has a left adjoint $V \mapsto Ae_i \otimes V$.

Theorem 2.11. *Let $\Psi\colon \mathcal{I} \to \mathrm{Cat}$ be a diagram such that the categories Ψ_i have coproducts and the functors $\Psi_a\colon \Psi_i \to \Psi_j$ preserve them for all $a\colon i \to j$ in $\mathcal{I}$. Then, for every $i \in \mathcal{I}$, the forgetful functor*

$$\sigma^*\colon \overline{\mathrm{Rep}}(\Psi) \to \Psi_i, \qquad X \mapsto X_i$$

has a left adjoint functor $\sigma_!\colon \Psi_i \to \overline{\mathrm{Rep}}(\Psi)$.

Proof. Given an object $M \in \Psi_i$, we define an object $Y = \sigma_!(M) \in \mathcal{R} = \overline{\mathrm{Rep}}(\Psi)$ as follows. For any $j \in \mathcal{I}$, define

$$Y_j = \sigma_!(M)_j = \bigoplus_{a \in \mathcal{I}(i,j)} a_* M \in \Psi_j.$$

Let us construct the map $Y_b\colon b_* Y_j \to Y_k$ for any $b \in \mathcal{I}(j,k)$. For any morphisms $a \in \mathcal{I}(i,j)$ consider the canonical embedding

$$b_* a_* M \simeq (ba)_* M \hookrightarrow Y_k.$$

These maps induce

$$Y_b\colon b_* Y_j = b_* \left(\bigoplus_{a \in \mathcal{I}(i,j)} a_* M \right) \simeq \bigoplus_{a \in \mathcal{I}(i,j)} b_* a_* M \to Y_k.$$

In this way we obtain an object $Y \in \mathcal{R}$. For any $X \in \mathcal{R}$ there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{R}}(Y, X) \simeq \mathrm{Hom}_{\Psi_i}(M, X_i) \tag{2}$$

Given $f \in \mathrm{Hom}_{\mathcal{R}}(Y, X)$, we define $g \in \mathrm{Hom}_{\Psi_i}(M, X_i)$ to be the composition $M \hookrightarrow Y_i \xrightarrow{f_i} X_i$ of the map f_i and an embedding corresponding to the identity $\mathrm{Id} \in \mathcal{I}(i,i)$. Conversely, given $g \in \mathrm{Hom}_{\Psi_i}(M, X_i)$, for every $j \in \mathcal{I}$ we construct $f_j\colon Y_j = \bigoplus_{a \in \mathcal{I}(i,j)} a_* M \to X_j$ such that the component $a_* M \to X_j$ is given by $a_* M \xrightarrow{a_*(g)} a_* X_i \xrightarrow{X_a} X_j$. □

Applying duality (1) we obtain

Theorem 2.12. *Let $\Phi\colon \mathcal{I}^{\mathrm{op}} \to \mathrm{Cat}$ be a diagram such that the categories Φ_i have products and the functors $\Phi_a\colon \Phi_j \to \Phi_i$ preserve them for all $a\colon i \to j$ in $\mathcal{I}$. Then for every $i \in \mathcal{I}$ the forgetful functor*

$$\sigma^*\colon \mathrm{Rep}(\Phi) \to \Phi_i, \qquad X \mapsto X_i$$

has a right adjoint functor $\sigma_* = \Phi_i \to \mathrm{Rep}(\Phi)$. Explicitly, given $M \in \Phi_i$, the object $\sigma_*(M)$ is defined by

$$\sigma_*(M)_j = \prod_{a \in \mathcal{I}(j,i)} \Phi_a M \in \Phi_j.$$

Corollary 2.13. *Under the above conditions*

1. *If $M \in \Psi_i$ is projective, then $\sigma_! M \in \overline{\mathrm{Rep}}(\Psi)$ is also projective.*
2. *If $M \in \Phi_i$ is injective, then $\sigma_* M \in \mathrm{Rep}(\Phi)$ is also injective.*

Proof. Let us prove the first statement. If $M \in \Psi_i$ is projective, then

$$\mathrm{Hom}_{\overline{\mathrm{Rep}}(\Psi)}(\sigma_! M, -) \simeq \mathrm{Hom}_{\Psi_i}(M, \sigma^*(-))$$

is an exact functor as a composition of two exact functors σ^* and $\mathrm{Hom}_{\Psi_i}(M, -)$. This implies that $\sigma_! M$ is projective. □

3. Standard resolutions and coresolutions

Recall that a quiver $Q = (Q_0, Q_1, s, t)$ is a directed (multi-) graph. Here Q_0, Q_1 are sets, called the sets of vertices and arrows respectively and $s, t\colon Q_1 \to Q_0$ are maps, called the source and target maps respectively. All quivers are assumed to be finite unless otherwise stated. We denote an arrow $a \in Q_1$ with $i = s(a)$ and $j = t(a)$ as $a\colon i \to j$. Define the category $\mathcal{P} = \mathcal{P}(Q)$ of paths in Q to have the set of objects Q_0 and the set of morphisms $\mathcal{P}(i,j) = \operatorname{Hom}_{\mathcal{P}}(i,j)$ consisting of all directed paths from $i \in Q_0$ to $j \in Q_0$. Define a Q-diagram of categories to be a functor $\Psi\colon \mathcal{P} \to \mathrm{Cat}$. It is uniquely determined by the categories Ψ_i for $i \in Q_0$ and the functors $a_* = \Psi_a\colon \Psi_i \to \Psi_j$ for arrows $a\colon i \to j$ in Q.

Definition 3.1. Given a Q-diagram Ψ, define the category of Q-representations (with respect to Ψ) to be the category $\overline{\operatorname{Rep}}(Q, \Psi) = \overline{\operatorname{Rep}}(\Psi)$. Its objects consist of data:

(1) An object $X_i \in \Psi_i$ for all $i \in Q_0$.
(2) A morphism $X_a\colon a_*(X_i) \to X_j$ in Ψ_j for every arrow $a\colon i \to j$ in Q.

Similarly, define a Q^{op}-diagram to be a diagram $\Phi\colon \mathcal{P}(Q)^{\mathrm{op}} \to \mathrm{Cat}$, which is uniquely determined by the categories Φ_i for $i \in Q_0$ and the functors $a^* = \Phi_a\colon \Phi_j \to \Phi_i$ for arrows $a\colon i \to j$ in Q. In this case we define $\operatorname{Rep}(Q, \Phi) = \operatorname{Rep}(\Phi)$.

Remark 3.2. To give a motivation of the construction of the next theorem, let us consider the case of quiver representations in vector spaces. Let $A = \Bbbk Q$ be the path algebra of a quiver Q over a field $\Bbbk$. Given a Q-representation X, there is a short exact sequence of Q-representations [1, p.7]

$$0 \to \bigoplus_{a:i\to j} Ae_j \otimes X_i \to \bigoplus_i Ae_i \otimes X_i \to X \to 0,$$

called the standard resolution of X. Our goal is to construct an analogue of this resolution in the category $\overline{\operatorname{Rep}}(Q, \Psi)$ as well as its dual version, called a coresolution.

Theorem 3.3. *Let $\Psi\colon \mathcal{P}(Q) \to \mathrm{Cat}$ be a diagram such that the categories Ψ_i have (countable) coproducts and the functors $\Psi_a\colon \Psi_i \to \Psi_j$ preserve them for all arrows $a\colon i \to j$ in Q. For any object $X \in \overline{\operatorname{Rep}}(\Psi)$, there is a short exact sequence*

$$0 \to \bigoplus_{a:i\to j} \sigma_!(\Psi_a X_i) \xrightarrow{\beta} \bigoplus_i \sigma_!(X_i) \xrightarrow{\gamma} X \to 0$$

called a standard resolution of X, where for any arrow $a\colon i \to j$ and any path $p\colon j \to k$,

$$\beta\colon \Psi_p\Psi_a X_i \xrightarrow{(\mathrm{Id}, -\Psi_p X_a)} \Psi_{pa}X_i \oplus \Psi_p X_j, \qquad \gamma\colon \Psi_p X_j \xrightarrow{X_p} X_k.$$

Proof. It is clear that $\gamma\beta = 0$. We need to verify exactness of the components for every vertex $k \in Q_0$. For every $n \geq 0$, define the degree n component

$$Z_n = \bigoplus_{\substack{p:i\to k\\ l(p)=n}} \Psi_p X_i \in \Psi_k,$$

where $k \in Q_0$ is fixed, $i \in Q_0$ varies, and $l(p)$ is the length of the paths p. We have to verify that the sequence

$$0 \to \bigoplus_{n\geq 1} Z_n \xrightarrow{\beta} \bigoplus_{n\geq 0} Z_n \xrightarrow{\gamma} Z_0 \to 0$$

is exact in Ψ_k. The matrices of β and γ are of the form

$$\beta = \begin{pmatrix} * & 0 & 0 & 0 & \dots \\ 1 & * & 0 & 0 & \dots \\ 0 & 1 & * & 0 & \dots \\ 0 & 0 & 1 & * & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1 & * & * & \dots \end{pmatrix}$$

We can assume that Ψ_k is a category of modules over an algebra. It is clear that γ is surjective. If $a \in \operatorname{Ker}\beta$ is nontrivial, let $a_n \in Z_n$ be its nonzero component of maximal degree. Then the restriction of $\beta(a)$ to Z_n is $a_n \neq 0$. A contradiction. Finally, given $a \in \bigoplus_{n\geq 0} Z_n$, let $\deg a = \max\{n \geq 0 \mid a_n \neq 0\}$. Choose $0 \neq a \in \operatorname{Ker}\gamma \backslash \operatorname{Im}\beta$ with the minimal possible $n = \deg a$. If $n \geq 1$, then $a' = a - \beta(a_n)$ has degree $< n$ and we still have $a' \in \operatorname{Ker}\gamma \backslash \operatorname{Im}\beta$ as $\operatorname{Im}\beta \subset \operatorname{Ker}\gamma$. A contradiction. Therefore a has degree zero and $\gamma(a) = a = 0$. A contradiction. □

Taking the opposite categories we obtain a dual version of the above theorem.

Theorem 3.4. *Let $\Phi\colon \mathcal{P}(Q)^{\mathrm{op}} \to \mathrm{Cat}$ be a diagram such that the categories Φ_i have (countable) products and the functors $\Phi_a\colon \Phi_j \to \Phi_i$ preserve them for all arrows $a\colon i \to j$ in Q. For any object $X \in \operatorname{Rep}(\Phi)$, there is a short exact sequence*

$$0 \to X \xrightarrow{\gamma} \bigoplus_i \sigma_*(X_i) \xrightarrow{\beta} \bigoplus_{a:j\to i} \sigma_*(\Phi_a X_i) \to 0$$

called a standard coresolution of X, where for any arrow $a\colon j \to i$ and any path $p\colon k \to j$,

$$\beta\colon \Phi_{ap}X_i \oplus \Phi_p X_j \xrightarrow{(\mathrm{Id}, -\Phi_p X_a)} \Phi_p\Phi_a X_i, \qquad \gamma\colon X_k \xrightarrow{X_p} \Phi_p X_j.$$

4. Homological properties

Let Q be a quiver and Ψ be a Q-diagram, that is, a collection of categories $(\Psi_i)_{i\in Q_0}$ and functors $a_* = \Psi_a\colon \Psi_i \to \Psi_j$ for arrows $a\colon i \to j$ in Q. We will assume that the categories Ψ_i are Grothendieck [9, §8.3] (in particular they have enough injective objects, admit small inductive and projective limits, and their small filtrant inductive limits and direct sums are exact), hence we don't assume Ψ_i to be small anymore. Note that any small abelian category $\mathcal{A}$ can be canonically embedded into the category $\operatorname{Ind}(\mathcal{A})$ of ind-objects which is a Grothendieck category [9, §8.6]. This category is equivalent to the Quillen abelian envelope of $\mathcal{A}$ (usually applied to exact categories [10, Appendix A]), consisting of additive left-exact functors $F\colon \mathcal{A}^{\mathrm{op}} \to \operatorname{Mod}\mathbb{Z}$ [9, §8.6]. For example, if X is a noetherian scheme and $\mathcal{A} = \operatorname{Coh} X$ is the category of coherent sheaves on X, then $\operatorname{Ind}(\mathcal{A})$ is equivalent to the category of quasi-coherent sheaves on X [7, Appendix]. On the level of derived categories there is a natural equivalence $D^b(\mathcal{A}) \simeq D^b_{\mathcal{A}}(\operatorname{Ind}(\mathcal{A}))$ [9, §15.3], hence the Ext-groups are unchanged.

Theorem 4.1. *Let Φ be a (left exact) Q^{op}-diagram of Grothendieck categories such that Φ_a preserve products for all arrows a in Q. Then the category $\operatorname{Rep}(Q, \Phi)$ has enough injectives.*

Proof. Using the standard coresolution (Theorem 3.4), we can embed $X \in \operatorname{Rep}(Q, \Phi)$ into $\bigoplus_i \sigma_*(X_i)$. Therefore we have to show that every $\sigma_*(X_i)$ can be embedded into an injective object. Let $X_i \hookrightarrow I$ be an embedding into an injective object in Φ_i. Then the induced map $\sigma_*(X_i) \to \sigma_*(I)$ is a monomorphism as the functor σ_* is right adjoint, hence left exact. The object $\sigma_*(I)$ is injective by Corollary 2.13. □

Remark 4.2. We will often encounter the following situation. Let $L\colon\mathcal{A}\to\mathcal{B}$ be a functor left adjoint to a functor $R\colon\mathcal{B}\to\mathcal{A}$. If $\mathcal{A},\mathcal{B}$ are abelian categories with enough injectives and L,R are exact functors, then

$$\operatorname{Ext}^k_{\mathcal{B}}(LX,Y)\simeq\operatorname{Ext}^k_{\mathcal{A}}(X,RY)\qquad\forall X\in\mathcal{A},\,Y\in\mathcal{B},\,k\ge 0.$$

Indeed, the functor R maps injective objects to injective objects because of the exactness of L. As R is itself exact, it maps an injective resolution of Y to an injective resolution of RY. Applying the functors $\operatorname{Hom}(LX,-)$ and $\operatorname{Hom}(X,-)$ to these resolutions, we obtain the above isomorphism.

Theorem 4.3. *Let Φ be a Q^{op}-diagram of Grothendieck categories that admits an exact left adjoint diagram $\Psi\colon\mathcal{P}(Q)\to\mathrm{Cat}$. Then for any two objects $X,Y\in\mathcal{R}=\operatorname{Rep}(\Phi)$, there is a long exact sequence*

$$\begin{aligned}0\to\operatorname{Hom}_{\mathcal{R}}(X,Y)\to\bigoplus_i\operatorname{Hom}_{\Phi_i}(X_i,Y_i)\to\bigoplus_{a:i\to j}\operatorname{Hom}_{\Phi_j}(\Psi_aX_i,Y_j)\\ \to\operatorname{Ext}^1_{\mathcal{R}}(X,Y)\to\bigoplus_i\operatorname{Ext}^1_{\Phi_i}(X_i,Y_i)\to\bigoplus_{a:i\to j}\operatorname{Ext}^1_{\Phi_j}(\Psi_aX_i,Y_j)\to\dots\end{aligned}$$

Proof. Applying the functor $\operatorname{Hom}(-,Y)$ to the standard resolution of X (see Theorem 3.3)

$$0\to\bigoplus_{a:i\to j}\sigma_!(\Psi_aX_i)\to\bigoplus_i\sigma_!(X_i)\to X\to 0$$

we obtain a long exact sequence. The statement of the theorem will follow from

$$\operatorname{Ext}^k_{\mathcal{R}}(\sigma_!X_i,Y)\simeq\operatorname{Ext}^k_{\Psi_i}(X_i,\sigma^*Y)$$

for any $X_i\in\Psi_i$ and $Y\in\mathcal{R}$. According to Remark 4.2 it is enough to show that both $\sigma^*,\sigma_!$ are exact. The functor σ^* is exact as a forgetful functor. The functor $\sigma_!\colon\Psi_i\to\mathcal{R}$ is exact because of its construction (see Theorem 2.11), as Ψ is exact and coproducts preserve exactness. $\square$

Remark 4.4. Our next result relies on the exactness of products which appear in the construction of the functor $\sigma_*\colon\Phi_i\to\operatorname{Rep}(\Phi)$ (see Theorem 2.12). It is known that products are not exact in Grothendieck categories in general (see e.g. [11, Example 4.9]). To make our argument work we will assume that our quiver is acyclic, so that the path category $\mathcal{P}(Q)$ is finite. Then all products appearing in the construction of σ_* are finite, hence exact. Note also that the requirements of Theorem 2.12 and Theorem 3.4 on the preservation of products by Φ_a can be omitted in this case.

Theorem 4.5. *Let Q be an acyclic quiver and Φ be an exact Q^{op}-diagram of Grothendieck categories. Then for any two objects $X,Y\in\mathcal{R}=\operatorname{Rep}(\Phi)$, there is a long exact sequence*

$$\begin{aligned}0\to\operatorname{Hom}_{\mathcal{R}}(X,Y)\to\bigoplus_i\operatorname{Hom}_{\Phi_i}(X_i,Y_i)\to\bigoplus_{a:i\to j}\operatorname{Hom}_{\Phi_i}(X_i,\Phi_aY_j)\\ \to\operatorname{Ext}^1_{\mathcal{R}}(X,Y)\to\bigoplus_i\operatorname{Ext}^1_{\Phi_i}(X_i,Y_i)\to\bigoplus_{a:i\to j}\operatorname{Ext}^1_{\Phi_i}(X_i,\Phi_aY_j)\to\dots\end{aligned}$$

Proof. Applying the functor $\operatorname{Hom}_{\mathcal{R}}(X,-)$ to the standard coresolution of Y (see Theorem 3.4)

$$0\to Y\to\bigoplus_i\sigma_*(Y_i)\to\bigoplus_{a:i\to j}\sigma_*(\Phi_aY_j)\to 0$$

we obtain a long exact sequence. The statement of the theorem will follow from

$$\operatorname{Ext}^k_{\Phi_i}(\sigma^*X,Y_i)\simeq\operatorname{Ext}^k_{\mathcal{R}}(X,\sigma_*(Y_i))$$

for any $X \in \mathcal{R}$ and $Y_i \in \Phi_i$. According to Remark 4.2 it is enough to show that both σ^*, σ_* are exact. The functor σ^* is exact as a forgetful functor. The functor $\sigma_*\colon \Phi_i \to \mathcal{R}$ is exact because of its construction (see Theorem 2.12), as Φ is exact and finite products preserve exactness. □

Remark 4.6. According to Remark 4.2, we have $\operatorname{Ext}^k_{\Phi_j}(\Psi_a X_i, Y_j) \simeq \operatorname{Ext}^k_{\Phi_i}(X_i, \Phi_a Y_j)$ if both Φ and Ψ are exact diagrams. But this is not true in general.

5. Applications

5.1. **Parabolic vector bundles.** Let X be a projective curve over an algebraically closed field $\Bbbk$ and $w\colon X \to \mathbb{N}$ be a map such that the set $S = \{p \in X \mid w_p > 1\}$ is finite. A (quasi-) parabolic vector bundle over X of type w consists of a vector bundle E together filtrations of the fibers

$$E_p = E_{p,0} \supset E_{p,1} \supset \ldots \supset E_{p,w_p} = 0 \qquad \forall p \in S.$$

We define $E^i_p = E_p/E_{p,i}$. A morphism between parabolic vector bundles $\mathbf{E} = (E, E_*)$ and $\mathbf{F} = (F, F_*)$ is a morphism $f\colon E \to F$ that preserves filtrations. The category of parabolic vector bundles can be embedded into an abelian category of parabolic coherent sheaves, which is hereditary [4, 17, 8, 12].

Theorem 5.1. *Given two parabolic vector bundles $\mathbf{E} = (E, E_*)$ and $\mathbf{F} = (F, F_*)$, there is an exact sequence*

$$0 \to \operatorname{Hom}(\mathbf{E}, \mathbf{F}) \to \operatorname{Hom}(E, F) \oplus \bigoplus_{1\le i<w_p} \operatorname{Hom}(E^i_p, F^i_p) \to \bigoplus_{1\le i<w_p} \operatorname{Hom}(E^{i+1}_p, F^i_p)$$
$$\to \operatorname{Ext}^1(\mathbf{E}, \mathbf{F}) \to \operatorname{Ext}^1(E, F) \to 0$$

Proof. We can interpret parabolic vector bundles as quiver representations as follows. For simplicity let us assume that $S = \{p\}$ and let $n = w_p - 1$. Let $\mathfrak{m}_p \subset \mathcal{O}_X$ be the maximal ideal of the point $p \in X$ and let $\Bbbk_p = \mathcal{O}_X/\mathfrak{m}_p$ be the corresponding skyscraper sheaf. For any coherent sheaf $E \in \operatorname{Coh} X$, define its fiber $E_p = E/\mathfrak{m}_p E \simeq E \otimes_{\mathcal{O}_X} \Bbbk_p$. For any $V \in \operatorname{Vect}$ we have

$$\operatorname{Hom}_{\operatorname{Vect}}(V, (E_p)^*) \simeq \operatorname{Hom}_{\operatorname{Vect}}(E_p, V^*) \simeq \operatorname{Hom}_{\operatorname{Coh} X}(E, V^* \otimes \Bbbk_p).$$

Therefore the functor

$$\Psi_a\colon \operatorname{Coh} X \to \operatorname{Vect}^{\mathrm{op}}, \qquad E \mapsto (E_p)^* \simeq \operatorname{Hom}_{\operatorname{Coh} X}(E, \Bbbk_p)$$

is left adjoint to the exact functor $\Phi_a\colon \operatorname{Vect}^{\mathrm{op}} \to \operatorname{Coh} X$, $V \mapsto V^* \otimes \Bbbk_p$. Consider the quiver Q

$$0 \xrightarrow{a} 1 \xrightarrow{a_1} 2 \to \cdots \to n$$

and the Q-diagram with $\Psi_0 = \operatorname{Coh} X$, $\Psi_i = \operatorname{Vect}^{\mathrm{op}}$ for $i \ge 1$, Ψ_a defined as above and $\Psi_{a_i} = \operatorname{Id}$ for $i \ge 1$. It has a right adjoint Q^{op}-diagram Φ with Φ_a defined as above. A representation is given by $E \in \operatorname{Coh} X$, $V_i \in \operatorname{Vect}$ for $i \ge 1$ and a chain of morphisms in Vect

$$(E_p)^* = V_0 \leftarrow V_1 \leftarrow V_2 \leftarrow \ldots \leftarrow V_n.$$

Given a parabolic vector bundle $\mathbf{E} = (E, E_*)$, we have a chain of epimorphisms $E_p = E^{n+1}_p \to E^n_p \to \cdots \to E^1_p \to E^0_p = 0$ which induces a chain of monomorphisms

$$(E_p)^* = V_0 \leftarrow V_1 \leftarrow V_2 \leftarrow \ldots \leftarrow V_n, \qquad V_i = (E^{n+1-i}_p)^*.$$

We consider this data as an object of $\overline{\operatorname{Rep}}(\Psi)$. In this way we embed the category of parabolic vector bundles into the abelian category $\overline{\operatorname{Rep}}(\Psi) \simeq \operatorname{Rep}(\Phi)$.

Given two representations $\mathbf{E} = (E, V_*)$ and $\mathbf{F} = (F, W_*)$ with locally free $E \in \operatorname{Coh} X$, we obtain from Theorem 4.5 an exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}(\mathbf{E},\mathbf{F}) \to \operatorname{Hom}(E,F) \oplus \bigoplus_{i=1}^{n} \operatorname{Hom}(W_i, V_i) \to \bigoplus_{i=1}^{n} \operatorname{Hom}(W_i, V_{i-1}) \\ \to \operatorname{Ext}^1(\mathbf{E},\mathbf{F}) \to \operatorname{Ext}^1(E,F) \to \operatorname{Ext}^1(E, \Phi_a W_1) = 0 \end{aligned}$$

The last equality follows from Serre duality $\operatorname{Ext}^1(E, \Bbbk_p) \simeq \operatorname{Hom}(\Bbbk_p, E)^*$ and the assumption that E is locally free. If $\mathbf{E}$ and $\mathbf{F}$ correspond to parabolic bundles (E, E_*) and (F, F_*) respectively, we have $\operatorname{Hom}(W_i, V_j) \simeq \operatorname{Hom}(E_p^{n+1-j}, F_p^{n+1-i})$, hence the statement of the theorem. ☐

5.2. **Nested sheaves.** Let X be an algebraic surface and $p \in X$ be a point with the maximal ideal $\mathfrak{m}_p \subset \mathcal{O}_X$ and the skyscraper sheaf $\Bbbk_p = \mathcal{O}_X/\mathfrak{m}_p$. Consider the category of pairs (F, F') of coherent sheaves such that $\mathfrak{m}_p F \subset F' \subset F$. The moduli spaces of such pairs are studied in [16, §4] under the name of moduli spaces of perverse coherent sheaves on a blow-up. Such pairs are determined by F together with an epimorphism $F_p = F/\mathfrak{m}_p F \twoheadrightarrow F/F'$.

We can represent such pairs as quiver representations as follows. Consider the quiver $Q = [0 \xrightarrow{a} 1]$ and the Q-diagram with $\Psi_0 = \operatorname{Coh} X$, $\Psi_1 = \operatorname{Vect}^{\mathrm{op}}$ and

$$\Psi_a \colon \operatorname{Coh} X \to \operatorname{Vect}^{\mathrm{op}}, \qquad F \mapsto (F_p)^* \simeq \operatorname{Hom}_{\operatorname{Coh} X}(F, \Bbbk_p). \tag{3}$$

It has an exact right adjoint Q^{op}-diagram Φ with $\Phi_a \colon \operatorname{Vect}^{\mathrm{op}} \to \operatorname{Coh} X$, $V \mapsto V^* \otimes \Bbbk_p$. A representation is given by a triple (F, V, s), where $F \in \operatorname{Coh} X$, $V \in \operatorname{Vect}$ and $s \in \operatorname{Hom}_{\operatorname{Vect}}(V, (F_p)^*) \simeq \operatorname{Hom}_{\operatorname{Coh} X}(F, V^* \otimes \Bbbk_p)$. Given a pair (F, F') as above, we consider $V^* = F/F'$ and s corresponding to $F \to F/F'$. In this way we embed the category of pairs (F, F') as above into $\overline{\operatorname{Rep}}(\Psi)$. Given two representations $\mathbf{E} = (E, V, s)$ and $\mathbf{F} = (F, W, s')$ with a torsion free sheaf E we obtain from Theorem 4.5, an exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}(\mathbf{E},\mathbf{F}) \to \operatorname{Hom}(E,F) \oplus \operatorname{Hom}(W,V) \to (E_p)^* \otimes W^* \\ &\to \operatorname{Ext}^1(\mathbf{E},\mathbf{F}) \to \operatorname{Ext}^1(E,F) \to \operatorname{Ext}^1(E,\Bbbk_p) \otimes W^* \to \operatorname{Ext}^2(\mathbf{E},\mathbf{F}) \to \operatorname{Ext}^2(E,F) \to 0, \end{aligned} \tag{4}$$

where we used the fact that $\operatorname{Ext}^2(E, \Bbbk_p) \simeq \operatorname{Hom}(\Bbbk_p, E)^* = 0$.

5.3. **Framed sheaves.** Let X be an algebraic variety and let $P \in \operatorname{Coh} X$. The mapping cylinder of the left exact functor $\Phi_a \colon \operatorname{Coh} X \to \operatorname{Vect}$, $E \mapsto \operatorname{Hom}(P, E)$, is a category with object consisting of triples (E, V, s), where $E \in \operatorname{Coh} X$, $V \in \operatorname{Vect}$ and $s \colon V \to \operatorname{Hom}(P, E)$. It can be interpreted as the category of representations of the quiver $Q = [0 \xrightarrow{a} 1]$ and the Q^{op}-diagram $\Phi_0 = \operatorname{Vect}$, $\Phi_1 = \operatorname{Coh} X$ and $\Phi_a \colon \Phi_1 \to \Phi_0$ defined as above. This diagram has an exact left adjoint Q-diagram Ψ with $\Psi_a \colon \operatorname{Vect} \to \operatorname{Coh} X$, $V \mapsto V \otimes P$. Given two representations $\mathbf{E} = (E, V, s)$ and $\mathbf{F} = (F, W, s')$, we obtain from Theorem 4.3 a long exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}(\mathbf{E},\mathbf{F}) \to \operatorname{Hom}(E,F) \oplus \operatorname{Hom}(V,W) \to \operatorname{Hom}(V \otimes P, F) \\ \to \operatorname{Ext}^1(\mathbf{E},\mathbf{F}) \to \operatorname{Ext}^1(E,F) \to \operatorname{Ext}^1(V \otimes P, F) \to \dots \end{aligned} \tag{5}$$

Remark 5.2. More generally, consider a (small) abelian category $\mathcal{A}$ and a left exact functor $\Phi_a \colon \mathcal{A} \to \operatorname{Vect}$. Its mapping cylinder can be identified with the category of quiver representations in the same way as above, where a representation $\mathbf{E} = (E, V, s)$ consist of $E \in \mathcal{A}$, $V \in \operatorname{Vect}$ and $s \colon V \to \Phi_a E$. If Φ_a is an exact functor, we obtain for two representations

$\mathbf{E}=(E,V,s)$ and $\mathbf{F}=(F,W,s')$ an exact sequence from Theorem 4.5

$$0\to\operatorname{Hom}(\mathbf{E},\mathbf{F})\to\operatorname{Hom}(E,F)\oplus\operatorname{Hom}(V,W)\to\operatorname{Hom}(V,\Phi_aF)\\ \to\operatorname{Ext}^1(\mathbf{E},\mathbf{F})\to\operatorname{Ext}^1(E,F)\to 0 \tag{6}$$

and $\operatorname{Ext}^i(\mathbf{E},\mathbf{F})\simeq\operatorname{Ext}^i(E,F)$ for $i\ge 2$. This means that the homological dimension of the mapping cylinder coincides with the homological dimension of $\mathcal{A}$.

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School of Mathematics, Trinity College Dublin, Ireland
Hamilton Mathematics Institute, Ireland
E-mail address: `mozgovoy@maths.tcd.ie`