Menelaus' Theorem, Weil Reciprocity, and a Generalisation to Algebraic Curves

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Contents

1 Overview

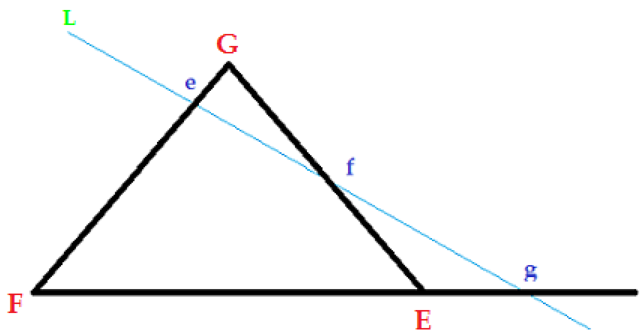
This article is meant to provide a leisurely and expository overview of some classical results, along with some modern theory. Topics from Euclidean geometry, projective geometry and more recent geometric results are discussed with the principal section being a generalisation of Menelaus' theorem originally due to the French politician and mathematician Lazare Carnot (1753-1823).

2 A Classical Result from Euclidean Geometry

2.1 Menelaus' Theorem

Here we prove a classical Euclidean theorem due to *Menelaus of Alexandria* (c. 70-140 CE).

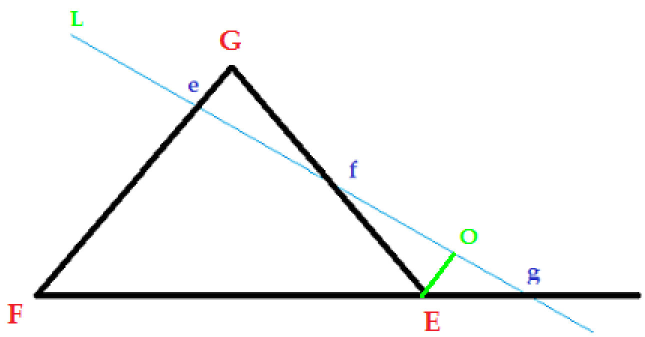
Proposition 2.1 *Menelaus' Theorem*
Let EFG be a triangle (with sides suitably extended) in the plane. Let L be any line cutting EF at g, FG at e and EG at f. Suppose L is not parallel to any side of EFG and that it does not pass through any of the vertices of EFG[1]*. Then the following identity holds:*



$$\frac{|Ge|}{|eF|} \cdot \frac{|Fg|}{|gE|} \cdot \frac{|Ef|}{|fG|} = 1 \tag{1}$$

Proof: Draw a line segment from the point E parallel to the line segment GF so that it intersects the line L at O.

[1]These two conditions on L ensure that the equation 1 is well defined



Note that the triangles Gef and EOf are similar, thus the ratios of the lengths of their sides are in proportion:

$$\frac{|EO|}{|Ef|} = \frac{|Ge|}{|Gf|} \tag{2}$$

Similarly the triangles EOg and Feg are similar and we have:

$$\frac{|OE|}{|gE|} = \frac{|eF|}{|gF|} \tag{3}$$

Subbing (2) and (3) into equation (1) gives us the required identity:

$$\frac{|Ge|}{|eF|} \cdot \frac{|Fg|}{|gE|} \cdot \frac{|Ef|}{|fG|} = \frac{|EO|}{|Ef|} \cdot \frac{|Fg|}{|eF|} \cdot \frac{|Ef|}{|gE|} = \frac{|OE|}{|gE|} \cdot \frac{|Fg|}{|eF|} = \frac{|eF|}{|Fg|} \cdot \frac{|Fg|}{|eF|} = 1$$

Where we have used the fact that $|fG| = |Gf|$, $|EO| = |OE|$ and $|Fg| = |gF|$.

□

There is also a version of Menelaus' theorem allowing signed lengths. In this case, each side of the triangle EFG is assigned an arbitrary direction. The length of a line segment $|ab|$ from point a to b is considered positive if the direction of the vector $\vec{ab}$ is the same as the direction assigned to the side of the triangle. Conversely the length is considered to be negative if the directions are opposite, that is to say $|ab| = -|ba|$. In this version of Menelaus' theorem, the product of the ratios is as follows:

$$\frac{|Ge|}{|eF|} \cdot \frac{|Fg|}{|gE|} \cdot \frac{|Ef|}{|fG|} = -1$$

To see this we follow the same line of reasoning as in the proof above, but note that the ratio $\frac{|Ge|}{|eF|}$ (for example) is positive if e lies inside the line segment GF, and negative if e lies outside. It is clear from inspection that the line L will either cut the triangle EFG twice between the vertices of the triangle and once outside, or will cut all of the sides outside the triangle. In either case the product of the ratios will be -1.

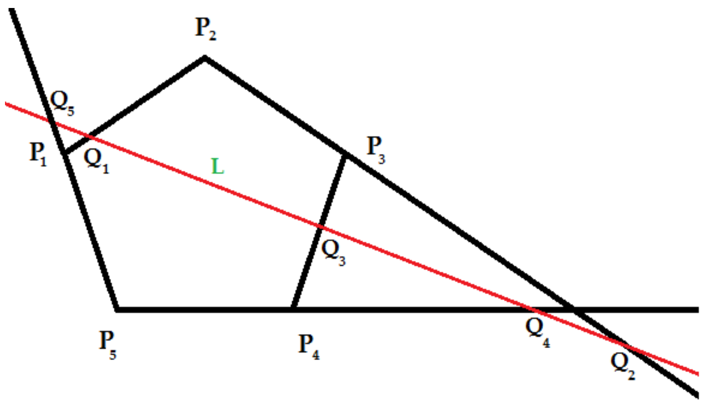
2.2 A Generalisation to Polygons

There is also a painless generalisation from triangles to convex n-gons with $n \geq 2$:

Proposition 2.2 *Let P be a convex n-gon. Label these vertices by $P_1, P_2, ..., P_n$ in the clockwise direction. Let L be a line which is not parallel to any side of the polygon nor passes through any of the vertices P_i. Denote the point which L cuts the side P_iP_{i+1} (taking $P_{n+1} = P_1$) by the point Q_i. Then the following relation holds:*

$$\prod_{i=1}^{n} \frac{|P_iQ_i|}{|Q_iP_{i+1}|} = (-1)^n$$

where the ratio $\frac{|P_iQ_i|}{|Q_iP_{i+1}|}$ is considered positive if Q_i lies between P_i and P_{i+1}, and negative otherwise.



Proof: Triangulate the polygon by joining P_1 to each other point. Denote by r_i the point where L cuts the line joining P_1 to P_i for $2 \leq i \leq n$ (note that in this notation $r_2 = Q_1$ and $r_n = Q_n$. Consider the triangles $P_1P_iP_{i+1}$ for $2 \leq i \leq n-1$. Let the line segments P_iP_{i+1} and P_1P_i be assigned arbitrary directions, as in the discussion following Menelaus' theorem. Apply Menelaus' theorem to each triangle $P_1P_iP_{i+1}$ for $2 \leq i \leq n-1$ to get:

$$\frac{|P_1r_i|}{|r_iP_i|} \cdot \frac{|P_iQ_i|}{|Q_iP_{i+1}|} \cdot \frac{|P_{i+1}r_{i+1}|}{|r_{i+1}P_1|} = -1$$

So multiplying all the expressions together we get:

$$\prod_{i=2}^{n-1} \frac{|P_1 r_i|}{|r_i P_i|} \cdot \frac{|P_i Q_i|}{|Q_i P_{i+1}|} \cdot \frac{|P_{i+1} r_{i+1}|}{|r_{i+1} P_1|} = (-1)^{n-2} = (-1)^n$$

$$\prod_{i=2}^{n-1} \frac{|P_1 r_i|}{|r_i P_i|} \cdot \frac{|P_{i+1} r_{i+1}|}{|r_{i+1} P_1|} \cdot \prod_{i=2}^{n-1} \frac{|P_i Q_i|}{|Q_i P_{i+1}|} = (-1)^n$$

$$\prod_{i=2}^{n-1} \frac{|P_1 r_i|}{|P_1 r_{i+1}|} \cdot \frac{|P_{i+1} r_{i+1}|}{|P_i r_i|} \cdot \prod_{i=2}^{n-1} \frac{|P_i Q_i|}{|Q_i P_{i+1}|} = (-1)^n$$

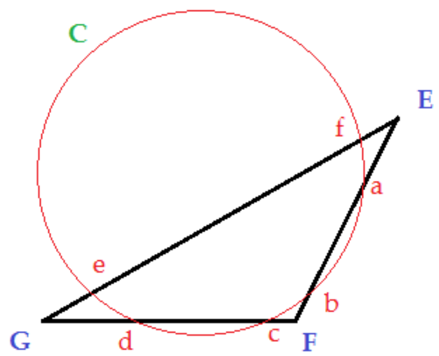
$$\frac{|P_1 Q_1|}{|Q_1 P_2|} \cdot \frac{|P_n Q_n|}{|Q_n P_1|} \cdot \prod_{i=2}^{n-1} \frac{|P_i Q_i|}{|Q_i P_{i+1}|} = (-1)^n$$

$$\prod_{i=1}^{n} \frac{|P_i Q_i|}{|Q_i P_{i+1}|} = (-1)^n$$

□

3 Carnot's Theorem

Carnot's theorem is a generalisation of Menelaus' theorem which arises when we attempt to loosen the hypothesis to admit arbitrary curves C rather than just a line L. It is too much to hope for that a canonical relation exists between the points of intersection of C with a triangle. However it turns out that one does hold for algebraic curves, that is curves which are described by an algebraic equation such as $x^2 + y^2 = 1$ or $x^3 - 3y^5 = 0$.

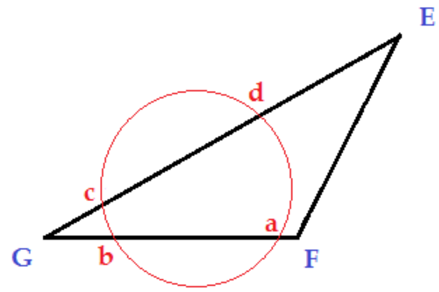


Taking for example the case where the curve C is a circle, then a relation very similar to the relation in Menelaus' theorem holds:

$$\frac{|Ea|}{|aF|} \cdot \frac{|Eb|}{|bF|} \cdot \frac{|Fc|}{|cG|} \cdot \frac{|Fd|}{|dG|} \cdot \frac{|Ge|}{|eE|} \cdot \frac{|Gf|}{|fE|} = 1$$

An algebraic curve C of degree n will cut a side of the triangle at at most n points. To see this, suppose that C is given by the equation the polynomial in 2 variables of degree n p(x,y) and that the side of the triangle is given by y=mx+c. Upon substitution of the equation of the line in p(x,y) it should be clear that we are left with a polynomial in one variable of degree n, which by the fundamental theorem of algebra has at most n real roots. Note if the polynomials were taken over $\mathbb{C}$ then there would be exactly n complex roots.

In generalising Menelaus to Carnot, we cannot simply substitute algebraic curve for line, and replace the relation with one along the lines of the above. A problem we run into is that not all algebraic curves of degree n will cut each side of the triangle at n points (counting multiplicity), and it appears that this is a necessary condition. Taking the above example, the degree of the algebraic curve C in the above example is 2 (as C is described by the polynomial equation of degree 2: $x^2 + y^2 = r^2$). In this case C cuts each side of the triangle in two places, so there are six ratios in the relation. If C does not cut each side of the triangle at the maximal of n points, then a relation similar to the one above does not necessarily hold, as would be the case for the following:



In light of the proceeding remark on the degree of the curve, we formulate a naive version of Carnot's theorem. In this formulation we consider the lengths of line segments to be signed, in exactly the same manner as the signed version of Menelaus' theorem.

Theorem 3.1 *Carnot's Theorem (Naive Version)*
Let EFG be a triangle (with sides suitably extended) in the plane. Let C be an algebraic curve of degree n cutting side EF at the points g_i, FG at e_i and EG at f_i. Suppose C cuts each side of EFG at n points (counting multiplicity) and that it does not pass through any of the vertices of EFG. Then the following identity holds:

$$\prod_{i=1}^{n} \frac{|Eg_i|}{|g_iF|} \frac{|Fe_i|}{|e_iG|} \frac{|Gf_i|}{|f_iE|} = (-1)^n$$

Let us compare the statements of Menelaus' theorem and Carnot's theorem. They are almost identical. The line L becomes an arbitrary algebraic curve C, while the condition *suppose L is not parallel to any side of EFG* is replaced by *suppose C cuts each side of EFG at n points (counting multiplicity)*. Finally the identity in Carnot's theorem is clearly a direct generalisation of that of Menelaus' theorem.

We are not going to prove this verion of Carnot's theorem, but in fact a more general version which dispenses with the hypothesis that C cut each side of EFG at n points. It should be evident to the reader that the requirement that C cut each side n times results from the fact that we are considering algebraic curves over $\mathbb{R}$. If we consider algebraic curves over $\mathbb{C}$, then C will always intersect each side of EFG at n points, by consequence of the fundamental theorem of algebra as outlined before. Glancing back at the preceding two images should convince the reader that it is not possible to remove the condition that C cuts each side n times while working in the real plane $\mathbb{R}^2$, and that we must instead work in $\mathbb{C}^2$.

We will however go slightly further, and will instead opt to work in the slightly larger space of $\mathbb{P}^2(\mathbb{C})$ which we will define in the following subsection 3.1 *The Projective Plane*. For now let us just say that it is the way to glue to our usual plane some "infinite part". If we start with the usual real plane $\mathbb{R}^2$ we can extend it to the real projective plane $\mathbb{P}^2(\mathbb{R})$. If we already passed from $\mathbb{R}^2$ to $\mathbb{C}^2$, $\mathbb{C}^2$ is contained in the complex projective plane $\mathbb{P}^2(\mathbb{C})$. The following diagram of inclusions summerises what we have said on the "extension" of the usual plane $\mathbb{R}^2$:

$$\begin{array}{ccc} \mathbb{R}^2 & \subset & \mathbb{P}^2(\mathbb{R}) \\ \cap & & \cap \\ \mathbb{C}^2 & \subset & \mathbb{P}^2(\mathbb{C}) \end{array}$$

This discussion is technical, but we are going to give the assumption-free formulation of Carnot's Theorem and prove it in $\mathbb{P}^2(\mathbb{C})$. The naive version of Carnot's theorem will follow when we restrict back to the plane, and the assumptions come simply from the fact that we want to see all our objects (the triangle, the curve and all n points of intersection with each of the three sides) in the original plane.

3.1 The Projective Plane

In this section we describe the space $\mathbb{P}^2(F)$, the projective plane over a field F. We take the field F to be either $\mathbb{R}$ or $\mathbb{C}$.

Definition: *(Projective Plane* $\mathbb{P}^2(F)$*)*
The projective plane over a field F is the set of all lines over F passing through the origin in F^3.

There is a natural way to define a coordinate system on $\mathbb{P}^2(F)$, called homogenous coordinates.We define an equivalence relation of $F^3 \setminus \{(0,0,0)\}$ as follows $(t_0, t_1, t_2) \sim (s_0, s_1, s_2)$ if and only if there exists a nonzero $\lambda \in F$ such that $t_i = \lambda s_i$ for i=0,1,2

We denote each equivalence class by $[t_0 : t_1 : t_2]$, and it is clear that $[t_0 : t_1 : t_2] = [\lambda t_0 : \lambda t_1 : \lambda t_2]$ for each nonzero $\lambda \in F$. There is a one to one correspondence between equivalence classes and lines in F^3 passing through the origin, and indeed $\mathbb{P}^2(F)$ is the quotient space of $F^3 \backslash (0,0,0)$ by $\sim$.

It was remarked in the previous section that $\mathbb{P}^2(F)$ can be seen to be an extension of F^2. F^2 can be embedded in $\mathbb{P}^2(F)$ through the map

$$\mu : F^2 \hookrightarrow \mathbb{P}^2(F) \tag{4}$$

$$\mu(x_1, x_2) = [1 : x_1 : x_2] \tag{5}$$

μ is thus a homeomorphism from F^2 to $\{[t_0 : t_1 : t_2] : t_0 \neq 0)\} \subset \mathbb{P}^2(F)$, and this set $\{[t_0 : t_1 : t_2] : t_0 \neq 0)\}$ is sometimes referred as a 'copy of F^2' sitting in $\mathbb{P}^2(F)$.

3.2 Lines and Curves in the Projective Plane

From the previous section we know that points in the projective plane $\mathbb{P}^2(F)$ are lines in F^3 passing through the origin. Further to this lines in $\mathbb{P}^2(F)$ are given by planes through the origin in F^3. In homogeneous coordinates lines in $\mathbb{P}^2(F)$ are given by homogeneous linear equations of the form

$$t(s_0, s_1, s_2) = as_0 + bs_1 + cs_2 = 0$$

where t is a nonzero linear functional on F^3. Analogously curves of degree n are given as the zero sets of irreducible homogeneous polynomials of degree n. Irreducible means that the homogeneous polynomial can not be factorised into two homogeneous polynomials of degree strictly less than n. If it were possible to factorise a homogeneous polynomial of degree n in such a way, we would see that its zero set would consist of the finite union of curves as defined above.

Each curve in the projective plane $\mathbb{P}^2(F)$ has a corresponding curve in the affine plane F^2 which can be seen from the following example of a generic quadric curve in $\mathbb{P}^2(F)$ given by

$$a_0s_0^2 + a_1s_1^2 + a_2s_2^2 + b_0s_1s_2 + b_1s_0s_2 + b_2s_0s_1 = 0 \tag{6}$$

The corresponding affine curve is given by

$$a_0 + a_1x^2 + a_2y^2 + b_0xy + b_1y + b_2x = 0 \tag{7}$$

which is a generic quadric curve in F^2. This affine curve is arrived at by restricting the projective curve to a copy of $F^2 \subset \mathbb{P}^2(F)$, through the isomorphism μ in (4). Indeed to arrive at (7) we divide (6) across by s_0^2 and set

$$x = \frac{s_1}{s_0}, y = \frac{s_2}{s_0}$$

We note that this procedure is well defined since the particular copy of $F^2 \subset \mathbb{P}^2(F)$ given by μ is given by $\{[s_0 : s_1 : s_2] : s_0 \neq 0\}$. Also it should be noted that since there are many different copies of F^2 sitting in $\mathbb{P}^2(F)$ there are many different ways of arriving at an affine curve, all of which however will be generic quadric curves in F^2.

3.3 Statement of Carnot's Theorem

A curve C of degree n is the set in $\mathbb{P}^2(F)$ given by the equation $f(s_0, s_1, s_2) = 0$ where f is an irreducible homogeneous polynomial of degree n.

Let L be a line given by a linear equation $t(s_0, s_1, s_2) = as_0 + bs_1 + cs_2 = 0$. Then for each point $P \in C \cap L$ we define the multiplicity ν_P as follows. To find all points of intersection we substitute the equation of the line into the equation of the curve, e.g. when $a \neq 0$ one does

$$s_0 = -\frac{b}{a}s_1 - \frac{c}{a}s_2,$$
$$f\Big(-\frac{b}{a}s_1 - \frac{c}{a}s_2, s_1, s_2\Big) = 0.$$

The latter equation is homogeneous in two variables and of degree n. If we divide it by s_1^n we will get a polynomial equation in $x = \frac{s_2}{s_1}$ of degree n, and its solutions correspond to points $P \in C \cap L$, i.e. homogeneous coordinates of P could be recovered from the above computation. The multiplicity of the respective root is called ν_P, and since we are over $\mathbb{C}$ then

$$\sum_{P \in C \cap L} \nu_P = n.$$

Indeed, every polynomial of degree n has n complex roots if one counts them with multiplicities.

We now give the general statement of Carnot's theorem, a special case of which was first proved by the famous French politician and mathematician Lazare Carnot in his 1803 work *Géométrie de Position*[1].

Theorem 3.2 *(Carnot's Theorem)*
Consider a triangle in the projective plane $\mathbb{P}^2(\mathbb{C})$ whose sides are given by the equations $\{t_i = 0\}$, $i = 0, 1, 2$ where t_i are linear functionals of homogeneous coordinates. Let C be a non-singular irreducible algebraic curve of degree n which does not pass through the vertices of the triangle. Then we have the following identity:

$$\prod_{P \in \{t_0=0\} \cap C} \left(\frac{t_2(P)}{t_1(P)}\right)^{\nu_P} \cdot \prod_{P \in \{t_1=0\} \cap C} \left(\frac{t_0(P)}{t_2(P)}\right)^{\nu_P} \cdot \prod_{P \in \{t_2=0\} \cap C} \left(\frac{t_1(P)}{t_0(P)}\right)^{\nu_P} = (-1)^n$$

4 The Weil Reciprocity Law

In this section we discuss the Weil reciprocity law, followed by the introduction of Weil symbols, due the French mathematician André Weil (1906-1998). The properties of the Weil symbol will allow us to prove Carnot's theorem.

In what follows we shall consider X to be a connected, compact Riemann surface. Let $\mathbb{C}(X)$ denotes the field of meromorphic functions on X. We define $\mathbb{C}(X)^\times = \mathbb{C}(X) \setminus \{0\}$ to be all meromorphic functions on X except the zero one. If f $\in \mathbb{C}(X)^\times$ then we shall denote by (f) set of zeros and poles of f. Let us begin with the formulation of the Weil reciprocity law:

4.1 Formulation of the Weil Reciprocity Law

Theorem 4.1 *(Weil's Reciprocity Law)*
Let X be a compact Riemann surface. Let Let $f, g, \in \mathbb{C}(X)^\times$ be two meromorphic

functions with (f) disjoint from (g), then

$$\prod_{P \in X} f(P)^{ord_P(g)} = \prod_{P \in X} g(P)^{ord_P(f)} \tag{8}$$

where $ord_P(h)$ means the order of the zero of $h \in \mathbb{C}(X)$ at P and is negative in the case the P is a pole and zero if P is neither a pole nor a zero of h.

We check the Weil reciprocity law for meromorphic functions on the Riemann sphere. Let us take $X = \mathbb{P}^1(\mathbb{C})$,the Riemann sphere, as our compact Riemann surface. Let $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}$ all be distinct, and let $A, B \in \mathbb{C}$ be arbitrary nonzero constants. Define the meromorphic functions f and g as follows

$$f(z) = A \prod_{i=1}^{m} \frac{(z - \alpha_i)}{(z - \beta_i)}, \quad g(z) = B \prod_{j=1}^{k} \frac{(z - \gamma_j)}{(z - \delta_j)} \tag{9}$$

Then an application of Weil reciprocity would establish the following identity:

$$\prod_{j=1}^{k} \prod_{i=1}^{m} \frac{(\gamma_j - \alpha_i)}{(\gamma_j - \beta_i)} \cdot \frac{(\delta_j - \beta_i)}{(\delta_j - \alpha_i)} = \prod_{i=1}^{m} \prod_{j=1}^{k} \frac{(\alpha_i - \gamma_j)}{(\alpha_i - \delta_j)} \cdot \frac{(\beta_i - \delta_j)}{(\beta_i - \gamma_j)} \tag{10}$$

However is not hard to see that expressions on both sides of (10) are in fact equal.

Indeed this check in fact proves theorem (4.1) for the Riemann sphere since it is always possible to choose a coordinate system z on $\mathbb{P}^1(\mathbb{C})$ such that f and g take the forms in (9). The coordinate system z is chosen so that ∞ is neither a pole nor a zero of f or g. In this case $f(\infty) = A$ and $g(\infty) = B$. Thus f and g take the forms in (9), and the identity follows from the preceding discussion.

4.2 Weil Symbols

Definition: (*Weil Symbol*)
To each pair $f, g \in \mathbb{C}(X)^\times$ and to each point P on our Riemann surface X we assign $(f, g)_P \in \mathbb{C}^\times$ which we call the Weil symbol of f,g at P and which is defined as follows:

$$(f, g)_P := (-1)^{ord_P(f) ord_P(g)} \left. \frac{f^{ord_P(g)}}{g^{ord_P(f)}} \right|_P$$

We remark that this symbol is well-defined at every point P on X. To see this consider the meromorphic function:

$$h(Q) = \left. \frac{f^{ord_P(g)}}{g^{ord_P(f)}} \right|_Q$$

We have to check that P is neither a zero nor a pole of h, i.e. that h assumes a finite non-zero value at P. We expand f and g as a Laurent series in a local coordinates around P. The first term in each Laurent series will have the same degree ($ord_P(f).ord_P(g)$) and so their ratio will have a finite limit at P.

Let $n = ord_P(f).ord_P(g)$ and a_i and b_i be the Laurent coefficients of $f^{ord_P(g)}$ and $g^{ord_P(f)}$ respectively in the coordinate system z, noting that a_n and b_n are nonzero. Then we have:

$$\begin{aligned} h(Q) &= \frac{a_n(z(Q)-z(P))^n + a_{n+1}(z(Q)-z(P))^{n+1} + ...}{b_n(z(Q)-z(P))^n + b_{n+1}(z(Q)-z(P)) + ...} \\ &= \frac{a_n + a_{n+1}(z(Q)-z(P)) + ...}{b_n + b_{n+1}(z(Q)-z(P)) + ...} \end{aligned}$$

and so in the limit as Q tends towards P we have the finite limit:

$$h(P) = \frac{a_n}{b_n}$$

The reader could also check that this value is independent of the choice of the local coordinate while a_n and b_n depend on this choice. The following are some elementary properties of the Weil symbol:

Proposition 4.2 *(Properties of the Weil symbol)*
i) The Weil symbol is multiplicative:

$$(f, g_1 \cdot g_2)_P = (f, g_1) \cdot (f, g_2) \textit{ for every } g_1, g_2 \in \mathbb{C}(X)^\times$$

ii) The Weil symbol is anti-symmetric:

$$(f, g)_P = (g, f)_P^{-1}$$

iii) The Weil symbol is trivial away from $(f) \cup (g)$:

$$(f, g)_P = 1 \textit{ when } P \notin (f) \cup (g)$$

iv) The Weil symbol takes a simple form away from (f):

$$(f, g)_P = f(P)^{ord_P(g)} \textit{ when } P \notin (f)$$

The previous proposition deals with a fixed point $P \in X$, and arbitrary meromorphic functions $f, g \in \mathbb{C}(X)^\times$. In the following, it is the functions $f, g \in \mathbb{C}(X)^\times$ which are fixed, giving a relation between their Weil symbols as P varies over X. It is essential that X be a compact Riemann surface for this relation to hold.

Proposition 4.3 *(Product over all Points)*
Let $f, g \in \mathbb{C}(X)^\times$ be two arbitrary meromorphic functions, then the following relation holds:

$$\prod_{P \in X} (f, g)_P = 1 \tag{11}$$

Note that in the above relation, although we are taking the product over an infinite number of points in X, $(f, g)_P = 1$ at all but finitely many points by property iii) above. The reader can find proofs for both above propositions in Jean-Pierre Serre's *Algebraic Groups and Class Fields* [2]. In the case of the Weil reciprocity law, it can be readily seen to follow directly from Proposition 4.3 and property iv) of the Weil symbol.

5 Proof of Carnot's Theorem

We will give a proof of Carnot's theorem using Weil symbols.

Proof: *Carnot's Theorem*

We can choose the functionals t_0, t_1, t_2 to be the new homogeneous coordinates in the projective plane. There will be n points of intersection of C with every line $L_i = \{t_i = 0\}$ (counting points with multiplicity). Choose another line L which doesn't pass through any of the $3n$ points of intersection of C with L_i's. Let equation of L be given by $at_0 + bt_1 + ct_2 = 0$. Consider three rational functions on C given by

$$f_i(t_0 : t_1 : t_2) = \frac{t_i}{at_0 + bt_1 + ct_2}$$

As a non-singular projective algebraic curve, C is also a compact connected Riemann surface, hence Proposition 4.3 of the Weil symbol tells us that for every i, j

$$\prod_{P \in C} (f_i, f_j)_P = 1.$$

Multiplying out these identities for each of the three pairs we get

$$\prod_{P \in C} (f_2, f_0)_P \cdot \prod_{P \in C} (f_0, f_1)_P \cdot \prod_{P \in C} (f_1, f_2)_P = 1. \tag{12}$$

For each i=0,1,2 the function f_i on C has zeros at the points P $\in C \cap L_i$, and for each such P the order of the zero is equal to the multiplicity of the intersection of C with the line L_i at P, that is $ord_P(f_i) = \nu_P$. The poles of f_i are located at the points $P \in C \cap L$ and in this case $ord_P(f_i) = -\nu_P$, that is the multiplicity of intersection at P taken with the negative sign.

Since C does not pass through any vertices of the triangle then the zeros of f_i and f_j are disjoint on C, so we may split the product in (12) into the ratio of two products over points on $t_i = 0$ and $t_j = 0$ separately. Furthermore the term $(-1)^{ord_P(f_i) ord_P(f_j)} = 1$ at every point since either one or both of $ord_P(f_i)$ and $ord_P(f_i)$ is zero. Thus we have:

$$\prod_{P \in C} (f_i, f_j)_P = \frac{\prod_{P \in \{t_j = 0\} \cap C} f_i(P)^{\nu_P}}{\prod_{P \in \{t_i = 0\} \cap C} f_j(P)^{\nu_P}}$$

The terms multiplied over the zeros then give:

$$\frac{\prod_{P \in \{t_0=0\} \cap C} f_2(P)^{\nu_P}}{\prod_{P \in \{t_2=0\} \cap C} f_0(P)^{\nu_P}} \cdot \frac{\prod_{P \in \{t_1=0\} \cap C} f_0(P)^{\nu_P}}{\prod_{P \in \{t_0=0\} \cap C} f_1(P)^{\nu_P}} \cdot \frac{\prod_{P \in \{t_2=0\} \cap C} f_1(P)^{\nu_P}}{\prod_{P \in \{t_1=0\} \cap C} f_2(P)^{\nu_P}}$$
$$= \prod_{P \in \{t_0=0\} \cap C} \left(\frac{f_2(P)}{f_0(P)} \right)^{\nu_P} \cdot \prod_{P \in \{t_1=0\} \cap C} \left(\frac{f_0(P)}{f_1(P)} \right)^{\nu_P} \cdot \prod_{P \in \{t_2=0\} \cap C} \left(\frac{f_1(P)}{f_2(P)} \right)^{\nu_P}$$
$$= \prod_{P \in \{t_0=0\} \cap C} \left(\frac{t_2(P)}{t_0(P)} \right)^{\nu_P} \cdot \prod_{P \in \{t_1=0\} \cap C} \left(\frac{t_0(P)}{t_1(P)} \right)^{\nu_P} \cdot \prod_{P \in \{t_2=0\} \cap C} \left(\frac{t_1(P)}{t_2(P)} \right)^{\nu_P}$$

Recall that the poles of f_i and f_j coincide. Noting that the parity of ν_P^2 is the same as ν_P, the terms multiplied over the poles gives:

$$\prod_{P\in L\cap C}(-1)^{\nu_P}\left(\frac{f_2(P)}{f_0(P)}\right)^{\nu_P}\cdot\prod_{P\in L\cap C}(-1)^{\nu_P}\left(\frac{f_0(P)}{f_1(P)}\right)^{\nu_P}\cdot\prod_{P\in L\cap C}(-1)^{\nu_P}\left(\frac{f_1(P)}{f_2(P)}\right)^{\nu_P}$$
$$=\prod_{P\in L\cap C}(-1)^{\nu_P}\left(\frac{t_2(P)}{t_0(P)}\right)^{\nu_P}\cdot\prod_{P\in L\cap C}(-1)^{\nu_P}\left(\frac{t_0(P)}{t_1(P)}\right)^{\nu_P}\cdot\prod_{P\in L\cap C}(-1)^{\nu_P}\left(\frac{t_1(P)}{t_2(P)}\right)^{\nu_P}$$
$$=\prod_{P\in L\cap C}(-1)^{3\nu_P}\left(\frac{t_2(P)}{t_0(P)}.\frac{t_0(P)}{t_1(P)}.\frac{t_1(P)}{t_2(P)}\right)^{\nu_P}=\prod_{P\in L\cap C}(-1)^{3\nu_P}$$
$$=\prod_{P\in L\cap C}(-1)^{\nu_P}=(-1)^{\sum_{P\in L\cap C}\nu_P}=(-1)^n$$

Where we have used that $\sum_{P\in L\cap C}\nu_P=n$ due to the fact that the curve C will cut L exactly n times counting multiplicities. So altogether we have:

$$\prod_{P\in C}(f_2,f_0)_P.\prod_{P\in C}(f_0,f_1)_P.\prod_{P\in C}(f_1,f_2)_P=1$$
$$\prod_{P\in\{t_0=0\}\cap C}\left(\frac{t_2(P)}{t_0(P)}\right)^{\nu_P}\cdot\prod_{P\in\{t_1=0\}\cap C}\left(\frac{t_0(P)}{t_1(P)}\right)^{\nu_P}\cdot\prod_{P\in\{t_1=0\}\cap C}\left(\frac{t_1(P)}{t_2(P)}\right)^{\nu_P}.(-1)^n=1$$

Or

$$\prod_{P\in\{t_0=0\}\cap C}\left(\frac{t_2(P)}{t_0(P)}\right)^{\nu_P}\cdot\prod_{P\in\{t_1=0\}\cap C}\left(\frac{t_0(P)}{t_1(P)}\right)^{\nu_P}\cdot\prod_{P\in\{t_2=0\}\cap C}\left(\frac{t_1(P)}{t_2(P)}\right)^{\nu_P}=(-1)^n$$

Which is precisely what we wanted to prove.

□

References

[1] Lazare Carnot, *Géométrie de Position*, Paris 1803.

[2] Jean-Pierre Serre, *Algebraic Groups and Class Fields*, Springer 1987.