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One Monopole with k Singularities

Chris D. A. Blair
School of Mathematics,
Trinity College, Dublin
cblair@maths.tcd.ie

Sergey A. Cherkis
School of Mathematics and
Hamilton Mathematics Institute,
Trinity College, Dublin
cherkis@maths.tcd.ie

Abstract

We obtain all charge one monopole solutions of the Bogomolny equation with k prescribed Dirac singularities for the gauge groups $U(2), SO(3)$, or $SU(2)$. We analyze these solutions comparing them to the previously known expressions for the cases of one or two singularities.

1 Introduction

The Dirac magnetic monopole [1] is a solution of the $U(1)$ gauge group Bogomolny equation

$$\mathbf{B} + \nabla\phi = 0, \tag{1}$$

where ϕ is a scalar field and $\mathbf{B} = (B_1, B_2, B_3)^{\mathrm{t}}$ is the magnetic field with the one-form potential ω, so that $\epsilon_{abc}B^a dt^b dt^c = d\omega$, where ϵ_{abc} is the Levi-Civita symbol. The basic monopole solution is

$$\phi(\vec{t}\,) = \frac{1}{2|\vec{t}\,|}, \qquad \omega(\vec{t}\,) = \begin{cases} \omega^N_{\vec{T}}(\vec{t}\,) & \text{for } t_3 > 0, \\ \omega^S_{\vec{T}}(\vec{t}\,) & \text{for } t_3 < 0, \end{cases} \tag{2}$$

with

$$\omega^N_{\vec{T}}(\vec{t}\,) = \frac{\big(\vec{T}\times\vec{t}\,\big)\cdot d\vec{t}}{2t\big(Tt + \vec{T}\cdot\vec{t}\,\big)}, \qquad \omega^S_{\vec{T}}(\vec{t}\,) = -\frac{\big(\vec{T}\times\vec{t}\,\big)\cdot d\vec{t}}{2t\big(Tt - \vec{T}\cdot\vec{t}\,\big)}, \tag{3}$$

for any given vector $\vec{T}$. Clearly $\omega^N_{\vec{T}}$ (and $\omega^S_{\vec{T}}$) extend from its domain to the complement of the semi-infinite line $\{\vec{t} = -r\vec{T} | r > 0\}$ (and $\{\vec{t} = r\vec{T} | r > 0\}$ respectively). Since Eq. (1) is linear, it is straightforward to write a solution with k Dirac monopoles with positions $\vec{\nu}_j \in \mathbb{R}^3,\ j = 1, \ldots, k$. If we denote by $\vec{t}_j = \vec{t} - \vec{\nu}_j$ the position relative to the j^{th} point and let $t_j = |\vec{t}_j|$, then the solution is $\phi = \sum_j \frac{1}{2t_j}$ and $\omega = \sum_j \omega\big(\vec{t}_j\big)$ with the vector potentials ω of Eq. (2). Clearly these solutions are singular only at the points $\vec{\nu}_j$.

The first nonabelian monopole solution was found by 't Hooft and Polyakov in [2] and [3]. It is a nonabelian generalization of the Dirac monopole and in the Bogomolny-Prasad-Sommerfield (BPS) limit [4, 5] it can be written exactly:

$$\Phi(\vec{z}\,) = \left(\lambda\coth 2\lambda z - \frac{1}{2z}\right)\frac{\not{z}}{z}, \tag{4}$$

$$A(\vec{z}\,) = \left(\frac{\lambda}{\sinh(2\lambda z)} - \frac{1}{2z}\right)\frac{i[\not{z}, d\not{z}]}{z}, \tag{5}$$

where Φ is the Higgs field and A is the gauge field for the $SU(2)$ gauge group. It is the solution of the Bogomolny equation

$$F_{ab} + \sum_{c=1}^{3}\epsilon_{abc}[D_c, \Phi] = 0, \tag{6}$$

where F the field strength of the gauge field A. As opposed to the abelian Dirac monopole of Eq (2), which is singular, the 't Hooft-Polyakov monopole (4,5) is everywhere smooth. The Bogomolny equation (6) is nonlinear and superimposing its solutions becomes an interesting nonlinear problem.

In this brief note we present solutions to the Bogomolny equation that can be thought of as nonlinear superpositions of one 't Hooft-Polyakov monopole (4,5) with k minimal Dirac singularities (2) embedded into the gauge group.

A general formalism for constructing BPS monopoles was discovered by Nahm in [6, 7, 8]. Singular monopoles were introduced in [9], where their twistor theory and moduli spaces were studied. They play a significant role in quantum gauge theory as first pointed out in [10] and explored in various contexts in e.g. [11],[12], and [13]. Their significance in the geometric Langlands program became apparent after [14].

First singular monopole solutions with nonabelian charge were found in [15] and [16]. These solutions were derived using the conventional Nahm transform of the Nahm data described in [17][1] This Nahm transform technique was limited however to the cases of one or two singularities at most. The reason for this limitation is that the conventional Nahm data for one monopole with k singularities is defined on a real line which is divided by two points $\pm\lambda$ into a finite interval $[-\lambda, \lambda]$ and left and right semi-infinite intervals $(-\infty, -\lambda)$ and $(\lambda, +\infty)$. The Nahm data over the finite interval is rank one is easy to work with, while the Nahm data over the left and right semi-infinite intervals is of respective ranks k_- and k_+ with $k_- + k_+ = k$. For $k_\pm > 1$ such data has not yet been constructed explicitly, and even if found, would be difficult to work with when performing the Nahm transform. Until now this difficulty precluded any derivation of a singular monopole with more than two singularities.

We circumvent this limitation by employing bow diagrams and a generalization of the Nahm transform presented in [18, 19, 20] and in particular their Cheshire representations [21]. Our method is based on the observation of Kronheimer [9] that an instanton on a multi-Taub-NUT space that is invariant under the isometry of the Taub-NUT is equivalent to a singular monopole. The bow formalism of [18, 19] was developed to construct all instantons on multi-Taub-NUT space. In [21] we identify the bow representations that give rise to the instantons invariant under the multi-Taub-NUT isometry. Since these are the representations which have one of the ranks equal to zero we call them Cheshire representations. The detailed derivation

[1] The notion of the nonabelian charge of the singular monopole is also defined in [17].

of the results we present here shall appear in [21]. In this letter we limit ourselves to giving the explicit general one monopole solution with any number of singularities for the gauge groups $U(2), SO(3)$, and $SU(2)$.

2 Solutions

We place the singularities at some k distinct points with $\vec{t} = \vec{\nu}_j,\ j = 1, 2, \ldots, k$. The position relative to the j^{th} singularity is $\vec{t}_j = \vec{t} - \vec{\nu}_j$. The nonabelian monopole position parameter is $\vec{T}$, which approximately corresponds to the negative of the monopole position. Let $\vec{T}_j = \vec{T} + \vec{\nu}_j$ and $T_j = |\vec{T} + \vec{\nu}_j|$. By $\vec{z} = \vec{t} + \vec{T}$ we denote the position relative to the monopole.

For any three-vector $\vec{a}$ we use its projection $\vec{a}_\perp$ on the plane orthogonal to $\vec{z}$, that is $\vec{a}_\perp \equiv \vec{a} - \frac{\vec{a}\cdot\vec{z}}{z}\frac{\vec{z}}{z}$, and we denote the length of $\vec{a}$ by $a = |\vec{a}|$. We also use the conventional notation $\not{a}$ to denote $\vec{a} \cdot \vec{\sigma} = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$, where σ_1, σ_2, and σ_3 are the Pauli matrices. Since one common combination that enters our solution is $T_j + t_j + \not{z}$, we introduce the following functions

$$\mathcal{P}_j = \sqrt{(t_j + T_j)^2 - z^2} = \sqrt{2\big(T_j t_j - \vec{T}_j \cdot \vec{t}_j\big)} \quad \text{and} \quad \alpha_j = \frac{1}{4z} \ln \frac{T_j + t_j + z}{T_j + t_j - z}, \tag{7}$$

so that $T_j + t_j \pm \not{z} = \mathcal{P}_j e^{\pm 2\alpha_j \not{z}}$. Also let the sum of all the α_j functions be $\alpha = \alpha(\vec{t}) = \sum_{j=1}^k \alpha_j$.

2.1 $U(2)$ and $SO(3)$ Monopoles

For a $U(2)$ singular monopole each minimal singularity has a sign associated to it [13], so that its charge e_j is $+1$ or -1 depending on whether one of the Higgs field eigenvalues approaches $+$ or $-$ infinity respectively as one approaches the singularity $\vec{\nu}_j$. For a singularity at $\vec{t} = \vec{\nu}_j$ let $\omega_j = \omega^S_{\vec{T}_j}(\vec{t}_j)$, with the one-form $\omega^S_{\vec{T}}(\vec{t})$ given in Eq. (3). The one $U(2)$ monopole solution derived in [21] can easily be generalized to the case of minimal singularities of arbitrary charge $e_j = \pm 1$ giving

$$\Phi = \sum_{j=1}^k \frac{e_j}{4t_j} + \vec{\Phi} \cdot \vec{\sigma}, \qquad\qquad A = \sum_{j=1}^k \frac{e_j}{2}\omega_j + \vec{A} \cdot \vec{\sigma}, \tag{8}$$

where

$$\vec{\Phi} = \left(\left(\lambda + \sum_{j=1}^{k} \frac{1}{4t_j} \right) \coth 2(\lambda+\alpha)z - \frac{1}{2z} \right) \frac{\vec{z}}{z} + \frac{z}{\sinh 2(\lambda+\alpha)z} \sum_{j=1}^{k} \frac{1}{2t_j \mathcal{P}_j^2} \vec{T}_{j\,\perp}. \quad (9)$$

$$\begin{aligned} \vec{A} &= \frac{1}{z} \left(\frac{1}{\sinh 2(\lambda+\alpha)z} \left[\lambda + \sum_{j=1}^{k} \frac{T_j + t_j}{2\mathcal{P}_j^2} \right] - \frac{1}{2z} \right) \vec{z} \times d\vec{t} \\ &+ \sum_{j=1}^{k} \frac{\omega_j}{2} \coth 2(\lambda+\alpha)z \frac{\vec{z}}{z} - \frac{z}{\sinh 2(\lambda+\alpha)z} \sum_{j=1}^{k} \frac{1}{2\mathcal{P}_j^2 t_j} \left(\vec{t}_j \times d\vec{t} \right)_{\perp}. \end{aligned} \quad (10)$$

Stripping off the trace part of this solution one obtains a solution $\Phi = (\Phi_{ab})$ and $A = (A_{ab})$ for the singular monopole with the $SO(3)$ gauge group with

$$\Phi_{bc} = \epsilon_{abc} \Phi^c, \qquad A_{ab} = \epsilon_{abc} A^c. \quad (11)$$

Here Φ^c and A^c denote the components of the vectors $\vec{\Phi}$ and $\vec{A}$ of Eqs. (9) and (10) above.

2.2 $SU(2)$ Monopole

By bringing the singular points of opposite charges together in pairs in the $U(2)$ solution (8), we obtain the singular monopole solution for the $SU(2)$ gauge group

$$\Phi = \left(\left(\lambda + \sum_{j=1}^{k} \frac{1}{2t_j} \right) \coth 2(\lambda+2\alpha)z - \frac{1}{2z} \right) \frac{\not{z}}{z} + \frac{z}{\sinh 2(\lambda+2\alpha)z} \sum_{j=1}^{k} \frac{1}{t_j \mathcal{P}_j^2} \not{T}_{j\,\perp}. \quad (12)$$

$$\begin{aligned} A &= \frac{i}{2z} [\not{z}, d\not{t}] \left(- \frac{1}{\sinh 2(\lambda+2\alpha)z} \left[\lambda + \sum_{j=1}^{k} \frac{T_j + t_j}{\mathcal{P}_j^2} \right] + \frac{1}{2z} \right) \\ &+ \sum_{j=1}^{k} \omega_j \frac{\not{z}}{z} \coth 2(\lambda+2\alpha)z + \frac{z}{\sinh 2(\lambda+2\alpha)z} \sum_{j=1}^{k} \frac{i}{2\mathcal{P}_j^2 t_j} [\not{t}_j, d\not{t}]_{\perp}. \end{aligned} \quad (13)$$

3 Exploring the solutions

Here we study various limits and special points of our solutions verifying the expected behavior and comparing to the solutions known earlier.

3.1 At the location of the monopole

Let us begin by establishing the regularity of our solutions at $z = 0$. Since the term $z/\sinh 2(\lambda+\alpha)z$ has a regular limit, the only potentially divergent terms are

$$\left(\lambda + \sum_{j=1}^{k} \frac{1}{4t_j}\right) \coth 2(\lambda+\alpha)z - \frac{1}{2z} \tag{14}$$

and

$$\frac{1}{\sinh 2(\lambda+\alpha)z} \left[\lambda + \sum_{j=1}^{k} \frac{T_j + t_j}{2\mathcal{P}_j^2}\right] - \frac{1}{2z}. \tag{15}$$

Since $T_j = t_j - \vec{z}\cdot\vec{t}_j/t_j + O(z^2)$ we conclude from the definition of α_j that $\alpha_j = \frac{1}{4t_j} + O(z)$. Thus in all of the above solutions the $\frac{1}{2z}$ terms is canceled by the singular term in the expansion of term containing coth or sinh and the whole expression is regular, as expected.

3.2 At the singularities

Since $\vec{t}_j = \vec{t} - \vec{\nu}_j$ and $\vec{z} = \vec{t}_j + \vec{T}_j$ we have

$$4z\alpha_j = \log \frac{2T_j + O(t_j)}{t_j - \vec{T}_j\cdot\vec{t}_j/T_j + O(t_j^2)}, \tag{16}$$

and

$$\coth 2(\lambda+\alpha)z = \frac{1+e^{-4(\lambda+\alpha)z}}{1-e^{-4(\lambda+\alpha)z}} = 1 + \frac{t_jT_j - \vec{T}_j\cdot\vec{t}_j}{T_j^2} e^{-4\left(\lambda+\sum_{i\neq j}\alpha_i\right)z} + O(t_j^2). \tag{17}$$

Thus the singularity of the Higgs field as $\vec{t} \to \vec{\nu}_j$ is

$$U(2): \qquad \Phi = \frac{1}{4t_j}\left(1 + \frac{\vec{T}_j}{T_j}\right) + O(t_j^0), \tag{18}$$

$$SO(3): \qquad \Phi_{ab} = \frac{1}{4t_j}\epsilon_{abc}\frac{T_j^c}{T_j} + O(t_j^0), \tag{19}$$

$$SU(2): \qquad \Phi = \frac{1}{2t_j}\frac{\vec{T}_j}{T_j} + O(t_j^0). \tag{20}$$

3.3 Apparent Dirac String

Since our expressions for the monopole solutions contain terms with $\mathcal{P}_j^2 = (T_j + t_j)^2 - z^2 = 2(t_j T_j - \vec{t}_j \cdot \vec{T}_j)$ one can expect them to be singular along the line $L_j : \left\{ \vec{t}_j | \vec{t}_j = r\vec{T}_j, r > 0 \right\}$. For concreteness let us consider the term

$$\frac{z}{\sinh 2(\lambda + 2\alpha)z} \frac{1}{2t_j \mathcal{P}_j^2} \vec{T}_{j\,\perp}, \tag{21}$$

in the expression for the $SU(2)$ monopole. As we approach the line L_j we have $\mathcal{P}_j \to 0, |\vec{T}_{j\,\perp}| \to 0$, and $\sinh 2(\lambda + 2\alpha)z \to \infty$. To find the leading behavior of there terms use

$$\sinh 2(\lambda + 2\alpha)z = \frac{1}{2} \left(e^{2\lambda z} \prod_j \frac{T_j + t_j + z}{T_j + t_j - z} - e^{-2\lambda z} \prod_j \frac{T_j + t_j - z}{T_j + t_j + z} \right) \tag{22}$$

$$\to 2e^{2\lambda z} \left(\frac{T_j + t_j}{\mathcal{P}_j} \right)^2 \prod_{\substack{i=1 \\ i \neq j}}^{k} \frac{T_i + t_i + T_j + t_j}{T_i + t_i - T_j - t_j}. \tag{23}$$

This leads to a regular limit along L_j.

All of our solutions are written in a gauge that is partial to the non-abelian monopole; this results in the appearance of apparent Dirac strings L_j. There is a simple gauge transformation that is more democratic making the solutions everywhere regular except at the points $\vec{\nu}_j$.

Focussing on one pure singularity, in the Dirac form it is

$$\Phi_D = \phi(\vec{t}_j) \frac{\not{T}_j}{T_j}, \qquad A_D = \omega(\vec{t}_j) \frac{\not{T}_j}{T_j}, \tag{24}$$

with ϕ and ω given by Eq. (2), while in the Wu-Yang form [22, 23], which makes sense globally and has no Dirac strings,

$$\Phi_{WY} = -\frac{1}{2t_j} \frac{\not{t}_j}{t_j}, \qquad A_{WY} = -i \frac{[\not{t}_j, d\not{t}_j]}{2t_j^2}. \tag{25}$$

The gauge transformation relating these two solutions is

$$g_j = \frac{\sqrt{T_j t_j}}{\mathcal{P}_j} \left(\frac{\not{t}_j}{t_j} - \frac{\not{T}_j}{T_j} \right). \tag{26}$$

This g_j is both unitary and Hermitian and thus $g_j = \vec{n}_j \cdot \vec{\sigma}$ with the unit vector $\vec{n}_j = \frac{\sqrt{T_j t_j}}{\mathcal{P}_j} \left(\vec{t}_j / t_j - \vec{T}_j / T_j \right)$. So it has the form $ig_j = \exp(i\frac{\pi}{2} g_j)$. Thus if we find some vector-valued function $\vec{h}$ such that as $\vec{t} \to \vec{\nu}_j$ we have $\vec{h} \to \vec{n}_j$ then the gauge transformation

$$g = \exp(i\frac{\pi}{2}\not{h}), \tag{27}$$

puts the solutions we have in a nonsingular form with Wu-Yang form of the singularities.

For example let $\vec{h} = \vec{H}/f$, with any function f satisfying $\lim_{\vec{t}\to\vec{\nu}_j} f = \frac{\sqrt{T_j t_j}}{\mathcal{P}_j}$. Some possible choices are $\vec{H} = \frac{\vec{z}}{z} - \nabla\frac{1}{\sum\frac{1}{t_j}}$, or $\vec{H} = \frac{\vec{z}}{z} - \frac{1}{\sum\frac{1}{t_j}}\sum\frac{1}{t_j}\frac{\vec{t}_j}{t_j}$, and $f = \left(\sum_j \frac{1}{\mathcal{P}_j}\sqrt{\frac{T_j}{t_j}}\right)\Big/\sum\frac{1}{t_j}$.

3.4 Charges measured at infinity

As $\vec{t}$ tends to infinity $\coth 2(\lambda+\alpha)z$ and $\coth 2(\lambda+2\alpha)z$ tend to one up to terms exponentially small terms containing $\exp(-4\lambda|\vec{t}|)$, while $\sinh 2(\lambda+\alpha)z$ and $\sinh 2(\lambda+2\alpha)z$ grow exponentially. Thus the $U(2)$ Higgs field at infinity has the form

$$U(2): \qquad \Phi = \sum_{j=1}^{k}\frac{e_j}{4t_j} + \left(\lambda - \frac{1}{2z} + \sum_{j=1}^{k}\frac{1}{4t_j}\right)\frac{\not{z}}{z} + o(e^{-4\lambda z}), \tag{28}$$

with the eigenvalues behavior $\mathrm{EigVal}(\Phi) = \left(\lambda - \frac{1-k_+}{2t}, -\lambda + \frac{1-k_-}{2t}\right)$, with k_- and k_+ the number of singularities with $e_j = -1$ and $e_j = 1$ respectively. This exactly corresponds to the nonabelian charge one configuration as defined in [13].

For the remaining two cases

$$SO(3): \qquad \Phi_{ab} = \left(\lambda - \frac{1}{2z} + \sum_{j=1}^{k}\frac{1}{4t_j}\right)\epsilon_{abc}\frac{z^c}{z} + o(e^{-4\lambda z}) \tag{29}$$

$$= \left(\lambda + \frac{k-2}{4t}\right)\epsilon_{abc}\frac{t^c}{t} + O(t^{-2}), \tag{30}$$

$$SU(2): \qquad \Phi = \left(\lambda - \frac{1}{2z} + \sum_{j=1}^{k}\frac{1}{2t_j}\right)\frac{\not{t}}{t} + o(e^{-4\lambda z}) \tag{31}$$

$$= \left(\lambda + \frac{k-1}{2t}\right)\frac{\not{t}}{t} + O(t^{-2}). \tag{32}$$

so the total charge measured at infinity is $\frac{1}{2}k-1$ for the $SO(3)$ case and $k-1$ for the $SU(2)$ case and, since we have k charge $\frac{1}{2}$ minimal singularities in $SO(3)$ and k charge 1 minimal singularities in $SU(2)$ the nonabelian charge equals to one, as expected.

3.5 Removing the Singular Points

If we remove one of the singularities by sending $\vec{\nu}_k \to \infty$, then T_k and $t_k \to \infty$ and $\alpha_k \to 0$. As a result α reduces to the expression for the case with $k-1$ singularity, while all the terms associated with the removed singularity vanish. This procedure relates a solution with k singularities to the solutions with any lower number of singularities. In particular, removing all of the singularities one recovers the original BPS limit of the 't Hooft-Polyakov monopole.

In order to compare to the solutions with one singularity [16] or two singularities [15] it suffices to observe that in general

$$\sinh 2\alpha z = \frac{1}{2}\frac{1}{\mathcal{P}_1 \ldots \mathcal{P}_k}\left(\prod_{j=1}^{k}(T_j + t_j + z) - \prod_{j=1}^{k}(T_j + t_j - z)\right), \qquad (33)$$

$$\cosh 2\alpha z = \frac{1}{2}\frac{1}{\mathcal{P}_1 \ldots \mathcal{P}_k}\left(\prod_{j=1}^{k}(T_j + t_j + z) + \prod_{j=1}^{k}(T_j + t_j - z)\right). \qquad (34)$$

Using these our solutions with $k=1$ or 2 reduce to those of [16] and [15].

4 Conclusions

The moduli space of the $U(2)$ and $SO(3)$ singular monopoles we found here is the k-centered Taub-NUT space, while in the case of $SU(2)$ singular monopole it is the $2k$-centered Taub-NUT space with these centers arranged into k degenerate pairs. As a result this space is singular with k A_1 singularities. Even though the moduli spaces of singular monopoles were well studied explicit singular monopole solutions were scarce. The conventional Nahm transform for singular monopoles was effective in obtaining one monopole solutions with at most two singularities. It is substantially more difficult to use it in order to obtain a monopole solution with arbitrary number of singularities. We are able to circumvent these difficulties by employing the bow formalism. The resulting explicit singular monopole solutions for $U(2), SO(3)$, and $SU(2)$ gauge groups are presented here and their properties analyzed. Our technique can be used to find explicitly the charge $(1, 1, \ldots, 1)$ monopole in $U(n)$ with any number of minimal singularities.

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