Topological Quantum Field Theories
and
Some Modern Problems of Mathematics

PATHWAYS LECTURE SERIES IN MATHEMATICS

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Plan

• Formulate mathematical, enumerative, problem
• Define physical, QFT, problem which computes same thing
• Explain physics role of this QFT and relation to Math
• Formulate answers (in some cases but not all)
• Discuss the results

Examples

LECTURE I
1. Gromov-Witten theory - Top. Sigma Model, type A

2. Deformation theory - complex structures: Top. B model

3. Donaldson Theory - twisted N=2 SYM on four manifold.

LECTURE II

4. Intersection theory on moduli space of flat connections over Riemann surface - Topological YM theory on Riemann surface ⇒ CS and $G/G$ WZW.

5. Intersection theory on moduli space of Higgs Bundles (Hitchin system) - Topological YM-Higgs theory ⇔ Nonlinear Schrödinger theory and representation theory of Quantum Groups ⇒ CS and $G/G$ WZW for complexified $G$. 

Main Math application can be summarized by:

**Mirror formula**

- Type A sigma model on $V = $ Type B sigma model on $\tilde{V}$
- Relates GW on $V$ to (gen)deformations of cmplx str on $\tilde{V}$
- Manifolds $V$ and $\tilde{V}$ are called mirrors.
- *For Kähler manifolds:* $h^{p,q}(V) = h^{-p,q}(\tilde{V})$
- The concept extends to: $V$ symplectic and $\tilde{V}$ complex.
- Mirror exchanges kähler (A) and complex (B) deformations.

$$\sum_{n;\{k_1,\ldots,k_n\}} \frac{T^{k_1} \cdots T^{k_n}}{n!} \left\langle \mathcal{O}_{a}^{(0)} \mathcal{O}_{b}^{(0)} \mathcal{O}_{c}^{(0)} \int_{\Sigma} \mathcal{O}_{k_1}^{(2)} \cdots \int_{\Sigma} \mathcal{O}_{k_n}^{(2)} \right\rangle_A = \frac{\partial^3 \mathcal{F}_B (T)}{\partial T^a \partial T^b \partial T^c}$$
Type A sigma models: Gromov – Witten theory

Two dimensional sigma model - maps

\[ \Phi : \Sigma \to V \]

\( \Sigma \) - two dimensional manifold, world-sheet

\( V \) - some Riemannian manifold.

Let \( V \) be complex manifold.

- Mathematical reformulation of what physicists call partition function in the topological type A sigma model:

Given a set of submanifolds \( C_1, \ldots, C_k, C_i \subset V \), compute the number \( N_{C_1, \ldots, C_k; \beta} \) of rigid genus \( g \) holomorphic curves \( \Sigma \subset V \), \( [\Sigma] = \beta \in H_2(V; \mathbb{Z}) \) passing through them

The cycles in \( H_*(V) \) represented by \( C_1, \ldots, C_k \) are Poincare dual to some cohomology classes \( \omega_1, \ldots, \omega_k \in H^*(V) \).
(Supersymmetric) Sigma model - defined through action functional ⇔ functional of “fields”: $\phi^I, \psi_+, \psi_-$. 

$\Phi$ - a map: ($\Sigma$ - Riemann surface) $\rightarrow$ ($V$ - Riemannian manifold of metric $g_{IJ}$).

Pick local coordinates: on $\Sigma$ - $z, \bar{z}$, on $V$ - $\phi^I$.

- $\phi^I$: Map $\Phi$ - locally described by $\phi^I(z, \bar{z})$.

$K$ ($\overline{K}$) - the canonical (anti-canonical) line bundles of $\Sigma$ (the bundle of one forms of types $(1,0)$ ($(0,1)$))

$TV$ - complexified tangent bundle of $V$.

To get supersymmetry $\Rightarrow$ add Grassmann variables:

- $\psi^I_+$ - a section of $K^{1/2} \otimes \Phi^*(TV)$
- $\psi^I_-$ - a section of $\overline{K}^{1/2} \otimes \Phi^*(TV)$. 
Physical Sigma Model:

\[ S_0(\phi, \psi_-, \psi_+) = \frac{1}{f^2} \int \Sigma \left( \frac{1}{2} g_{IJ}(\phi) \partial_z \phi^I \partial_\bar{z} \phi^J + \frac{i}{2} g_{IJ} \psi_+^I D_z \psi_-^J \right) + \frac{i}{2} g_{IJ} \psi_-^I D_z \psi_+^J + \frac{1}{4} R_{IJKL} \psi_-^I \psi_+^J \psi_+^K \psi_-^L \right) \]

\( f^2 \) - coupling constant

\( R_{IJKL} \) - Riemann tensor of \( V \).

\( D_z \) - \( \bar{\partial} \) operator on \( K^{1/2} \otimes \Phi^*(TV) \) constructed using the pullback of the Levi-Civita connection on \( TV \).

- **Now suppose** \( V \) is Kähler

Sigma model has extended SUSY: \( \mathcal{N} = 2 \).

Map \( \Phi \rightarrow \) local coordinates: \( \phi^i, \bar{\phi}^i = \phi^i \).

Decompose: \( TV = T^{1,0}V \oplus T^{0,1}V \).

\( \psi_+^i (\psi_{+}^i) \) - the projection of \( \psi_+ \) in:

\[ K^{1/2} \otimes \Phi^*(T^{1,0}V) \quad (K^{1/2} \otimes \Phi^*(T^{0,1}V)) \]

\( \psi_-^i (\psi_{-}^i) \) - the projections of \( \psi_- \) in:

\[ \bar{K}^{1/2} \otimes \Phi^*(T^{1,0}V) \quad (\bar{K}^{1/2} \otimes \Phi^*(T^{0,1}V)) \]
Action has more parameters:

$$S_0 = i\theta \int_\Sigma \frac{1}{2} g_{ij} \left( \partial_z \phi^i \partial_{\bar{z}} \phi^j - \partial_{\bar{z}} \phi^i \partial_z \phi^j \right) + \frac{1}{f^2} \int_\Sigma \frac{1}{2} g_{IJ} \partial_z \phi^I \partial_{\bar{z}} \phi^J +$$

$$+ i\bar{\psi}_- D_z \psi_- g_{ii} + i\bar{\psi}_+ D_{\bar{z}} \psi_+ g_{\bar{i}\bar{i}} + R_{i\bar{i}j\bar{j}} \psi^i \psi^j\psi_+\psi_- + \theta - \text{another parameter, theta-angle.}$$

Twist:

$$+$$ : $\psi_+^i$ and $\psi_+^\bar{i}$ - sections of $\Phi^*(T^{1,0}X)$ and $K \otimes \Phi^*(T^{0,1}X)$.  

$$-$$ : $\psi_+^i$ and $\psi_+^\bar{i}$ - sections of $K \otimes \Phi^*(T^{1,0}X)$ and $\Phi^*(T^{0,1}X)$.  

A Model: $+$ twist of $\psi_+$ and a $-$ twist of $\psi_-$.  

B Model: $-$ twists of both $\psi_+$ and $\psi_-$. 

Locally the twisting does nothing at all, since locally $K$ and $\overline{K}$ are trivial.
• $\chi$ - section of $\Phi^*(TX)$ ( $\chi^i = \psi_+^i$, and $\chi^i = \psi_-^i$);

• $\psi_+^i$ - (1, 0) form on $\Sigma$ with values in $\Phi^*(T^{0,1}X)$; $\psi_+^i = \psi_z^i$.

• $\psi_-^i$ - (0, 1) form with values in $\Phi^*(T^{1,0}X)$; $\psi_-^i = \psi_z^i$.

Topological transformation laws:

$$
\delta \Phi^I = i\chi^I
$$

$$
\delta \chi^I = 0
$$

$$
\delta \psi_z^i = -\partial_z \phi^i - i\chi^j \Gamma_{j\bar{m}} \psi_z^m
$$

$$
\delta \psi_z^i = -\partial_{\bar{z}} \phi^i - i\chi^j \Gamma_{jm} \psi_z^m.
$$

$\delta^2 = 0$ - on the space of solutions of equations of motion (minimizing the action). Can be made ”off-shell” by introducing auxiliary fields.

Let $t = \theta + \frac{i}{f^2}$.

**Action:**

$$
S_0 = \frac{1}{f^2} \int \Sigma d^2 z \delta R + t \int \Sigma \Phi^*(\omega)
$$

$$
R = g_{i\bar{j}} \left( \psi_z^i \partial_z \phi^j + \partial_z \phi^i \psi_z^j \right),
$$

$$
\int \Sigma \Phi^*(\omega) = i \int \Sigma d^2 z \left( \partial_z \phi^i \partial_z \phi^j g_{i\bar{j}} - \partial_{\bar{z}} \phi^i \partial_{\bar{z}} \phi^j g_{i\bar{j}} \right)
$$

– the integral of the pullback of the Kähler form $\omega = -ig_{i\bar{j}} dz^i d\bar{z}^j$.

$$
\int \Phi^*(\omega) - \text{depends only on the cohomology class of } \omega \text{ and the homology class } \beta \in H_2(V) \text{ of the image of the map } \Phi.
$$
In physics one computes correlation functions of some operators (observables) in given theory.

**Definition.** Observable \( \{O_i\} \) – a functional of the fields, s.t. \( \delta O_i = 0 \).

**Definition.** Physical observable = a \( \delta \) - cohomology class, \( O_i \sim O_i + \delta \Psi_i \).

**Definition.** Correlator - path integral:

\[
\langle \prod_a O_a \rangle_\beta = e^{-2\pi t} \int_\omega \int_{B_\beta} D\phi \ D\chi \ D\psi \ e^{-\frac{1}{t^2} \delta \int R} \ \prod_a O_a .
\]

\( B_\beta \) - the component of the field space for maps of degree \( \beta = [\Phi(\Sigma)] \in H_2(V, \mathbb{Z}) \), and \( \langle \rangle_\beta \) - degree \( \beta \) contribution to the expectation value.

**Correlators of the observables depend only on their \( \delta \)-cohomology class, in particular — independent of the complex structure of \( \Sigma \) and \( V \), and depend only on the cohomology class of the Kähler form \( \omega \).**
Standard argument: $\delta \sim$ exterior derivative on the field space $\mathcal{B} \to \langle \delta \Psi \rangle_\beta = 0$ for any reasonable $\Psi$. Thus, the $\mathcal{O}_i$ should be considered as representatives of the $\delta$-cohomology classes.

Thus, correlator is independent of $f^2$. If $f^2 \to 0$ - Gaussian model.

Bosonic part of the Action

$$it \int \Phi^*(\omega) + \frac{1}{f^2} \int \Sigma g_{i\bar{j}}(\phi) \partial_z \phi^j \partial_{\bar{z}} \phi^i$$

for given $\beta$ is minimized by holomorphic map:

$$\partial_z \phi^i = \partial_{\bar{z}} \phi^\bar{i} = 0.$$

The entire path integral, for maps of degree $\beta$, reduces to an integral over the space of degree $\beta$ holomorphic maps $\mathcal{M}_\beta$. 
• Descend procedure

Pick an $n$-form $W = W_{I_1 I_2 \ldots I_n} (\phi) d\phi^{I_1} \wedge d\phi^{I_2} \wedge \ldots \wedge d\phi^{I_n}$ on $V \Rightarrow$ a local functional

$$\mathcal{O}_W (P) = W_{I_1 I_2 \ldots I_n} (\Phi(P)) \chi^{I_1} \ldots \chi^{I_n}(P).$$

$$\delta \mathcal{O}_W = -\mathcal{O}_{dW},$$

d the exterior derivative on $V$.

$\Rightarrow W \mapsto \mathcal{O}_W$ - natural map from the de Rham cohomology of $V$ to the space of physical observables, $\delta$-cohomology, of quantum field theory $A(V)$. For local operators - isomorphism.

Let $d$ be the DeRham differential on $\Sigma$. We have descend equations:

$$d\mathcal{O}_W = \delta \mathcal{O}_W^{(1)}, \quad \oint_C \mathcal{O}_W^{(1)}$$ - 1-observable. The physical observable depends on the homology class of $C$ in $H_1(\Sigma)$.

$$d\mathcal{O}_W^{(1)} = \delta \mathcal{O}_W^{(2)}, \quad \int_\Sigma \mathcal{O}_W^{(2)}$$ - 2-observable.

**Deformations of the theory:** change the action as follows:

$$S_A(T) = S_0 + T^a \int_\Sigma \mathcal{O}_W^a$$

$T^a$ are the formal parameters (nilpotent). The path integral with the action $S_T$ computes the generating function $\mathcal{F}_A(T)$ of the correlation functions of the two-observables:

$$\mathcal{F}_A(T) = \langle e^{-\int_\Sigma S(T)} \rangle$$

$$S(0) = S_0, \quad \frac{\partial S}{\partial T^a} \bigg|_{T=0} = \int_\Sigma \mathcal{O}_W^a$$
Reduction to the enumerative problem

$C$ - submanifold of $V$ (only its homology class matters).

The "Poincaré dual" to $C$ - cohomology class that counts intersections with $C$. Represent by a differential form $W(C)$ that has delta function support on $C$:

$$W(C) = \delta_C$$

Conclude:

Correlators of topological observables $O_{W(C_1)} \ldots O_{W(C_k)}$ are integrals over $M_\beta$ of the products of delta functions which pick out the holomorphic maps whose image intersects the submanifolds $C_1, \ldots, C_n$:

Let $C_1, \ldots, C_k \subset V$ - complex submanifolds, $\dim C_l = d_l$.

$\omega_m = W(C_m) \in H^*(V)$ - their Poincare duals.

Let $z_1, \ldots, z_m \in \Sigma$, $m \leq k$ be the marked points.

For a complex submanifold $C \subset V$ and for $1 \leq l \leq m$ define the following submanifolds $M_{C,l}^0 \subset M$, $M_C^2 \subset M$:

**Definition.** $M_{C,l}^0 = \{ \Phi : \Sigma \to V | \Phi \in M, \Phi(z_l) \in C \}$

**Definition.** $M_C^2 = \{ \Phi : \Sigma \to V | \Phi(\Sigma) \cap C \neq \emptyset \}$
The correlation functions in the type A sigma model are simply the intersection numbers:

$$\langle \mathcal{O}_{C_1}(z_1) \cdots \mathcal{O}_{C_m}(z_m) \int_{\Sigma} \mathcal{O}_{C_{m+1}}^{(2)} \cdots \int_{\Sigma} \mathcal{O}_{C_k}^{(2)} \rangle =$$

$$\# \mathcal{M}_{C_1,1}^0 \cap \ldots \mathcal{M}_{C_m,m}^0 \cap \mathcal{M}_{C_{m+1}} \cap \ldots \cap \mathcal{M}_{C_k}^2$$

$$\sum \dim \mathcal{M}_{C_i,i}^0 + \sum \dim \mathcal{M}_{C_i}^2 = \dim \mathcal{M}_{\beta}$$

otherwise $$\langle \ldots \rangle$$ vanishes,

$$\dim \mathcal{M}_{\beta} = \int_{\beta} c_1(V) + (1 - g) \dim V$$
**Problem:** $\mathcal{M}_\beta$ is non-compact. Need to compactify it in order to get a nice intersection theory.

**Compactification is not unique.**

Option I. Kontsevich stable maps.

Option II. Freckleds – in case where $V$ is a symplectic quotient of a $G$-equivariant submanifold of a vector (affine) symplectic space $A$: $V \subset A//G$.

<table>
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<th><strong>Compactification of $\mathcal{M}$ - Regularization</strong></th>
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Non-compactness of $\mathcal{M}$ comes from ultraviolet non-compactness of the fields space $\mathcal{B}$. ($\text{UV} = ||d\Phi||^2 \to \infty$)

**Physical picture**

Option I = coupling to topological gravity $\approx$ averaging over conformal structures on $\Sigma$.

Option II = gauged linear sigma model with target $A$ and gauge group $G$ (and perhaps superpotential).
Type B sigma models: Kodaira-Spencer theory.

Consider the space $S$ of generalized (in the sense of Kontsevich-Witten) deformations of complex structures of variety $\tilde{V}$ ($\tilde{V}$ - mirror to $V$).

The tangent space to $S$ at some point $s$ represented by a variety $V'_s$ is given by:

$$T_sS = \bigoplus_{p,q} H^p(\tilde{V}_s, \Lambda^q T_{V_s}) \equiv \bigoplus_{p,q} H^{-q,p}(\tilde{V}_s)$$

Let $T$ denote special coordinates on this space.

The right-hand side of the mirror formula - essentially a partition function in type B sigma model expressed in terms of special coordinates, whose choice is absolutely necessary for the formulation of mirror symmetry.

**Note:** genus dependence doesn’t enter in this definition. Precise mathematical definition of $\mathcal{F}_g^B(T)$ is not known.
Physical Picture

$\psi^i_{\pm}$ - sections of $\Phi^*(T^{0,1}\tilde{V})$

$\psi^i_+$ - section of $K \otimes \Phi^*(T^{1,0}\tilde{V})$

$\psi^i_-$ - section of $\overline{K} \otimes \Phi^*(T^{1,0}\tilde{V})$.

$\rho$ - one form with values in $\Phi^*(T^{1,0}\tilde{V})$; $\rho^i_z = \psi^i_+$, $\rho^i_{\bar{z}} = \psi^i_-$. 

*all fields above are valued in Grassmann algebra*

Denote:

\[
\eta^i = \psi^i_+ + \psi^i_-
\]

\[
\theta_i = g_{i\bar{i}} \left( \psi^i_+ - \psi^i_- \right).
\]

Transformations:

\[
\delta \phi^i = 0
\]

\[
\delta \phi^i = i \eta^i
\]

\[
\delta \eta^i = \delta \theta_i = 0
\]

\[
\delta \rho^i = -d\phi^i.
\]

nilpotent symmetry: $\delta^2 = 0$ on-shell, on the solutions of the equations of motion (minimizing the action functional). Can be made off-shell by introducing extra fields.
Action:

\[
S = \frac{1}{f^2} \int_{\Sigma} d^2 z \left( g_{IJ} \partial_z \phi^I \partial_{\bar{z}} \phi^J + i \eta^i (D_z \rho_z^i + D_{\bar{z}} \rho_{\bar{z}}^i) g_{i\bar{i}} 
+ i \theta_i (D_{\bar{z}} \rho_{\bar{z}}^i - D_z \rho_z^i) + R_{i\bar{i}j\bar{j}} \rho_z^i \rho_{\bar{z}}^j \eta^i \theta_k g^{k\bar{j}} \right).
\]

Again one can rewrite the action using \( \delta \):

\[
S = \frac{1}{f^2} \int \delta U + S_0
\]

\[
U = g_{ij} \left( \rho_z^i \partial_z \phi^j + \rho_z^j \partial_z \phi^i \right)
\]

\[
S_0 = \int_{\Sigma} \left( -\theta_i D \rho^i - \frac{i}{2} R_{i\bar{i}j\bar{j}} \rho^i \wedge \rho^j \eta^i \theta_k g^{k\bar{j}} \right).
\]

As in A model - define the observables as:

**Definition.** Observable \( \{ \mathcal{O}_i \} \) – a functional of the fields, s.t. \( \delta \mathcal{O}_i = 0 \).

**Definition.** Physical observable = a \( \delta \) - cohomology class, \( \mathcal{O}_i \sim \mathcal{O}_i + \delta \Psi_i \).
Correlators

\[ \langle \prod_a O_a \rangle = \int_{B_\beta} D\phi \ D\rho \ D\eta \ e^{-\frac{1}{i^2} \delta \int U - S_0} \cdot \prod_a O_a. \]

\( B \) theory is independent of the complex structure of \( \Sigma \) and the Kähler metric of \( \tilde{V} \). Change of complex structure of \( \Sigma \) or Kähler metric of \( \tilde{V} \) - Action changes by irrelevant terms of the form \( \delta(\ldots) \).

**The theory depends on the complex structure of \( \tilde{V} \), which enters \( \delta \)**

\( B \) model is independent of \( f^2 \); take limit \( f^2 \to 0 \); In this limit, one expands around minima of the bosonic part of the Action = constant maps \( \Phi : \Sigma \to \tilde{V} \):

\[ \partial_z \phi_i = \partial_{\bar{z}} \phi_i = 0 \]

The space of such constant maps is a copy of \( \tilde{V} \); the path integral reduces to an integral over \( \tilde{V} \).

All above can be demonstrated by considerations similar to those in \( A \)-model.
Observables:

Consider \((0, p)\) forms on \(\tilde{V}\) with values in \(\wedge^q T^{1,0} \tilde{V}\), the \(q^{th}\) exterior power of the holomorphic tangent bundle of \(\tilde{V}\).

\[
W = d\bar{z}^i d\bar{z}^j \ldots d\bar{z}^p W_{\bar{i}_1 \bar{i}_2 \ldots \bar{i}_p \bar{j}_1 \bar{j}_2 \ldots \bar{j}_q} \frac{\partial}{\partial z_{j_1}} \ldots \frac{\partial}{\partial z_{j_q}}
\]

\(W\) is antisymmetric in the \(j\)'s as well as in the \(\bar{i}\)'s.

Form local operator:

\[
\mathcal{O}_W = \eta^{\bar{i}_1} \ldots \eta^{\bar{i}_p} W_{\bar{i}_1 \ldots \bar{i}_p \bar{j}_1 \ldots \bar{j}_q} \psi_{j_1} \ldots \psi_{j_q}.
\]

\[
\delta \mathcal{O}_W = -\mathcal{O}_{\bar{\partial}W},
\]

\(\mathcal{O}_W\) is \(\delta\)-invariant if \(\bar{\partial}W = 0\) and \(\delta\)-exact if \(W = \bar{\partial}S\) for some \(S\).

\(W \mapsto \mathcal{O}_W\) - natural map from \(\oplus_{p,q} H^p(V, \wedge^q T^{1,0} V)\) to the \(\delta\)-cohomology of the B model. It is isomorphism for local operators.

The story of Correlators in B model, Descend Equations, Deformation of the action by 2-observables, Generating function \(\mathcal{F}_B(T)\) is completely parallel to that in A-model:

\[
S_B(T) = S + T^a \int \mathcal{O}_{W_a}
\]

\[
\mathcal{F}_B(T) = < e^{-S_B(T)} >
\]
• Interesting examples of the deformations:

\[ W = \tilde{A}_i^j \frac{\partial}{\partial z^j} d\tilde{z}^i \] - deformation of the complex structure of \( \tilde{V} \)

\[ W = W(z) \] - holomorphic function (for non-compact \( \tilde{V} \)) - singularity (Landau-Ginzburg in physical terminology) theory

\[ W = \frac{1}{2} \pi^{i j} \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^j} \] - non-commutative deformation

• **Complex structure deformations:**

\( \tilde{V}_s \) – family of \( d \) complex dimensional projective varieties with \( c_1(\tilde{V}_s) = 0 \) - CY.

Calibrated CY manifold - \((\tilde{V}, \Omega)\); \( \tilde{V} \) - CY supplied with the holomorphic \((d, 0)\) form \( \Omega \). Holomorphic \((d, 0)\) form - unique up to the multiplication by a non-zero complex number.

\( \mathcal{M} \) - moduli of cmplx structures \( \tilde{V}_{s_0} \):

\[ T_{s_0} \mathcal{M} \approx H^{d-1,1}(\tilde{V}_{s_0}) \]

The moduli space \( \widehat{\mathcal{M}}_{\tilde{V}_{s_0}} \) of the calibrated CY manifolds is a \( C^* \)-bundle over \( \mathcal{M}_{\tilde{V}_{s_0}} \). The normalized holomorphic \((d, 0)\) from \( \Omega_0 \) defines locally a section of the bundle.

The choice of the complex structure provides the decomposition of the external derivative \( D = D^{1,0} + D^{0,1} = \partial + \bar{\partial} \).

Let \((z^i, \bar{z})\) be local coordinates on \( \tilde{V} \) and let \( \tilde{A} \in \Omega^{-1,1}(\tilde{V}) \) be a \((-1, 1)\) differential, locally: \( \tilde{A} = \sum \tilde{A}_i^j dz^i \frac{\partial}{\partial z^j} \).
The deformation of the complex structure may be described in terms of the deformation of the operator $D^{0,1} = \bar{\partial}$

$$\bar{\partial} \to \bar{\partial}_{\tilde{A}} = \bar{\partial} + \tilde{A} = \sum d\bar{z}^{i}(\frac{\partial}{\partial \bar{z}^{i}} + \bar{A}^{\bar{j}}_{i} \frac{\partial}{\partial z^{j}})$$

subjected to the integrability condition $\bar{\partial}^{2}_{\tilde{A}} = 0$ (Kodaira-Spencer equation).

$I_{KS}(\tilde{A})$ - functional with critical points KS-equation. For 3-complex dimensions can be written as function of $\Omega^{(3)}$ via identification $\Omega^{(2,1)} = A \vdash \Omega$.

Special coordinates on $\hat{M}$: $T^{i}, i = 0, \ldots, h^{d-1,1}(\tilde{V}_{s})$:

Let $\alpha_{I}(s), \beta^{I}(s), I = 0, \ldots, h^{d-1,1}(Y)$ be a symplectic basis in $H^{d}(\tilde{V}_{s}, \mathbb{Z})$:

$$\alpha_{I} \cap \alpha_{J} = \beta^{I} \cap \beta^{J} = 0, \quad \alpha_{I} \cap \beta^{J} = \delta^{J}_{I}$$

On the $\hat{M}$ this basis is defined uniquely once it is chosen at some marked point $p_{0} \in \hat{M}$.

$$A^{I}(s) = \int_{\alpha_{I}(s)} \Omega, \quad A_{D,I}(s) = \int_{\beta^{I}(s)} \Omega$$

$\Omega$ - defined uniquely up to a constant. Let us fix this freedom by choosing a distinguished cycle $\alpha_{0}$ and demanding $A^{0} = 1$. Then

$$T^{i} = A^{i}, \quad i = 1, \ldots, \dim M$$
There exists a function $\mathcal{F}_{(0)} B$ on $\hat{\mathcal{M}}$ such that

$$d\mathcal{F}_{(0)} = \sum_i A_{D_i} dA^i$$

Locally $\mathcal{F}_0$ can be viewed as a function of $T^i$ - generating function of Lagrangian sub-manifold in $H^d(\tilde{V}, \mathbb{C})$ which coincides with $\hat{\mathcal{M}}$.

Form a function of one extra variable $\lambda \in H^{(d,0)}$ (normalization of $(d,0)$ - form - coordinate in fibre):

$$Z(\lambda, T) = e^{-\sum g \lambda^{2g-2} \mathcal{F}_g(T)} = e^{-\mathcal{F}(\lambda, T)}$$

If we denote base complex structure as $(t, t^*)$, one can show that $Z(t, t^*) (\lambda, T)$ depends on base complex structure $t^*$ which is captured by differential equation is of heat-kernel type, Holomorphic Anomaly equation.

A. Gerasimov & S.Sh. 2004: value of Kodaira-Spencer action $I(\bar{A})$ at critical points coincides with $\mathcal{F}_{(0)}$ - generating function of Lagrangian sub-manifold introduced above.

Higher genus corrections to $Z(\lambda, T)$ - quantization of symplectomorphism relating polarization defined by Lagrangian submanifold $\hat{\mathcal{M}}$ to linear polarization at given base point $(t, t^*) \rightarrow$ corrections in coupling constant $\lambda$ (volume $\Leftrightarrow$ holomorphic three form).

**Mirror symmetry:** $A = B$

not only for CY, but more general
Special case of CY threefolds: physical intuition

As $\mathcal{N} = 2$ SCFT’s the theories $A$ and $B$ don’t differ (internal automorphism of the $\mathcal{N} = 2$ algebra maps $A$ to $B$ and vice versa)

SCFT has different large volume limits - the same theory looks as different sigma models with different target spaces $V$ and $\tilde{V}$ in different limits.

T-duality - the simplest example.
FOUR DIMENSIONAL THEORY

DONALDSON-WITTEN THEORY

• $X - 4$ dimensional compact smooth Riemannian manifold
• $b_i = b_i(X)$ – Betti numbers.
• On $H^*(X)$: intersection form $(,)$; metric $\langle , \rangle$:

$$ (\omega_1, \omega_2) = \int_X \omega_1 \wedge \omega_2, \quad \langle \omega_1, \omega_2 \rangle = \int_X \omega_1 \wedge \ast \omega_2 $$

$\ast$ - the Hodge star operation.

$b^\pm_2$ – dim’s of the positive and negative subspaces of $H^2(X)$.

$\omega \in H^2(X)$: $\omega^\pm$ – orthogonal projections to the spaces of self- and antiselfdual classes: $H^{2,\pm}(X) - (\omega^\pm, \cdot) = \pm \langle \omega^\pm, \cdot \rangle$, $\omega = \omega^+ + \omega^-.$

$\chi = \sum_{i=0}^{4} (-1)^i b_i$, – the Euler characteristics of $X$

$\sigma = b^+_2 - b^-_2$ the signature of $X$
• \( e_\alpha \) is a basis in \( H_*(X, \mathbb{C}) \),

• \( e^\alpha \) the dual basis in \( H^*(X, \mathbb{C}) \):

\[
(e^\alpha, \omega) = \int_{e_\alpha} \omega
\]

for any \( \omega \in H^*(X) \).

\( G' = SU(r+1) \), \( G = G'/Z \), \( Z \cong \mathbb{Z}_{r+1} \), \( g = \text{Lie}G \).

\( T = U(1)^r \) – maximal torus of \( G \), \( W = S_{r+1} \) the Weyl group,

\( g = \text{Lie}(G), t = \text{Lie}(T) \).

\( h = r + 1 \) – dual Coxeter number.

\( \ell = (w_2; k), k \in \mathbb{Z}, w_2 \in H^2(X, \mathbb{Z}) \) – generalized Stiefel-Whitney class.

\( P_\ell \) - a principal \( G \) bundle over \( X \) and \( E_\ell \) the associated vector bundle with \( w_2(E_\ell) = w_2 \),

\[ c_2(E_\ell) + \frac{1}{2} w_2 \cdot w_2 = k. \]
\( \mathcal{A}_\ell \) - the space of connections in \( \mathcal{P}_\ell \).

\( \mathcal{G}_\ell \) - the group of gauge transformations of \( \mathcal{P}_\ell \).

The Lie algebra of \( \mathcal{G}_\ell \) - the algebra of sections of the associated adjoint bundle \( \mathfrak{g}_\ell = \mathcal{P}_\ell \times \text{Ad} \mathfrak{g} \). \( \phi \) - an element of \( \text{Lie} \mathcal{G}_\ell \).

For the connection \( A \) (= the gauge field) let \( F_A \) denote its curvature (it is a section of \( \Lambda^2 T^*_X \otimes \mathfrak{g}_\ell \)).

**Definition.** \( \mathcal{G} \)-instanton is the solution to the equation

\[
F^+_A = F + \ast F = 0
\]

where \( + \) acts on the \( \Lambda^2 T^*_X \) part of \( F_A \).

**Definition.** a \( \mathcal{G} \)-instanton \( A \) is called irreducible if there are no infinitesimal gauge transformations, preserving \( A \). This condition is equivalent to the absence of the solutions to the equation

\[
d_A \phi = 0, \quad 0 \neq \phi \in \Gamma(\mathfrak{g}_\ell)
\]

where \( d_A \) is the connection on \( \mathfrak{g}_\ell \) associated with \( A \).

**Definition.** a \( \mathcal{G} \)-instanton is called unobstructed if there are no solutions to the equation \((d^+_A)' \chi = 0, \quad 0 \neq \chi \in \Gamma(\Lambda^{2,+} T^*_X \otimes \mathfrak{g}_\ell)\).

**Definition.** The moduli space \( \mathcal{M}_\ell \) of \( \mathcal{G} \)-instantons is the space of all irreducible unobstructed \( \mathcal{G} \)-instantons modulo action of \( \mathcal{G}_\ell \). For the instanton \( A \) let \([A]\) denote its gauge equivalence class - a point in \( \mathcal{M}_\ell \).
The tangent space to $\mathcal{M}_\ell$ at $A$ is the middle cohomology group of the Atiyah-Hitchin-Singer (AHS) complex of bundles over $X$:

$$0 \to \Lambda^0 T_X^* \otimes g_\ell \to \Lambda^1 T_X^* \otimes g_\ell \to \Lambda^2, + T_X^* \otimes g_\ell \to 0$$

the first arrow is $d_A$, the second is $d_A^+ = P_+ d_A$.

$P_+$ - the projection $\Lambda^2 T_X^* \otimes g_\ell \to \Lambda^2, + T_X^* \otimes g_\ell$.

$d_A^+ \circ d_A = F_A^+ = 0 \to$ the sequence is the complex.

$H^0(\text{AHS}) = 0$ for irred. instantons. $H^2(\text{AHS}) = 0$ - obstruction space; absent for unobstructed instantons.

**Lemma.** The dimension of the moduli space $\mathcal{M}_\ell$:

$$\dim \mathcal{M}_\ell = 4hk - \dim G \frac{\chi + \sigma}{2}$$

**Proof:** index theorem applied to the AHS complex.
Remark. $\mathcal{M}_\ell$ is non-compact. Sometimes it can be compactified (Donaldson-Uhlenbeck) by adding the point-like instantons:

$$\overline{\mathcal{M}}_\ell = \mathcal{M}_\ell \cup \mathcal{M}_{\ell-(0;1)} \times X \cup \ldots \cup \mathcal{M}_{\ell-(0;k)} \times S^k X$$

For $A$ from class $[A] \in \mathcal{M}_\ell$ the space $T_{[A]}\mathcal{M}_\ell$ can be identified with the space of solutions $\alpha$:

$$d_A^+ \alpha = 0, \quad d_A^* \alpha = 0$$

$\alpha \in \Gamma \left( \Lambda^1 T^* X \otimes g_\ell \right)$. 
Consider the product $\mathcal{M}_\ell \times X$ and form the universal bundle $\mathcal{E}_\ell$ - the bundle whose restriction onto $[A] \times X \subset \mathcal{M}_\ell \times X$ coincides with $E_\ell$.

d be the differential in the DeRham complex on $\mathcal{M}_\ell \times X$ and $d_m, d$ be its components along $\mathcal{M}_\ell, X$ respectively.

**Definition.** The universal connection is the $G$-connection $a$ in $\mathcal{E}_\ell$ with the following properties:
1. $a|_{[A] \times X} \in [A]$ 
2. $a|_{\mathcal{M}_\ell \times \{x\}} = \frac{1}{\Delta_A} d_A^* d_m A$ with $\Delta_A = d_A^* d_A$

**Lemma.** The curvature of the universal connection can be expanded as:
$$F_a = F_A + \psi + \phi$$

$\psi$ is the fundamental solution to the equations:
$$d_A^+ \psi = 0, \quad d_A^* \psi = 0$$

$\phi$ is given by:
$$\phi = \frac{1}{\Delta_A} [\psi, \star \psi]$$

**Comments.** We view $\psi$ as the mixed $(\mathcal{M}_\ell, X)$ component of the curvature of $a$. It means that locally we view $\psi$ as one-form on $\mathcal{M}_\ell$ with values in $g$. Using metric on $X$ and the induced metric on $\mathcal{M}_\ell$ we identify $T_{[A]} \mathcal{M}_\ell$ with $T_{[A]}^* \mathcal{M}_\ell$. 
Similarly $\phi$ is the $(\mathcal{M}_\ell, \mathcal{M}_\ell)$ component of the curvature of $a$.

$\{I_k\}$ - additive basis in the space of invariants: $\text{Fun}(g)^G \approx \text{Fun}(t)^W$.

$d_k$ - the degree of $I_k$.

$O_n^\alpha = \int_{e_\alpha} I_n \left( \frac{\phi + \psi + F_A}{2\pi i} \right)$.

**Examples.** $I_1(\phi) = \text{Tr} \phi^2, d_1 = 2, I_2(\phi) = \text{Tr} \phi^3, I_3 = \text{Tr} \phi^4, I_4 = (\text{Tr} \phi^2)^2, d_2 = 3, d_3 = d_4 = 4$.

Denote $\mathcal{M} = \Pi_\ell \mathcal{M}_\ell$, $\mathcal{E} = \Pi_\ell \mathcal{E}_\ell$. There is a a characteristic class $c_I(\mathcal{E})$ associated to each invariant $I \in \text{Fun}(g)^G$.

Let $\Omega_n^\alpha$ be the slant product $\int_{e_\alpha} c_{I_n}(\mathcal{E}) \in H^{2d_n - \text{dim}_\alpha}(\mathcal{M})$. 
\textbf{Definition.} The following integral over $\mathcal{M}$ is the attempt to define the intersection theory of $\Omega_{n}^{\alpha}$

$$\langle \Omega_{n_1}^{\alpha_1} \ldots \Omega_{n_k}^{\alpha_k} \rangle = \sum_{\ell} \int_{\mathcal{M}_{\ell}} O_{n_1}^{\alpha_1} \wedge \ldots \wedge O_{n_k}^{\alpha_k}$$

\textbf{Definition.} The prepotential of the refined Donaldson-Witten theory is the generating function:

$$Z_{A}(T) = \langle \exp (T_{\alpha}^{k} \Omega_{k}^{\alpha}) \rangle \equiv$$

$$\sum \frac{1}{k!} T_{\alpha_1}^{n_1} \ldots T_{\alpha_k}^{n_k} \langle \Omega_{n_1}^{\alpha_1} \ldots \Omega_{n_k}^{\alpha_k} \rangle$$
Physical Picture

**The fields:** twisted $\mathcal{N} = 2$ vector multiplet

**Bosons:** gauge field $A = A_\mu dx^\mu$, the complex scalar $\phi$ and its conjugate $\bar{\phi}$, self-dual two form $H$

**Fermions:** the one-form $\psi$, the scalar $\eta$ and the self-dual two-form $\chi$.

All fields take values in the adjoint representation.

**Nilpotent Symmetry:**

\[
\begin{align*}
\delta \phi &= 0, & \delta \bar{\phi} &= \eta, & \delta \eta &= [\phi, \bar{\phi}] \\
\delta \chi &= H, & \delta H &= [\phi, \chi] \\
\delta A &= \psi, & \delta \psi &= D_A \phi
\end{align*}
\]

$\delta^2 = \mathcal{L}_\phi = $ infinitesimal gauge transformation generated by $\phi \Rightarrow$ nilpotent on the gauge invariant functionals of the fields (equivariant cohomology).

**Definition.** Observables - gauge invariant functionals of the fields, annihilated by $\delta$.

The correlation functions of observables do not change under a small variation of metric on the four-manifold $X$. 
**Observables:** Invariant polynomial $\mathcal{P} = \sum_k t^k I_k$ on the algebra $\mathfrak{g}$, $\mathcal{O}^k, k = 0, \ldots 4$ – closed $k$-cycles on $X$. Their homology cycles are denoted as $[\mathcal{O}^k] \in H_k(X; \mathbb{C})$. The observables form the descend sequence:

$$\mathcal{O}^{(0)} = \mathcal{P}(\phi), \quad \delta\mathcal{O}^{(0)} = 0$$

$$d\mathcal{O}^{(0)} = -\delta\mathcal{O}^{(1)} \quad (\mathcal{O}^{(1)}, [\mathcal{C}^1]) \equiv \int_{\mathcal{C}^{(1)}} \mathcal{O}^{(1)} \equiv \int_{\mathcal{C}^1} \frac{\partial \mathcal{P}}{\partial \phi^a} \psi^a$$

$$d\mathcal{O}^{(1)} = -\delta\mathcal{O}^{(2)} \quad (\mathcal{O}^{(2)}, [\mathcal{C}^2]) = \int_{\mathcal{C}^2} \mathcal{O}^{(2)} =$$

$$\int_{\mathcal{C}^2} \frac{\partial \mathcal{P}}{\partial \phi^a} F^a + \frac{1}{2} \frac{\partial^2 \mathcal{P}}{\partial \phi^a \partial \phi^b} \psi^a \wedge \psi^b$$

\[\vdots\]

top degree observable: $\mathcal{O}^{(4)}_{\mathcal{P}} = \frac{1}{2} \frac{\partial^2 \mathcal{P}}{\partial \phi^a \partial \phi^b} F^a F^b +$

$$+ \frac{1}{3!} \frac{\partial^3 \mathcal{P}}{\partial \phi^a \partial \phi^b \partial \phi^c} F^a \psi^b \psi^c + \frac{1}{4!} \frac{\partial^4 \mathcal{P}}{\partial \phi^a \partial \phi^b \partial \phi^c \partial \phi^d} \psi^a \psi^b \psi^c \psi^d$$
Action $S$ equals the sum of the 4-observable, constructed out of the prepotential $\mathcal{F}$ and the $\delta$-exact term:

$$S = \mathcal{O}^{(4)}_{\mathcal{F}} + \delta R$$

The standard choice: $\mathcal{F} = \left(\frac{i\theta}{8\pi^2} + \frac{1}{e^2}\right) \text{Tr} \phi^2$,

$$R = \frac{1}{e^2} \text{Tr} \left(\chi F^+ - \chi H + D_A \bar{\phi} \times \psi + \eta \star [\phi, \bar{\phi}]\right),$$

$\text{Tr}$ denotes the Killing form.

The bosonic part of the action $S$ is then:

$$S = \int_X \tau \text{Tr} F \wedge F + \frac{1}{e^2} \left( \text{Tr} F \wedge \star F + \text{Tr} D_A \phi \wedge \star D_A \bar{\phi} + \text{Tr} [\phi, \bar{\phi}]^2 \right)$$

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$$

The $e^2$-dependence – only via $\delta(\ldots)$ terms:

$$S = \frac{\theta}{2\pi} \int_X F \wedge F + \frac{1}{e^2} \delta(\ldots)$$

$\Rightarrow$ can take $e^2 \to 0$ limit for correlators of observables: the path integral measure gets localized near solutions to $F^+ = 0, D_A \phi = 0$

Moral. The correlation functions of observables reduce to the integrals over $\mathcal{M}\ell$.
• Donaldson theory ($G = SU(2)$ or $G = SO(3)$): aim is to compute:

$$\langle \exp((\mathcal{O}^{(2)}_u, w) + \lambda \mathcal{O}^{(0)}_u) \rangle,$$

for $w \in H^2(X, \mathbb{R})$, $\mathcal{O}^{(0)}_u = u \equiv \text{Tr} \phi^2$,

$$(\mathcal{O}^{(2)}_u, w) = -\frac{1}{4\pi^2} \int_X \text{Tr}(\phi F + \frac{1}{2} \psi \psi) \wedge w$$

• Refinement: generating function of all correlators of all observables:

$$Z_A(T^k) = \langle e^{T^k,\alpha (\mathcal{O}^{4-d\alpha}_{I_k}, e_\alpha)} \rangle$$

$$T^k = T^k,\alpha e_\alpha \in \mathcal{V} = \oplus_{p=0}^4 H^p(X, \mathbb{C})$$

This is a physical definition of the four dimensional type A theory

Very important tool of computing infinite-dimensional path integral over all fields entering in the definition of correlators $\Rightarrow$ Abelianization.
Problem. $\mathcal{M}_\ell$ is non-compact. Need to compactify it in order to have a nice intersection theory.

- Donaldson compactification: add point-like instantons as above (for high enough instanton charges get a manifold, perhaps with orbifold singularities)

- For Kähler $X$ a refinement of the compactification above: Gieseker compactification:

Idea: On Kähler $X$ with Kähler form $\omega$:

$$F^+ = 0 \iff \bar{\partial}_A^2 = 0, \quad F \wedge \omega = 0$$

$\bar{\partial}_A$ defines a holomorphic bundle $\mathcal{E}$ over $X$: its local sections are annihilated by $\bar{\partial}_A$. Then $F \wedge \omega = 0$ is a stability condition.

Replace $\mathcal{E}$ by its (holomorphic) sheaf of sections. Consider the moduli space $\overline{\mathcal{M}}_\ell^G$ of sheaves which are torsion free as $\mathcal{O}_X$-modules. The latter has sheaves which are not locally free, i.e. which are not holomorphic bundles. However, for each such sheaf $\mathcal{E}'$ there is a zero-dimensional subscheme $Z \subset X$, such that on $X \setminus Z$, $\mathcal{E}'$ is a holomorphic bundle and has a connection.
**Problem.** Find an analogue of Kontsevich compactification.

**Problem.** Find a physical realization of all these compactifications.

**Partial answer to the last problem:** On $X = \mathbb{CP}^2$ the compactification by sheaves corresponds to the gauge theory on a non-commutative space.
Intersection theory in four dimensions

Take \( X = \mathbb{CP}^2 \), \( G = U(r) \), \( w \) - Kähler form.

\( p \in H^2(X, \mathbb{Z}) \), \( k \in H^4(X, \mathbb{Z}) \).

- Monad construction of the torsion free sheaves on \( X \): Let \( V_0, V_1, V_2 \) be the complex vector spaces of dimensions \( v_{0,1,2} \) respectively. Consider the complex of bundles over \( X \):

\[
0 \to V_0 \otimes \mathcal{O}(-1) \xrightarrow{a} V_1 \otimes \mathcal{O} \xrightarrow{b} V_2 \otimes \mathcal{O}(1) \to 0
\]

In down-to-earth terms this sequence has the following meaning. The maps \( a, b \) in the homogeneous coordinates \( (z^0 : z^1 : z^2) \) are the matrix-valued linear functions: \( a(z) = z^\alpha a_\alpha \), \( b(z) = z^\alpha b_\alpha \). The words “complex” mean that

\[
b(z) \cdot a(z) = z^\alpha z^\beta b_\alpha a_\beta = 0 \iff b_\alpha a_\alpha = 0, \alpha = 0, 1, 2, \quad b_\alpha a_\beta + b_\beta a_\alpha = 0, \alpha \neq \beta
\]

For the pair \( (b, a) \) of the maps between the sheaves obeying this condition we can define a sheaf \( \mathcal{F} \) over \( X \), whose space of sections over an open set \( U \) is

\[
\Gamma (\mathcal{F}|_U) = \text{Ker} b(z)/\text{Im} a(z), \quad \text{for} \quad (z^0 : z^1 : z^2) \in U
\]

\[
\beta^{ij}(z)\Psi^j(z) = 0, \quad \text{modulo} \quad \Psi^j(z) = a^{jk}(z)\tilde{\Psi}^k(z)
\]

**Definition:** The space of monads is the space \( M_{\text{mon}} \) of triples of matrices \( a_\beta \in \text{Hom}(V_0, V_1) \), \( b_\alpha \in \text{Hom}(V_1, V_2) \) obeying \( b(z)a(z) = 0 \). This space is acted on by the group

\[
G_{\text{mon}}^c = (\text{GL}(V_0) \times \text{GL}(V_1) \times \text{GL}(V_2)) / \mathbb{C}^*
\]
\[(b, a) \mapsto g \cdot (b, a) = (g_2 b g_1^{-1}, g_1 a g_0^{-1}), \text{ for } (g_0, g_1, g_2) \in G_{\text{mon}}^c\]

The sheaves defined by the pairs \((b, a)\) and \(g \cdot (b, a)\) are isomorphic. The maximal compact subgroup of \(G_{\text{mon}}^c\)

\[G_{\text{mon}} \approx (U(V_0) \times U(V_1) \times U(V_2)) / U(1)\]

acts in \(M_{\text{mon}}\) preserving its natural symplectic structure

\[\Omega = \frac{1}{2i} \sum_{\beta} \text{Tr} \delta a_\beta \wedge \delta a_\beta^\dagger + \frac{1}{2i} \sum_{\alpha} \text{Tr} \delta b_\alpha^\dagger \wedge \delta b_\alpha\]

Fix the real numbers \(r_0, r_1, r_2\), such that \(\sum_{\alpha} v_\alpha r_\alpha = 0\), \(r_0, r_2 > 0\). Write the moment maps:

\[\mu_1 = -r_0 1_{v_0} + \sum_{\beta} a_\beta^\dagger a_\beta\]

\[\mu_2 = -r_1 1_{v_1} + \sum_{\alpha} b_\alpha^\dagger b_\alpha - \sum_{\beta} a_\beta a_\beta^\dagger\]

\[\mu_3 = -r_2 1_{v_2} + \sum_{\alpha} b_\alpha b_\alpha^\dagger\]

Then the moduli space of the semistable sheaves is

\[\overline{M}_{c_*} = (\mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)) / G_{\text{mon}}\]

This is typical example of hyperkähler quotient (Integration over Higgs Branches - MNS’97). The compactness of the space is obvious: if we first perform a reduction with respect to the groups \(U(V_0) \times U(V_2)\) then the resulting space is the product of two Grassmanians: \(\text{Gr}(v_0, 3v_1) \times \text{Gr}(v_2, 3v_1)\) which is already compact. The subsequent reduction does not spoil this.
The Chern classes, \( c_* = \{ r, c_1, c_2 \} \), of the sheaf \( F \) determined by the pair \((b, a)\) are:

\[
\begin{align*}
    r &= v_1 - v_0 - v_2, \\
    c_1 &= (v_0 - v_2), \\
    c_2 &= \frac{1}{2} \left( (v_2 - v_0)^2 + v_0 + v_2 \right)
\end{align*}
\]

Let \((i\psi, i\phi, i\chi)\) denote the elements of the Lie algebra of \( G_{\text{mon}} \), i.e. \( i\psi \in u(V_0), i\phi \in u(V_1), i\chi \in u(V_2) \) and \((\psi, \phi, \chi) \sim (\psi + 1v_0, \phi + 1v_1, \chi + 1v_2)\). We are interested in computing certain integrals over \( \overline{M}_{c_*} \). This can be accomplished by computing an integral over \( M_{\text{mon}} \) with the insertion of the delta function in \( \mu_i\) and dividing by the volume of \( G_{\text{mon}} \) provided that the expression we integrate is \( G_{\text{mon}}\)-invariant:

\[
\int_{\overline{M}_{c_*}} \ldots = \frac{1}{\text{Vol}(G_{\text{mon}})} \int_{\text{Lie}G_{\text{mon}}} d\psi d\phi d\chi e^{i\text{Tr}_\psi \mu_1 + i\text{Tr}_\phi \mu_2 + i\text{Tr}_\chi \mu_3} \ldots
\]

The useful fact is that the observables of the gauge theory we are interested in are the gauge-invariant functions on \((\psi, \phi, \chi)\) only. More specifically, there is a universal sheaf \( \mathcal{U} \) over \( \overline{M}_{c_*} \times X \), defined again as \( \text{Ker}b(z)/\text{Im}a(z) \) but now the space of parameters contains \((b, a)\) in addition to \( z \). Its Chern character is given by:

\[
\text{Ch}(\mathcal{U}) = \text{Tr}\phi - \text{Tr}\psi - \omega - \text{Tr}\chi + \omega
\]

In particular:

\[
\mathcal{O}_{u_1}^{(0)} = \frac{1}{2} (\text{Tr}\chi^2 + \text{Tr}\psi^2 - \text{Tr}\phi^2); \int_X \omega \wedge \mathcal{O}_{u_1}^{(2)} = \text{Tr}\chi - \text{Tr}\psi
\]
Since the observables are expressed through $\psi, \phi, \chi$ only we can integrate out $a_\beta, b_\alpha$ to obtain:

$$
\langle \exp t_1 \mathcal{O}_{u_1}^{(0)} + T_1 \int_S \omega \wedge \mathcal{O}_{u_1}^{(2)} \rangle_{\text{torsion free}} = \oint \prod_{i,j,k} d\psi_i d\chi_j d\psi_k
$$

\[
\frac{\prod_{i' < i''} (\psi_{i'} - \psi_{i''})^2 \prod_{j' < j''} (\phi_{j'} - \phi_{j''})^2}{\prod_{i,j} (\phi_j - \psi_i + i0)^3}
\frac{\prod_{k' < k''} (\chi_{k'} - \chi_{k''})^2 \prod_{i,k} (\chi_k - \psi_i)^6}{\prod_{j,k} (\chi_k - \phi_j + i0)^3}
\]

\[
\times e^{t_1 \frac{1}{2} \left( \sum_k \chi_k^2 + \sum_i \psi_i^2 - \sum_j \phi_j^2 \right) + T_1 \left( \sum_k \chi_k - \sum_i \psi_i \right) + i r_1 \sum_i \psi_i + i r_2 \sum_j \phi_j + i r_3 \sum_k \chi_k}
\]
Abelianization - Theory B, Physical Picture

- Integrate out non-abelian components of all fields (quadratic, Gaussian, integral). Result - some abelieain theory, defined on Cartan subgroup of Gauge group with abelian fields: $\phi^i, \bar{\phi}^i, \eta^i, A^i, \psi^i, \chi^i$

- Again, on the space of fields $\delta$-operators acts (original topological, $\delta$, symmetry is preserved - not broken): $\delta^2 = 0$. Define observables for abelian theory as in original, non-abelian theory: $O^i$.

- Find for every observable in non-abelian theory corresponding observable after abelianization.

- Write the action in abelian theory as 4-observable descending from some function $F(u)$, where $u_1, ..., u_N$ are invariant polynomials of $\phi$, functions of $\phi^i$.

- From general principles the abelian action must have the form:

$$S_0 = O_F^{(4)} + \delta R$$

and deformed action is:

$$S = S_0 + t_i O^i$$

The generating function for correlators is given by partition function on $\mathbf{B}$ side by:

$$Z_B(t) = < e^{-S(t)} > = \int D\phi^i D\bar{\phi}^i D\eta^i DA^i D\psi^i DH^i e^{-S_0 - t_i O^i}$$

and finally:

$$Z_A(T) = Z_B(t(T))$$
This shows that one needs:

1. Explicit expression for \( \delta \) in terms of abelian fields,

2. Explicit form of \( \mathcal{F}(u)(u = Tr\phi^2 \text{ for } SU(2)) \)

3. Explicit relation between observables \( \mathcal{O}^i \) between non-abelian and abelian theories

4. Explicit relation between parameters \( T^i \) in non-abelian theory and \( t^i \) in abelian theory - \( t^i(T) \).

1. & 2. \( \delta \) and formula for prepotential \( \mathcal{F} \) was found by Seiberg & Witten in 1994 (for \( SU(2) \)). Other groups - various authors after SW found prepotential \( \mathcal{F} \) for all groups and all generalizations of 4d \( N = 2 \) SYM with matter.

3. & 4. Solution to these was found by Moore & Witten and by Losev, Nekrasov & S. Sh. in 1997 ("universal formula for contact terms" etc.).

Integral over abelian fields in theory \( \mathcal{B} \) is reduced to finite-dimensional integral via localization technique and is related to nice and simple symplectic geometry problem.

**Few words on prepotential \( \mathcal{F} \):**

In abelianized theory \( \phi = \text{diag}(a_1, .., a_r) \). Let \( (a_i, a^i_D) \) coordinates in \( C^{2r} \) with complex symplectic worm \( \omega = da_i \wedge da^i_D \).

\( \mathcal{F} \) - generating function of Lagrangian submanifold \( \Theta = a^i_D da_i = d\mathcal{F} \) invariant under certain discrete subgroup \( \Gamma \) of \( SP(2r, \mathbb{Z}) \).

Turning on couplings \( T \) corresponds to deformations of this Lagrangian submanifold - flows described explicitly in LNS.