Nodally 3-connected planar graphs and convex combination mappings

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Abstract

A barycentric mapping of a planar graph is a plane embedding in which every internal vertex is the average of its neighbours. A celebrated result of Tutte's [16] is that if a planar graph is nodally 3-connected then such a mapping is an embedding. Floater generalised this result to convex combination mappings in which every internal vertex is a proper weighted average of its neighbours. He also generalised the result to all triangulated planar graphs.

This has applications in numerical analysis (grid generation), and in computer graphics (image morphing, surface triangulations, texture mapping): see [6, 17].

White [17] showed that every chord-free triangulated planar graph is nodally 3-connected.

We show that (i) a nontrivial plane embedded graph is nodally 3-connected if and only if every face boundary is a simple cycle and the intersection of every two faces is connected; (ii) every convex combination mapping of a plane embedded graph G is an embedding if and only if (a) every face boundary is a simple cycle, (b) the intersection of every two *bounded* faces is connected, and (c) there are no so-called inverted subgraphs; (iii) this is equivalent to G admitting a convex embedding (see [13]); and (iv) any two such embeddings (with the same orientation) are isotopic.

1 Planar graphs and nodal 3-connectivity

We follow the usual definitions of graphs, including paths, simple paths, cycles, simple cycles, and connectivity: [9] is a useful source on the subject. The accepted definition of graph does not allow self-loops nor multiple edges nor infinite sets of vertices, so it is a finite simple graph in Tutte's language [16], and a graph G can be specified as a pair (V, E) giving its vertices and edges. E is a set of unordered pairs of distinct vertices in V. Two vertices u, v are *adjacent* or *neighbours* if $\{u, v\} \in E$.

Given $G = (V, E)$, when u is considered to be a vertex, $u \in G$ means $u \in V$, and when e is considered to be an edge, $e \in G$ means $e \in E$.

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(1.1) Subgraphs, etcetera. Given $G = (V, E)$ and $G' = (V', E')$, G' is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$.

Given G and given $S \subseteq V$, the *subgraph of G spanned by S* is the graph (S, E') where

$$E' = \{\{u, v\} \in E : \quad u, v \in S\}.$$

The *degree* (in G) $\deg(v)$ of a vertex v is the number of edges incident to it, or the number of neighbours it has. The word 'node' is reserved in [16] to denote vertices whose degree $\neq 2$.

A *path* in G is a sequence $u_0, \ldots, u_k$ of vertices where $k \geq 0$ and for $0 \leq j \leq k-1$, $\{u_j, u_{j+1}\} \in E$. It is *simple* if all the vertices u_j are distinct. The *inner vertices* in a simple path are $\{u_1, \ldots, u_{k-1}\}$.

A *cycle* is a path $u_0, \ldots, u_k, u_0$ (that is, its first and last vertices are the same). It is a *simple cycle* if $k = 0$ or the path $u_0, \ldots, u_k$ is a simple path.

If we write, say, $v_1, \ldots, v_n$ for a cycle, it is implied that v_n is the second-last vertex rather than a recurrence of the first, so properly the cycle is $v_1, \ldots, v_n, v_1$.

If $G_i = (V_i, E_i)$ are two graphs then we define

$$G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2) \quad \text{and} \quad G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2).$$

If $G = (V, E)$ and $S \subseteq V$ then $G \backslash S = (V', E')$ where

$$V' = V \backslash S \quad \text{and} \quad E' = \{\{u, v\} \in E : \ u \notin S \text{ and } v \notin S\}.$$

We extend this notation loosely but with little risk of confusion: if x is a vertex then $G \backslash x = G \backslash \{x\}$, and if H is a subgraph, or a path, or a cycle, then $G \backslash H$ is the same as $G \backslash S$ where S is the set of vertices in H.

G is *connected* if every two vertices are connected by a path in G. G is *biconnected* if it is connected and for every $u \in G$, $G \backslash u$ is connected. G is *triconnected* if it is biconnected and for any $u, v \in G$, $G \backslash \{u, v\}$ is connected. (Here $\{u, v\}$ is a pair of vertices, not necessarily an edge.)

A *path (graph)* is either a trivial graph or a connected graph in which two vertices have degree 1 and all others have degree 2. A *simple cycle (graph)* is a connected nonempty graph all of whose vertices have degree 2.

This paper is concerned with *nodal 3-connectivity* (defined in 1.22), which requires biconnectivity but is weaker than triconnectivity.

(1.2) Definition *Let $G = (V, E)$ be a graph.*

- *The unit interval $\{t \in \mathbb{R} : \ 0 \leq t \leq 1\}$ is denoted $[0, 1]$. Given distinct points x and y in $\mathbb{R}^2$, a* simple curve-segment *joining x to y is continuous, injective map $\pi : [0, 1] \to \mathbb{R}^2$ such that $\pi(0) = x$ and $\pi(1) = y$.*

- *Let f be a map taking each vertex u to a point $f(u)$ in the plane $\mathbb{R}^2$, and each edge $e = \{u, v\}$ to a simple curve-segment $f(e)$ joining $f(u)$ to $f(v)$.*

  The relative interior *of e, which depends on f, is the open curve-segment*

  $$\text{interior}(e) = f(e) \backslash \{f(u)\} \backslash \{f(v)\}.$$

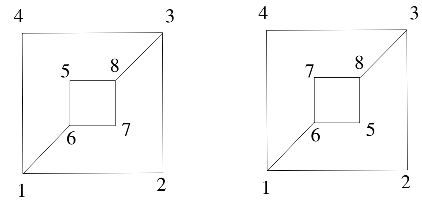

Figure 1: a graph with different plane embeddings. Also, the barycentric map is not an embedding.

- *The map f is a* plane embedding *of G if the points $f(u)$ are distinct and the relative interiors of any two edges are disjoint.*
- *A plane embedding f is* straight-edge *if $f(e)$ is a line-segment for every edge e.*
- *G is* planar *if a plane embedding exists.*

One often speaks of a planar graph G with a specific plane embedding of G in mind, so it really means a plane embedded graph. A very significant difference is that a plane embedded graph has a definite external face (Definition 1.10), whereas there is no notion of external face, nor perhaps even of face, in a planar graph without a prescribed embedding. Figure 1 shows a planar graph with two quite different embeddings.

Plane embeddings could somehow be pathological and they should be discussed in terms of the Jordan Curve Theorem mentioned below. However, the following proposition could be used to simplify the arguments.

(1.3) Proposition *Every planar graph admits a straight-edge embedding* [3, 11, 12]. ∎

(1.4) Topology in two dimensions. See [10, 14]. We assume the basic notions of open and closed sets, connectedness, and path-connectedness. If $x \in \mathbb{R}^2$ and $\varepsilon > 0$ then the ε-neighbourhood of x is

$$B(x, \varepsilon) = \{y \in \mathbb{R}^2 : \quad |y - x| < \varepsilon\}.$$

If S is any subset of $\mathbb{R}^2$ then its *closure*, written $\overline{S}$, is

$$\overline{S} = \{x \in \mathbb{R}^2 : \quad (\forall \varepsilon > 0) B(x, \varepsilon) \cap S \neq \emptyset\},$$

and its *boundary* ∂S is

$$\partial S = \overline{S} \cap \overline{\mathbb{R}^2 \backslash S}.$$

If S is open then $S \cap \partial S = \emptyset$. We are not concerned with connectedness, but with the rather stronger notion of path-connectedness: a set S is *path-connected* if for any $x, y \in S$ there exists a path from x to y, a continuous map $\pi : [0, 1] \to S$ such that $\pi(0) = x$ and $\pi(1) = y$.

(1.5) Jordan curves. A *Jordan curve* is a subset of $\mathbb{R}^2$ homeomorphic to the unit circle S^1. That is, J is a Jordan curve iff there exists a continuous injective map $h : S^1 \to \mathbb{R}^2$ whose range is the set J.

(1.6) Proposition *Let x and y be two vertices in a plane embedding f of a graph G. Then they are in the same component of G as a graph if and only if they are in the same path-component of G as a topological subspace of $\mathbb{R}^2$. Also if C is a simple cycle then its image under f is a Jordan curve. (Proof easy.)* $\blacksquare$

Part (i) of Proposition 1.7 below states the Jordan Curve Theorem, which is a difficult result. Proofs usually involve algebraic topology [8], but less advanced methods can be used [10, 14]. Actually for our purposes we need only consider polygonal Jordan curves, which makes the proofs much easier. Part (ii) is elementary.

(1.7) Proposition (i) *(Jordan Curve Theorem* [8, 10, 14]*). If J is a Jordan curve then $\mathbb{R}^2\backslash J$ is the union of two open, path-connected components,* interior(J) *and* exterior(J)*,* interior(J)*, the* inside*, is bounded, and* exterior(J)*, the* outside *or* exterior*, is unbounded, and* $\partial(\text{interior}(J)) = \partial(\text{exterior}(J)) = J$*.*

(ii) *If S is any path-connected open set such that $\partial S = J$, then $S =$* interior(J) *or $S =$* exterior(J)*.* $\blacksquare$

(1.8) Edges inside and outside Jordan curves. If J is a Jordan curve and $e = \{u, v\}$ an edge of a graph, and f an embedding such that $f(e)$ doesn't meet J except perhaps at $f(u)$ or $f(v)$, then the relative interior of e (Definition 1.2) satisfies

$$\text{interior}(e) \subseteq \text{interior}(C) \quad \text{or} \quad \text{interior}(e) \subseteq \text{exterior}(C).$$

In this case we say e is inside or outside J as appropriate. In Section 3 we shall need a certain refinement of the Jordan curve theorem:

(1.9) Proposition (Jordan-Schönflies Theorem). *Let D^1 be the unit disc in $\mathbb{R}^2$ and $S^1 = \partial D^1$, the unit circle. Then if J is a Jordan curve (a homeomorphic image of ∂D^1), the homeomorphism of ∂D^1 extends to a homeomorphism between D^1 and* $\overline{\text{interior}(J)}$*.*

More generally, if J and J' are two Jordan curves then the homeomorphism between J and J' extends to a homeomorphism between $\mathbb{R}^2$ and itself taking interior(J) *to* interior(J') *and* exterior(J) *to* exterior(J')*. (See* [10]*.)* $\blacksquare$

(1.10) Definition *Given a plane embedding f of a graph G, by abuse of notation let G also denote the union of points and curve-segments constituting its image in the plane. This is a closed and bounded set of points in the plane.*

A face *of G is a path-connected component of $\mathbb{R}^2\backslash G$.*

All faces except one are bounded. The unbounded face is called the external face *or* outer face*. Vertices on the external face are called* external*; the others are* internal.

The plane embedding is triangulated *if every bounded face is incident to exactly three edges, and* fully triangulated *if every face, bounded and unbounded, is incident to three edges.*

Faces are open sets in $\mathbb{R}^2$.

(1.11) Definition *Let f be a plane embedding of a graph $G = (V, E)$. A* triangulation *of the graph is a triangulated plane embedding f' of a graph $G' = (V', E')$ where $V' = V$ and $E' \supseteq E$, where $f'(u) = f(u)$ for all $u \in V$ and $f'(e) = f(e)$ for all $e \in E$.*

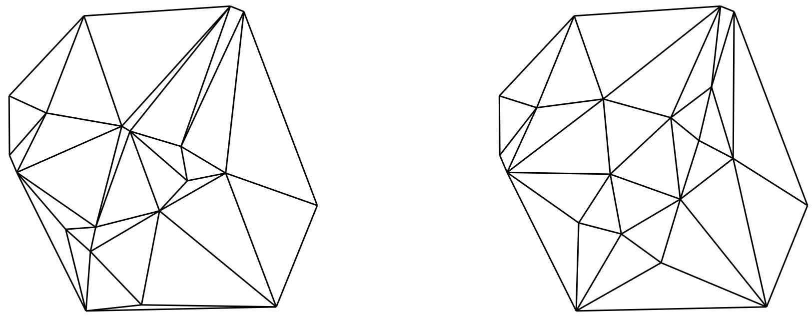

Figure 2: Delaunay triangulation of 20 points and barycentric embedding of the same graph with the same bounding polygon.

(1.12) Proposition *Every plane embedded graph can be triangulated* [9]. ∎

(1.13) Proposition (i) *If F is a face of a plane embedded graph G, then ∂F is a subgraph of G, and* (ii) $G = \bigcup_F \partial F$. *(Proof omitted.)* ∎

(1.14) Convex sets in the plane. We note the basic definitions and results (see [1]). A set A is *convex* if for any two points $a, b \in A$, the line-segment ab is entirely contained in A. Suppose S is a finite set of points in the plane. The *convex hull* hull(S) is the smallest convex set containing S, that is, the intersection of all convex sets containing S. It is also the intersection of all closed half-planes containing S. Either hull(S) is empty, or a point, or a line-segment, or it is bounded by a convex polygon whose corners are in S. In the latter case hull(S) is the intersection of those closed half-planes containing S whose boundaries contain sides of S.

(1.15) Proposition *If A is convex then its closure $\overline{A}$ is convex. (Proof easy.)* ∎

(1.16) Definition (convex combination maps) [7]. *A* convex embedding *of a planar graph G is a straight-edge embedding in which all bounded faces are convex, and the outer boundary is a simple polygon.*

Let G be a plane embedded graph whose external boundary is a simple cycle C. Another map f from its vertices to points in the plane is a convex combination map *if* (a) *there exist coefficients λ_{uv} (u, v vertices) such that*

- $\lambda_{uv} \geq 0$, *and* $\sum_v \lambda_{uv} = 1$.
- *If v is an external vertex then* $\lambda_{vv} = 1$.
- *If u and v are adjacent and u is internal then* $\lambda_{uv} > 0$.
- *Otherwise* $\lambda_{uv} = 0$.

(b) *the external vertices are mapped (in cyclic order) to the corners of a convex polygon, and* (c) *for every internal vertex u, that is, for every vertex $u \notin C$,*

$$f(u) = \sum_v \lambda_{uv} f(v) \tag{1.1}$$

The map is a barycentric map *if for each internal vertex u and neighbour v of u, $\lambda_{uv} = 1/\deg(u)$. If a barycentric map determines a straight-edge embedding of G then it is called a* barycentric embedding*.*

For example, Figure 2 shows a Delaunay triangulation with 20 vertices, and a barycentric embedding of the same graph.

The definition of convex embedding does not exclude the possibility that several edges on a face boundary be collinear. Tutte's definition of convex embedding [15] requires that the external boundary be a convex polygon, which would rule out most triangulated graphs. Hence we require that it be a simple polygon, though not necessarily convex.

In a barycentric map, every internal vertex is the average, centroid, or barycentre, of its neighbours. In a convex combination map every internal vertex is a proper weighted average of its neighbours.

The following simple lemma is very useful.

(1.17) Lemma *Let f be a convex combination map, H a closed convex set, and v an internal vertex such that for all neighbours u of v, $f(u) \in H$. If, for some neighbour u of v, $f(u) \in H^o$ (the topological interior of H), then $v \in H^o$.*

Proof. Fix a neighbour u such that $f(u) \in H^o$, and fix $\varepsilon > 0$ so for all points x in the plane, if $|x| < \varepsilon$, then $x + f(u) \in H$.

Since v is internal,

$$f(v) = \sum_w \lambda_{vw} f(w),$$

and $f(v) \in H$. The sum can be written as $\lambda_{vu} f(u) + (1 - \lambda_{vu})y$ where y is a proper weighted average of the other neighbours of v — or O if $\lambda_{vu} = 1$.

Since H is convex,

$$\{\lambda_{vu}(x + f(u)) + (1 - \lambda_{vu})y : \ |x| < \varepsilon\} \subseteq H.$$

This is the open disc around $f(v)$ of radius $\lambda_{vu}\varepsilon$, so $f(v) \in H^o$. **Q.E.D.**

(1.18) Lemma *If f is a convex combination map taking the external boundary of a connected plane embedded graph G to a convex polygon P, then all vertices and edges are mapped by f into* hull(P)*.*

Proof. Let $D = \text{hull}(P)$. Since D is convex, it is enough to show that for every vertex u, $f(u) \in D$. External vertices are mapped to corners of P, hence into D.

Suppose there is an internal vertex w such that $f(w) \notin D$. D is the intersection of finitely many closed half-planes, and one of them does not contain $f(w)$. By changing coordinates if necessary, it can be arranged that D is bounded above by the x-axis and there exist vertices u such that $f(u)$ is above the x-axis. Choose u so $f(u)$ has maximal y-coordinate, h, say, and let H be the close half-plane $y \leq h$.

Since G is connected, there is a path

$$u_0, \ldots, u_k = u$$

where u_0 is an external vertex. Since $f(u_0) \in D$, $f(u_0)$ is in the interior H^o of H, so without loss of generality, $f(u_{k-1}) \in H^o$ and by Lemma 1.17, $f(u) \in H^o$, a contradiction. **Q.E.D.**

(1.19) Lemma *If a convex combination map is an embedding, then its embedded faces are convex.*

Proof. Let F be a bounded face. Since f is a straight-edge embedding, $f(\partial F)$ is a simple polygon, and we need only show it has no concave corners. However, if $f(v)$ is a concave corner then v is an inner vertex and there is a convex wedge V such that $f(u) \in V$ for all neighbours u of v. Let H be a closed half-plane such that $V \subseteq H$ and $V \backslash H^o = \{f(v)\}$. By Lemma 1.17, $f(v) \in H^o$, a contradiction. **Q.E.D.**

(1.20) Matrix defining a convex combination map. Given a plane embedded graph G whose external boundary is a simple cycle C, convex combination maps are easily specified using a matrix A. Suppose that G has m vertices $v_1, \ldots, v_m$, the first n of them belonging to C, the last $m-n$ being internal vertices, and the coordinates of their images are $x_i, y_i, 1 \le i \le m$. Any map from vertices to points, including any straight-edge embedding, is equivalent to a column vector of height $2m$.

Let A be the $m \times m$ matrix whose first n rows are identical with those of the identity matrix, and whose last $m-n$ rows express the barycentric mapping equations (1.1). Equivalently, for $1 \le i, j \le m$, let

$$a_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \text{ and } j \le n, \text{ and} \\ -\lambda_{v_i v_j} & \text{if } i \neq j. \end{cases}$$

Equation 1.1 can be written in the form

$$\sum a_{ij} x_j = 0 \quad \text{and} \quad \sum a_{ij} y_j = 0, \quad (n < i \le m).$$

For any convex combination map f (with λ_{uv} given), let B_x be the column vector of height m whose first n entries give the x-coordinates of the corners of P and whose other entries are zero; similarly let B_y specify the y-coordinates. Then f is equivalent to column vectors X and Y satisfying

$$AX = B_x; \quad AY = B_y.$$

(1.21) Lemma (i) *If G is connected then the above matrix A is invertible.*

(ii) *If G is a connected plane embedded graph whose external boundary is a simple cycle, and whose external vertices are mapped in cyclic order to the corners of a convex polygon, and weights λ_{uv} are given, then this map extends to a unique convex combination map of G.*

Sketch of proof. (See [16, 2, 5, 17].) Tutte's proof of (i) [16, 2] says that the determinant of A (scaled up) is the number of spanning trees of a certain connected graph related to G. There is a much more transparent proof given in [5] and also in [17] saying that if A has nonzero kernel then one can follow a path from an external vertex to an internal vertex where the internal vertex cannot satisfy Equation 1.1. Part (ii) follows trivially. ∎

(1.22) Definition *A graph G is* nodally 3-connected *if it is biconnected and for every two subgraphs H and K of G, if $G = H \cup K$ and $H \cap K$ consists of just two vertices (and no edges), then H or K is a simple path.*

(1.23) Proposition *Every triconnected graph is nodally 3-connected, and every nodally 3-connected graph with no vertices of degree 2 is triconnected. (Proof omitted.)* ∎

(1.24) Definition *A* peripheral polygon *in a connected graph G is a simple cycle C such that $G\backslash C$ is connected.*

The following result of Tutte's is fundamental.

(1.25) Proposition (Tutte [16]). *If G is a nodally 3-connected planar graph*[1] *and C is a peripheral polygon, and the vertices of C are mapped (in cyclic order) onto the corners of a convex polygon P, then that map extends to a unique barycentric map which is a convex, straight-edge embedding of G.* ∎

It is easy to give a counterexample when G is not nodally 3-connected. For example, in Figure 1, any barycentric map must map the inner square face to a line-segment. The figure illustrates different plane embeddings of the same graph, which is not nodally 3-connected.

We shall rely more heavily on the following

(1.26) Proposition (Floater [7]). *If G is a triangulated (plane embedded) graph, then every convex combination map of G is an embedding.* ∎

Theorem 1.34 below shows that, except regarding the external face, a *planar* graph is nodally 3-connected if and only if barycentric maps are plane embeddings.

Lemmas 1.27 and 1.30 below are fairly obvious and well-known, but still worth mentioning.

(1.27) Lemma *A plane embedded graph G is connected if and only if for every face F, the boundary ∂F is (path-)connected.* ∎

(1.28) Proposition (Euler's Formula.) *If G is a plane (straight-edge) embedded graph then*

$$v - e + f = c + 1,$$

where $v, e, f,$ and c are the numbers of vertices, edges, faces, and components of G. (Proof omitted.) ∎

(1.29) Lemma *Let G be a straight-edge embedded plane graph in which all face boundaries are simple cycles, and let u be any vertex of G.*

Let $x_0, \ldots, x_k$ be a list of neighbours of u consecutive in anticlockwise order; possibly $x_0 = x_k$ but otherwise they are distinct. For $1 \leq j \leq k$ let F_j be the face occurring between the edges (line-segments) ux_{j-1} and ux_j in the anticlockwise sense. (The faces F_j are not necessarily distinct.)

Let B be the subgraph formed by the edges and vertices in $\bigcup_j \partial F_j$.

Then any two vertices in the list x_j are joined by a path in $B\backslash u$. See Figure 3.

Proof. $B\backslash u$ is also the subgraph consisting of all vertices and edges in $\bigcup_j(\partial F_j\backslash u)$. Since each face is a simple cycle, $\partial F_j\backslash u$ is a path joining x_{j-1} to x_j. Thus $B\backslash u$ contains paths joining all these vertices x_j. **Q.E.D.**

[1] with a few exceptions: see Figure 5. The result is phrased differently in [16].

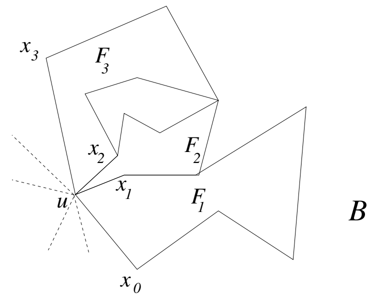

Figure 3: neighbours of u connected by paths avoiding u.

(1.30) Lemma *A plane straight-edge embedded graph G is biconnected if and only if the graph consists of a single vertex or a single edge, or the boundary of every face is a simple cycle.*

Sketch proof. (i): If. A single vertex or edge is biconnected, so we assume that the boundary of every face is a simple cycle. G is connected (Lemma 1.27).

For any vertex x and all neighbours x_j of x there exist paths connecting these neighbours which avoid x (Lemma 1.29). Therefore all these neighbours are in the same component of $G\backslash x$, and it follows that $G\backslash x$ is connected. Hence G is biconnected.

(ii): Only if. Suppose that G is connected, not a single vertex or edge, and there exists a face F whose boundary is not a simple cycle (graph): ∂F is connected but contains a node x whose degree (in ∂F, not in G) differs from 2. If ∂F contained a vertex of degree 0 then (since G is nontrivial) G would be disconnected. If it contained a vertex of degree 1, then G would be disconnected or not biconnected. Hence we can assume that all vertices on ∂F have degree ≥ 2 in ∂F.

Let $u \in \partial F$ be a vertex of degree ≥ 3 in ∂F. Let $x_1, \ldots, x_k$ be the vertices adjacent to u in anticlockwise order. For $1 \leq j \leq k$, $x_j u x_{j+1}$ ($x_{k+1} = x_1$) forms a clockwise part of the boundary of a face incident to u. Since u has degree ≥ 3 in ∂F, at least two of these paths are incident to F and there are fewer than k distinct faces incident to u.

Let $G' = G\backslash\{u\}$. All faces incident to u in G merge into a single face of G', and the other faces of G are preserved. The Euler formula gives

$$v - e + f = 2$$

for G, since G is connected. Correspondingly for G',

$$v' - e' + f' = 1 + c'.$$

Now $v' = v - 1$, and $e' = e - k$. Since in G' fewer than k faces are merged into a single face, $f' > f + 1 - k$. Therefore

$$v' - e' + f' > v - 1 - e + k + f + 1 - k = 2,$$

so $c' > 1$, G' is disconnected, and G is not biconnected. **Q.E.D.**

(1.31) Witnesses for a non-nodally 3-connected graph. Suppose G is not nodally 3-connected. We say that H, K, u, v are *witnesses* if $G = H \cup K$, $H \cap K$ contains just two vertices u, v and no edge, neither H nor K are path graphs, and neither H nor K equals G.

(1.32) Lemma (i) *Given witnesses H, K, u, v, if L is a path in G connecting $H\backslash K$ to $K\backslash H$, then L contains three consecutive vertices r, s, t where $\{r, s\} \in H$, and $\{s, t\} \in K$, $r \in H\backslash K$, $t \in K\backslash H$, and $s \in H \cap K$, so $s = u$ or $s = v$.*

(ii) *Any path (respectively, cycle) which avoids u and v except perhaps at its endpoints (respectively, perhaps once), is entirely in H or in K.*

Proof. (i) The first vertex in L is in $H\backslash K$, so the first edge is in H. Similarly the last edge is in K. Therefore there exist three consecutive vertices r, s, t on the path where $\{r, s\} \in H$ and $\{s, t\} \in K$. Then $s \in H \cap K$, so $s = u$ or $s = v$ and s is incident to edges from H and from K.

(ii) Now let P be a path which avoids u and v except perhaps at its endpoints. This includes the possibility of a cycle, viewed as a path which begins and ends at the same vertex w: we allow w, but no other vertex on the cycle, to equal u or v.

If the path is not entirely in H nor in K, then it contains a triple r, s, t where $s = u$ or $s = v$, a contradiction. **Q.E.D.**

The proof of Theorem 1.34 is long. To lighten it somewhat, we prove

(1.33) Lemma *Let G be a plane embedded graph in which all face boundaries are simple cycles. Then* (i) *either G is a simple cycle with two faces, or*
(ii) *for no two faces F, F' is $\partial F \cap \partial F'$ a simple cycle, and if there are 3 faces F_1, F_2, F_3 such that*

$$Q_1 = \partial F_1 \cap \partial F_2, Q_2 = \partial F_2 \cap \partial F_3, \quad \textit{and} \quad Q_3 = \partial F_3 \cap \partial F_1$$

are all nonempty and connected, therefore simple paths, and they all join the same two vertices u and v, then there are exactly three faces, and G consists of two nodes connected by three paths.

Proof. Since all face boundaries are simple cycles, G is biconnected, hence connected.

(i) Suppose $\partial F \cap \partial F' = \partial F$, that is $\partial F \cap \partial F'$ is a Jordan curve J. By Theorem 1.7 (ii), F is the inside of J and F' the outside or vice-versa, so G is a simple cycle with two faces.

(ii) W.l.o.g. F_1 and F_2 are bounded. Their intersection Q_1 is a simple path, which means that $X = \overline{F_1} \cup \overline{F_2}$ is simply connected, and $\partial X = \partial F_1 \cup \partial F_2 \backslash \mathrm{interior}(Q_1)$.

The only faces meeting the relative interior of Q_1 (respectively, Q_3) are F_1 and F_2 (respectively, F_3 and F_1), so $Q_1 \neq Q_3$. These are different paths joining u to v on ∂F_1, so $\partial F_1 = Q_1 \cup Q_3$. Again, $\partial F_2 = Q_1 \cup Q_2$, Thus $\partial X = Q_2 \cup Q_3 = \partial F_3$.

F_3 is either the inside or outside of ∂F_3 (Theorem 1.7), but $F_1 \cup F_2$ are inside, so it is the outside, and F_3 is the unbounded face. Thus there are three faces and G is the union of three paths $Q_1 \cup Q_2 \cup Q_3$ with two nodes in common. **Q.E.D.**

(1.34) Theorem *A plane (straight-edge) embedded graph is nodally 3-connected iff it is biconnected and the intersection of any two face boundaries is connected.*

Proof. We can assume G is biconnected, since that is required for nodal 3-connectivity. Since G is biconnected either it is empty or trivial, or a single edge, or every face is bounded by a simple cycle. In the first three cases the graph is obviously nodally 3-connected and biconnected with one face, so we need only consider the fourth case and can assume that every face is bounded by a simple cycle.

We can assume that G is straight-edge embedded. Therefore the boundary of every face is a simple polygon.

Only if: Suppose F_1 and F_2 are different faces and $\partial F_1 \cap \partial F_2$ is disconnected. R.T.P. G is not nodally 3-connected.

Let u and v be vertices in different components of $\partial F_1 \cap \partial F_2$. For $i = 1, 2$ there are two paths P_i and Q_i joining u to v in ∂F_i. These paths are polygonal.

One can also construct a path P_1' within F_1, loosely speaking by displacing P_1 slightly into F_1, and connecting its endpoints to u and v. The resulting path is in F_1 except at its endpoints. Similarly one can construct a path P_2' in F_2 except at its endpoints. These paths together form a (polygonal) Jordan curve J which meets G only at u and v. By construction, $P_1 \cup P_2$ is inside J and $Q_1 \cup Q_2$ is outside J.

Let H (respectively, K) be the subgraph consisting of all vertices and edges of G which lie inside or on J (respectively, outside or on J). The only vertices in $H \cap K$ are u and v, and $H \cap K$ contains no edge. H contains $P_1 \cup P_2$ and therefore is not a path graph, since otherwise $P_1 = P_2$ and u and v would be in the same component of $\partial F_1 \cap \partial F_2$. Similarly K is not a path graph. Therefore G is not nodally 3-connected.

If: Suppose G is biconnected but not nodally 3-connected, and H, K, u, v are witnesses. G has more than one face, so all face boundaries are simple cycles.

Claim 1. The subgraphs $H \backslash K$ and $K \backslash H$ are nonempty. If every vertex in K were also in H, then the vertices in K are in $H \cap K$, that is, u and v. Either K has no edges, in which case $H = G$, or it has the edge $\{u, v\}$ and is a path graph. Neither is possible. Therefore $H \backslash K$ and similarly $K \backslash H$ are nonempty.

Claim 2. Neither u nor v are isolated vertices in H nor in K.

Otherwise suppose u is isolated in K. Let L be any path joining $H \backslash K$ to $K \backslash H$. By Lemma 1.32, every path connecting $H \backslash K$ to $K \backslash H$ contains a vertex, u or v, incident to edges from H and from K. By hypothesis, u is not; so every such path contains v. By Claim 1, at least one such path exists, so $G \backslash v$ is not connected, and G is not biconnected.

Claim 3. Both u and v have neighbours both in $H \backslash K$ and in $K \backslash H$. Suppose all neighbours of u are in H. Since u is not isolated in K, there is an edge $\{u, t\}$ in K incident to u. But t is a neighbour of u, therefore $t \in H \cap K$, so $t = v$. The only edge in K incident to u is $\{u, v\}$.

Consider a path in G joining $H \backslash K$ to $K \backslash H$. Let t be the first vertex where the path meets $K \backslash H$, and let s be the vertex before t on the path. Since $\{s, t\} \in K$ and $s \notin K \backslash H$, $s \in H \cap K$: $s = u$ or $s = v$. However, if $s = u$, then, since $t \in K$, $t = v$ and $t \notin K \backslash H$. Therefore $s = v$. This implies that every path from $H \backslash K$ to $K \backslash H$ contains v. Again by Claim 1, such paths exist, so G is not biconnected.

This contradiction shows that not all neighbours of u are in H; neither are they in K, and the same goes for v.

Claim 4. The vertices u and v share a face in common. Otherwise let $x_1, \ldots, x_k$ be the neighbours of u. We know (Lemma 1.29) that they are all connected by paths in $B \backslash u$, where B is the union of boundaries of bounded faces incident to u. Assuming v is incident to none of these faces, these paths would also avoid v. This implies that all neighbours of u are in H or in K, contradicting Claim 3.

Claim 5. The vertices u and v have at least two faces in common. Let $F_1, \ldots$ be the faces incident to u in anticlockwise order around u. At least one of these faces, w.l.o.g. F_1, is incident to u and to v. Suppose no other face is.

There are two cases. If u or v, w.l.o.g. u, is an internal vertex, then all faces incident to u are bounded, and by Lemma 1.29, the subgraph $\bigcup_{i\geq 2}(\partial F_i \backslash u)$ would be connected and contain neither u nor v. Then all vertices in this subgraph would belong to H or to K. Since it includes all neighbours of u in G, it would contradict Claim 3.

If both u and v are external vertices, then F_1 is the external face, and all bounded faces incident to u avoid v. This time we consider the subgraph $\bigcup_{i\geq 2}(\partial F_i \backslash u)$. Again this is a connected subgraph containing all neighbours of u in G, and again it omits both u and v, so again all vertices in it are in H or in K, and again Claim 3 is contradicted.

Therefore u and v have at least two faces F and F' in common.

Claim 6. If u and v are incident to three faces F_1, F_2, and F_3, then the boundaries of at least two of these faces have disconnected intersection. Otherwise, by Lemma 1.33, G consists of two nodes u, v connected by three paths. If $G = H \cup K$ where $H \cap K = \{u, v\}$ then H or K is a path graph: G is nodally 3-connected.

This contradiction shows that the one of the pairs $\partial F_i \cap \partial F_j$ is disconnected, as claimed.

Claim 7. If there are exactly two faces F and F' incident to u and to v, then $\partial F \cap \partial F'$ is disconnected.

Otherwise $\partial F \cap \partial F'$ is a path Q' joining a vertex u' to another vertex v' and containing a subpath Q joining u to v. Not all of u', u, v, v' need be distinct, but it is assumed that they occur in that order in Q'.

By Lemma 1.32, all vertices in Q belong to H or to K: w.l.o.g. to H. The boundary cycles ∂F and $\partial F'$ include two other paths, Q_1 and Q_2, respectively, joining u' to v'. Let $J = Q_1 \cup Q_2$, a Jordan curve.

If $u' \neq u$ then J meets $H \cap K$ at v alone, or not at all, and by Lemma 1.32, all vertices on J, plus those in $Q' \backslash Q$, belong to H or to K.

If all vertices on J belong to H, then all vertices outside J also belong to H, because for any vertex y outside J, one can choose a shortest path joining y to a vertex in J. Neither u nor v occur as internal vertices on this path, so all vertices on the path are in H or K (Lemma 1.32), i.e., H, since the last vertex is in H.

We have counted all vertices in G: those outside J, those on J, and those on Q', and all are in H, so $H = G$, which is false.

On the other hand, if all vertices on J, and in $Q' \backslash Q$, belong to K, then all vertices outside J belong to K, and $H = Q$ is a path graph, which is false. This proves Claim 7 in the case $u \neq u'$, and by symmetry in the case $v \neq v'$.

If $u = u'$ and $v = v'$ then $Q = Q'$: let Q_1 and Q_2 be the other subpaths joining u to v in ∂F and $\partial F'$ respectively. By Lemma 1.32, each subpath Q_i is contained in H or in K. Again we have a Jordan curve $J = Q_1 \cup Q_2$.

If u and v are not both external vertices, w.l.o.g. u is an internal vertex, then F and F' are bounded faces incident to u, and since $\partial F \cap \partial F' = Q$, they are consecutive in cyclic order. Let u_1 (respectively, u_2) be the second vertex (following u) in Q_1 (respectively, Q_2). The only faces incident to u and to v are F and F', so u_1 and u_2 differ from v and u_1 and u_2 are connected by a path which avoids u and v (Lemma 1.29). Therefore, by Lemma 1.32, u_1 and u_2 are both

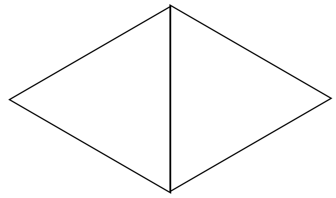

Figure 4: a nodally 3-connected but not triconnected triangulated planar graph

in H or in K, and so are all vertices on J. The same goes for all vertices outside J, so either $H = G$ or $H = Q$ is a path graph, a contradiction.

This leaves the case where u and v are external vertices with exactly two faces in common, F and F', whose boundaries have connected intersection. Since u and v are external vertices, one of these faces, F', say, is the external face. Since G is not nodally 3-connected, it is not a simple cycle, and $Q = \partial F \cap \partial F'$ is a simple path joining u to v (Lemma 1.33). Let Q_1 and Q_2 be the other paths joining u to v on ∂F (respectively, $\partial F'$). $\partial F' = Q \cup Q_2$ is the external cycle, a Jordan curve, and Q_1 separates its interior into two regions of which F is one. Let $J = Q_1 \cup Q_2$. It is a Jordan curve surrounding the other region.

Let $u_i,\ i = 1, 2$, be the second vertices on Q_i. Again there is a path joining u_1 to u_2 which avoids u and v, and all vertices on J are in H or K, and the same holds for all vertices inside J. If they are all in H then $H = G$, and if they are all in K then $H = Q$, a simple path. This contradiction finishes the proof of Claim 7.

Claims 6 and 7 taken together amount to the desired result. **Q.E.D.**

(1.35) Chord-free triangulated graphs. A triangulated plane embedded graph is one in which every bounded face is bounded by three edges. In a triangulated biconnected graph the external boundary is also a simple cycle. It can only fail to be nodally 3-connected if a bounded face meets the external boundary in a disconnected set. Equivalently, one of its edges is a chord joining two vertices on the external boundary, and the other two edges are not both on the external boundary [17].

The graph in Figure 4 is nodally 3-connected but not triconnected.

A fully triangulated planar graph is a triangulated planar graph in which there are three external edges. In other words, the external face also is bounded by a 3-cycle. Therefore the external cycle has no chords, so every fully triangulated planar graph is nodally 3-connected.

Also let G be a fully triangulated planar graph containing a vertex v of degree 2. Let u and w be the neighbours of v. There are only two faces incident to v and they are both incident to u, v, and w. One of them must be the external face. Thus u, v, and w are the three external vertices. They also bound the only bounded face. G is a 3-cycle, and therefore triconnected.

On the other hand, if G is fully triangulated then it is nodally 3-connected, so if it contains no vertex of degree 2 then it is triconnected (Proposition 1.23). Therefore

(1.36) Corollary *Every fully triangulated planar graph is triconnected.* ∎

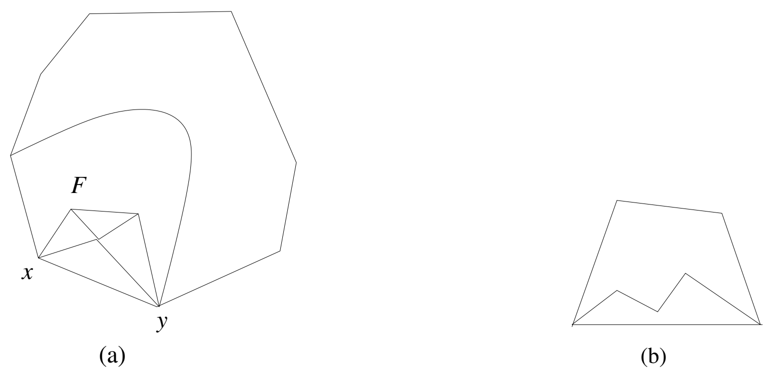

Figure 5: (a) inverted subgraph. (b) A nodally 3-connected graph which is not convex embeddable.

2 Conditions for a convex combination map to be an embedding

In this section we consider a plane embedded graph G whose external boundary is a simple cycle.

(2.1) If a convex combination map of G is an embedding, then it is a convex embedding (Lemma 1.19), so every face boundary is a simple cycle and the intersection of every two bounded faces is convex, hence connected. Also, if a bounded face meets both ends of an external edge, then it is incident to that edge. This gives three conditions necessary for the existence of a convex embedding, and hence for a convex combination map.

The first two conditions, and a weakened version of the third, were given by Stein [13], investigating the existence of convex embeddings. He allowed new vertices to be added within edges so effectively edges are mapped to polygonal curves, weakening the third condition in the following definition.

(2.2) Definition *Let G be a plane embedded biconnected graph. If a bounded face F meets both ends of an external edge, but F is not incident to that edge, then the subgraph between F and that edge is called an* inverted subgraph. *See Figure 5.*

G is convex embeddable *if G is nonempty, every face boundary is a simple cycle, the intersection of every two bounded faces is connected, and there are no inverted subgraphs.*

The phrase 'convex embeddable' suggests that G admits a convex embedding, and this will prove to be true (Theorem 2.23). The phrase 'inverted subgraph' is used because it is possible, by repeatedly reflecting inverted subgraphs through the external boundary, to produce an embedding in which there are no inverted subgraphs.

(2.3) Lemma *If some convex combination map of G is an embedding then G is convex embeddable (immediate from Paragraph 2.1).* ∎

There is one class of nodally 3-connected plane-embedded graphs which are not convex embeddable. These graphs have two nodes joined by three paths of which one is an external edge (Figure 5). Apart from these graphs, every nodally 3-connected graph is convex embeddable.

The aim of this section is to prove that if G is convex embeddable, then every convex combination map of G is an embedding. This has already been shown by Floater for triangulated planar graphs (1.26) and we shall depend heavily on that result. The point here is that we can consider limiting cases of convex combination maps, which would make no sense for barycentric maps. Rather than taking the more obvious approach and attempting induction on the number of faces of G, we can use Floater's result to describe a convex combination map f as a *limit* of straight-edge embeddings f^δ.

For the remainder of the section, G will be a plane embedded graph whose boundary is a simple cycle, and f a convex combination map of G. We shall use P to denote the convex polygon whose corners are the images of external vertices, Also, λ_{uv} are the coefficients associated with f.

(2.4) Lemma *If G is biconnected, and u is an external vertex, then for all vertices $v \neq u$, $f(v) \neq f(u)$.*

Proof. Since u is external, $f(u)$ is a corner of P, and it is not a proper convex combination of any other subset of hull(P).

Let S be the set of all vertices v such that $f(v) = f(u)$. Note that u is the only external vertex in S.

We assume that S contains some vertex besides u, or equivalently, S contains at least one internal vertex.

No internal vertex $v \in S$ can be adjacent to any vertex $w \notin S$. Otherwise v would have a neighbour w with $f(w) \neq f(v)$, $f(v)$ would be a proper convex combination of points in hull(P) including $f(w) \neq f(v) \in$ hull(P), and $f(v)$ would not be a corner of hull(P).

Let H be the subgraph of G spanned by S:

$$H = (S, \{\{u,v\} \in G : \ u, v \in S\}).$$

Claim that H is connected. Otherwise it has a connected component K not containing u. All vertices in K are internal vertices of G, so all vertices adjacent (in G) to vertices in K are also in K: K is a connected component of G not containing u, so G is disconnected, proving the claim.

Since H contains other vertices besides u, and is connected, it contains an internal vertex v adjacent to u.

Therefore u is adjacent in G to a vertex $v \in S$. Since u is adjacent to two external vertices, u is also adjacent to a vertex $w \notin S$.

Let Π be a path from v to w in G. There must be at least two consecutive vertices x, y in Π where $x \in S$ and $y \notin S$. Then x cannot be an internal node, so $x = u$. This shows that every path from v to w contains u, so $G \backslash \{u\}$ is disconnected and G is not biconnected. **Q.E.D.**

(2.5) Definition of the maps f^δ. Let G' be obtained by triangulating G (Proposition 1.12). G and G' have the same vertices, the same internal vertices, and the same external vertices. For each internal vertex u let Γ_u be its neighbours in G and $\Gamma'_u \supseteq \Gamma_u$ its neighbours in G'.

For any δ, $0 \leq \delta < 1$, internal vertex u and vertex v, let

$$\lambda^\delta_{uv} = \lambda_{uv} \quad \text{if } \Gamma'_u \backslash \Gamma_u = \emptyset.$$

Otherwise, $\Gamma'_u \neq \Gamma_u$:

$$\lambda^\delta_{uv} = \begin{cases} \frac{\delta}{|\Gamma'_u \backslash \Gamma_u|} & \text{if } v \in \Gamma'_u \backslash \Gamma_u, \\ (1-\delta)\lambda_{uv} & \text{if } v \in \Gamma_u, \\ 0 & \text{otherwise.} \end{cases}$$

If u is an external vertex, $\lambda^\delta_{uv} = 1$ if $u = v$ and 0 if $u \neq v$, just as with λ_{uv}. (See Definition 1.16.)

(2.6) Definition *With G, G', f and δ as just introduced, let f' be the convex combination map of G' with with coefficients λ^δ_{uv} and $f'(x) = f(x)$ for each external vertex x. This is a straight-edge embedding if $\delta > 0$ (Proposition 1.26).*

We define f^δ as the restriction of f' to G.

Recall (Paragraph 1.20) that f and f^δ can be identified with column vectors of height $2m$, which allows us to define the distance between them. It is most natural to define

$$||f - f^\delta|| = \max\{|f(v) - f^\delta(v)| : \; v \text{ a vertex}\}.$$

(2.7) Lemma $\lim_{\delta \to 0} f^\delta = f$.

Proof. The map f is the unique solution to $AX = B$, and f^δ is the unique solution to equations of the form $(A + \delta A')X = B$, where A is the matrix defining f. Also A is invertible (Lemma 1.21), so for small δ the map $\delta \mapsto (A+\delta A')^{-1}$ is well-defined and continuous. Therefore, as $\delta \to 0$, $f^\delta \to f$. **Q.E.D.**

(2.8) Remarks about the map f^δ.

- The map f^δ is a straight-edge embedding if $\delta > 0$, but is the restriction of a convex combination map f' of a triangulated graph, not itself a convex combination map of G. Face boundaries are mapped to simple polygons under f^δ. They are not necessarily convex.
- $f^0 = f$ is a convex combination map of G.
- Since $f = f^0 = \lim_{\delta \to 0} f^\delta$, even though f might not be an embedding, it fails to be only because edges may collapse to points and faces collapse to line-segments or points.
- The map f partially preserves the cyclic order of edges around a vertex, but edges may collapse to points or consecutive edges may overlap. The interpretation is that the face between them has collapsed under f.

(2.9) Extending f^δ to a homeomorphism. The graph G' is a plane embedded graph and all its bounded faces are bounded by 3-edge Jordan curves. It can be arranged that G' is embedded with straight edges, hence so is G. Fix δ, $0 < \delta < 1$. Let f' and f^δ be defined as above.

By Floater's result (Proposition 1.26), f' is a straight-edge embedding of G'. Let u, v, w be the three vertices on the boundary of a bounded (triangular) face of G'. The map f' can be extended in a piecewise-linear fashion to this face and all bounded faces. Let $\overline{G}$ be the complement of the unbounded face of G (and of G'). The map f' extended to the bounded faces of G' is a piecewise-linear homeomorphism from $\overline{G}$ onto hull(P). This homeomorphism can also be written as f^δ.

Thus f^δ means either a straight-edge embedding of G or a piecewise-linear homeomorphism from $\overline{G}$ onto hull(P).

(2.10) Definition *An edge e is* degenerate *if $f(e)$ is a single point.*

(2.11) Lemma *For any nondegenerate edges e_1 and e_2, $f(e_1)$ does not meet the interior of $f(e_2)$ transversally.*

Proof. Suppose otherwise. The interiors of $f(e_1)$ and $f(e_2)$ cannot intersect transversally, since otherwise for some $\delta > 0$ the interiors of $f^\delta(e_1)$ and $f^\delta(e_2)$ would intersect transversally. Suppose that $e_1 = \{u, v\}$ and $f(v)$ is interior to $f(e_2)$. Let L be the line through $f(e_2)$. The vertex u is a neighbour of v such that $f(u) \notin L$, and v cannot have another neighbour w such that $f(w)$ is on the other side of L, since otherwise for some $\delta > 0$ the line segment $f^\delta(e_2)$ and the broken line $f^\delta(u)f^\delta(v)f^\delta(w)$ would intersect in their interiors. Also, $f(v)$ is interior to $f(e_2)$, hence inside P, and v is an internal vertex. This contradicts Lemma 1.17. **Q.E.D.**

The following proposition is a simple corollary to the Jordan Curve Theorem.

(2.12) Proposition (interlacing property). *Let J be a Jordan curve and $a, b, c, d \in J$ be four points in cyclic order around J. If X and Y are paths inside J meeting J only at a and c, b and d, respectively, then X and Y intersect inside J.* ∎

(2.13) Lemma *If F is a (bounded) face where ∂F is a simple cycle, and p is a point such that for three or more edges e on ∂F, $f(e)$ is nondegenerate and incident to p, then all edge-images $f(e)$, which are incident to p, are collinear.*

Proof. Let $\partial F = v_1, \ldots, v_n$,

$$\eta \quad = \quad \frac{\min\{|f(v) - p| : \ v \text{ a vertex and } f(v) \neq p\}}{2},$$

and D be the closed disc with centre p and radius η.

For every vertex v, $f(v) \in D \iff f(v) = p$. Choose $\varepsilon > 0$ so that for all δ with $0 \leq \delta \leq \varepsilon$ and every vertex v, $f^\delta(v) \in D \iff f(v) = p$.

Given adjacent vertices u and v on ∂F such that $f(v) = p$ and $f(u) \neq p$, suppose $u = v_{i_1}$. Beginning with $u, v \ldots$, traverse ∂F in cyclic order until the next vertex v_{i_2} is reached such that $f(v_{i_2}) \neq p$. Continue the traversal in cyclic order until the next such pair u, v is found, hence identifying a subpath $v_{i_3}, \ldots, v_{i_4}$, and continue in this way until ∂F has been traversed fully. In this way we get a series $I_1 = v_{i_1}, \ldots, v_{i_2}$, I_2,$\ldots I_k$, of paths in ∂F, joining vertices v_{i_j} to $v_{i_{j+1}}$ ($j = 1, 3, 5 \ldots$) where $f(v_{i_j}) \notin D$ for all j, and all inner vertices (Paragraph 1.1) in each path I_j are mapped to p. By hypothesis, $k \geq 2$.

For $1 \leq j \leq k$ let V_j be the set of inner vertices in I_j, and let U_j consist of every vertex in G which is not in V_j but which has a neighbour in V_j. U_j can include vertices not in ∂F.

Since every two vertices in V_j are connected by a path in V_j, every two vertices a, b in U_j are connected by a (unique) simple path P_{ab} whose inner vertices are in V_j. The image $f(P_{ab})$ is the polygonal path $f(a)pf(b)$.

Claim: given $a, b \in U_1$ and $c, d \in U_2$, the paths $f(a)pf(b)$ and $f(c)pf(d)$ do not cross, meaning that given $\partial D \cap pf(a) = a'$, with b', c', d' similarly defined, the points

$$a', c', b', d'$$

are *not* in strict cyclic order around ∂D.

Otherwise let $X = D \cap f^{\varepsilon}(P_{ab})$ and $Y = D \cap f^{\varepsilon}(P_{cd})$. The endpoints of X and Y are alternating in cyclic order around ∂D. By Proposition 2.12, X and Y intersect in the interior of D. Since f^{ε} is an embedding, the intersection is contained in $f^{\delta}(V_1 \cap V_2)$, whereas $V_1 \cap V_2 = \emptyset$. This contradiction proves the claim.

Let C_j be the set of points on ∂D where edge-images $f(u)f(v)$, $u \in U_j, v \in V_j$, intersect ∂D. Let c_j be the smallest arc of ∂D containing C_j. This is ambiguous only when $k = 2$ and $C_1 = C_2$ contains two diametrically opposed points, in which case we may choose c_1 and c_2 either way (but different).

By the above claim, c_1 and c_2 do not overlap. Hence they cannot both subtend reflex angles at p. Without loss of generality, c_1 subtends an angle $\alpha \leq 180°$ at p. Let L be a line through p which does not intersect the relative interior of c_1. Then for all neighbours u of v_{i_1+1} in G, $f(u)$ is on L, or on the same side of L as is $f(v_{i_1})$. But since there exists more than one vertex v such that $p = f(v)$, v_{i_1+1} is an internal vertex (Lemma 2.4), and $f(v_{i_1+1})$ is a proper weighted average of its neighbours. By Lemma 1.17, $C_1 = \partial D \cap L$ and $\alpha = 180°$. Therefore c_2 does not subtend a reflex angle at p, and by the same argument $C_2 = \partial D \cap L$. Therefore $c_1 \cup c_2 = \partial D$, $k = 2$, and for all edges $\{u, v\}$ with $f(v) = p$, $f(u) \in L$, as claimed. **Q.E.D.**

As already mentioned, this section aims to prove that if G is convex embeddable then f is an embedding. We show that there are no degenerate edges, and therefore f is injective on faces. It will follow by Tutte's argument [16] that f is an embedding. We first study what happens if f collapses faces, and this leads us to consider the notion of monotone paths. The definition needs to allow for the possibility that f maps different vertices to the same point.

(2.14) Definition *Given $0 \leq \varepsilon < 1$ and a line V, V is* ε-vertex-avoiding *or simply* vertex-avoiding *when $\varepsilon = 0$, if, for all $\delta \leq \varepsilon$, and all vertices v, $f^{\delta}(v) \notin V$.*

Let V be a directed vertex-avoiding line. Given nondegenerate edges e_1, e_2, e_1 is above *e_2 on V if V intersects the relative interiors of $f(e_i)$, $i = 1, 2$, and for some ε such that V is ε-vertex-avoiding, V intersects the relative interiors of $f^{\varepsilon}(e_i)$ at points a_i where V is directed from a_2 to a_1.*

Let L be a directed line. A path $v_i, \ldots, v_k$ in G is monotone *(on L) if all points $f(v_i), \ldots, f(v_k)$ belong to L and are monotone non-decreasing or monotone non-increasing on L.*

Given two paths s_1 and s_2 which are monotone on L, and which have no vertices in common except perhaps at endpoints, we say that s_1 is above *s_2 if there exists a directed vertex-avoiding line V positively normal to L and edges $e_i \in s_i$ such that e_1 is above e_2 on V. (See [4], §11.2.)*

(2.15) Lemma *If s_1 and s_2 are monotone on L, and they are vertex-disjoint except perhaps at endpoints, and s_1 is above s_2, then s_2 is not above s_1.*

Proof. Given V and edges e_1 on s_1 and e_2 on s_2, and $\varepsilon > 0$ so that V is ε-vertex-avoiding, then the relative order of $V \cap f^{\delta}(e_1)$ and $V \cap f^{\delta}(e_2)$ is unchanged for $0 < \delta \leq \varepsilon$, since otherwise for some $\delta > 0$ $f^\delta(e_1) \cap f^\delta(e_2) \neq \emptyset$. So if $f^\delta(e_1)$ is above $f^\delta(e_2)$ on V for $\delta = \varepsilon$ then it holds for all positive $\delta \leq \varepsilon$.

Again, suppose that s_2 is also above s_1 according to different data $V', e_1', e_2', \varepsilon'$. We can replace ε and ε' by their minimum and assume $\varepsilon = \varepsilon'$. We could enclose these path-images by rectangles bounded on two sides by V and V'; the intersection points have the interlacing property so $f^\varepsilon(s_1)$ and $f^\varepsilon(s_2)$ would intersect in their interiors (Proposition 2.12), which is impossible. **Q.E.D.**

If $e \in \partial F$ and ∂F is a simple cycle (which is always true when G is biconnected), then for any $\varepsilon > 0$, $f^\varepsilon(F)$ is incident to e from just one side.

(2.16) Lemma *Let G be biconnected, F a face, and $e = \{u, v\}$ a nondegenerate edge in ∂F. Let E be the directed line-segment $f(u)f(v)$ and for any ε, $0 < \varepsilon < 1$, let $E^\varepsilon = f^\varepsilon(u)f^\varepsilon(v)$. Then if ε is sufficiently small, for $0 < \delta \leq \varepsilon$, $f^\delta(F)$ is always on the same side (right or left) of E^δ.*

Proof. Let V be a vertex-avoiding line intersecting E, and choose $\varepsilon > 0$ so that V is ε-vertex-avoiding. Given $0 < \delta \leq \varepsilon$, let $X^\delta = V \cap f^\delta(\partial F)$. If F is the external face then $f^\delta(\partial F) = P$ and the result is trivial. We may assume that F is bounded so for all $\delta > 0$ $f^\delta(\partial F)$ is a simple polygon containing $f^\delta(F)$.

X^δ divides V into open intervals alternately inside and outside $f^\delta(F)$. Also, $f(F)$ is to the right of E^δ if and only if the number of points in X^δ to the left of E^δ is even. By choice of ε this number is constant for $0 < \delta \leq \varepsilon$. **Q.E.D.**

(2.17) Definition *Let F be a bounded face with ∂F a simple cycle $v_1, \ldots, v_n$: $f(\partial F)$ is a possibly degenerate polygon, a union of k line-segments $p_i p_{i+1}$ (interpreting p_{k+1} as p_1): $p_1 = f(v_1)$; if for some $i \leq n$, $f(v_i) \neq p_1$, then $p_2 = f(v_{i_1})$ where i_1 is the least such i, and so on up to $p_k = f(v_n)$ (without loss of generality, either $k = 1$ or $p_k \neq p_1$).*

A reflex corner *is a triple $p_{\ell-1}, p_\ell, p_{\ell+1}$ of adjacent corners which are collinear, with $p_{\ell-1}$ and $p_{\ell+1}$ on the same side of p_ℓ. (Interpret p_{k+1} as p_1.)*

Next we show that reflex corners do not exist. Intuitively, if $f(\partial F)$ made a 180° turn at p_ℓ, then $f(F)$ would either be trapped in the line $p_\ell p_{\ell+1}$ or it would surround it. The latter is impossible since p_ℓ is a weighted average of neighbours (Figure 6). This means that a sequence of monotone paths spirals inwards, and the first edge to leave the line $p_\ell p_{\ell+1}$ crosses the spiral (Figure 7).

(2.18) Lemma *If G is biconnected, F a face, and $f(\partial F)$ is not collinear, then there are no reflex corners on $f(\partial F)$.*

Proof. Suppose otherwise. Let $S = v_i \ldots v_k$ be the longest subpath of ∂F such that $p_{\ell-1}, p_\ell, p_{\ell+1}$ is part of $f(S)$ and all of $f(S)$ is collinear. Since $f(\partial F)$ is not contained in a line, S is a proper subpath of ∂F. Let $I = f(S)$. I is a nondegenerate closed line-segment. **Claim** $I = f(v_i)f(v_k)$.

By definition of S, $f(v_{i-1})$ is not collinear with I. If $f(v_i)$ were interior to I then either for some other edge $e \in \partial F$ the edge $f(v_{i-1})f(v_i)$ would meet the relative interior of $f(e)$

transversally, which is impossible (Lemma 2.11), or there would be three or more edges e in ∂F such that $f(e)$ was nondegenerate and met $f(v_i)$, not all collinear, which is impossible (Lemma 2.13). The same arguments apply to v_k. Thus v_i and v_k are endpoints of I. Therefore either $I = f(v_i)f(v_k)$, or $f(v_i) = f(v_k) = p$, and for some other corner q, $I = pq$.

The latter is impossible since both edges $f(v_{i-1})f(v_i)$ and $f(v_k)f(v_{k+1})$ would be incident to p, and a third edge in ∂F, mapped into I, would be incident to p, and they would not be collinear, contradicting Lemma 2.13. Hence $I = f(v_i)f(v_k)$, as claimed.

We may assume that I is contained in the x-axis with $f(v_i)$ left of $f(v_k)$. Also, without loss of generality, we may assume that for all sufficiently small δ, $f^\delta(F)$ is to the right of $f^\delta(v_j)f^\delta(v_{j+1})$ for $i-1 \le j \le k$ (Lemma 2.16). If it is not, rotate the coordinate system through $180°$.

Let $s_1 = v_i, \ldots, v_{i_1}$ be a maximal monotone path (with respect to the x-axis), then let $s_2 = v_{i_1}, \ldots, v_{i_2}$ be a maximal monotone path (in the other direction), and continue until all of $v_i \ldots v_k$ has been subdivided into m monotone paths. Since $p_{\ell-1}p_\ell p_{\ell+1} \subseteq f(S)$, S is not monotone, so $m \ge 2$.

Claim: s_1 is above s_2. Let $\{v_{r-1}, v_r\}$ be the last nondegenerate edge in s_1 and $\{v_t, v_{t+1}\}$ the first in s_2: $f(v_r) = f(v_t)$. Let $q = f(v_r) = f(v_t)$.

Choose a vertex-avoiding vertical line V which intersects the interiors of $f(v_{r-1})q$ and $qf(v_{t+1})$.

For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all vertices v, $|f^\delta(v) - f(v)| < \varepsilon$, and V is δ-vertex-avoiding. Let q_1 and q_2 be the points where V intersects the interiors of $f^\delta(v_{r-1})f^\delta(v_r)$ and $f^\delta(v_t)f^\delta(v_{t+1})$. We want to show that q_1 is above q_2. Suppose otherwise, so q_1 is below q_2.

Suppose $V = \{(a, y) :\ y \in \mathbb{R}\}$.

There is a topological sub-path π of $f^\delta(s_1 \cup s_2)$ joining q_1 to q_2 and, since $f^\delta(s_1)$ and $f^\delta(s_2)$ cross V from left to right and right to left respectively, and V is δ-vertex-avoiding, π is contained in the half-plane $x \ge a$. By choice of ε, π is contained in the strip $-\varepsilon \le y \le \varepsilon$ and also in the open half-plane $x < b + \varepsilon$, where $q = (b, 0)$.

Thus $\pi \subseteq R_\varepsilon$ where R_ε is the rectangle

$$x < b + \varepsilon, -\varepsilon < y < \varepsilon.$$

For any edge e incident to any vertex on this path,

$$f^\delta(e) \cap \{(x, y) :\ x \ge a\} \subseteq R_\varepsilon.$$

Allowing $\varepsilon \to 0$, we deduce that for every such edge e, $f(e)$ lies in the x-axis and its right-hand end is q. See Figure 6. In particular, v_r must be an internal vertex, since every corner of P has non-collinear incident edges.

Since $f(v_{r-1})$ is left of q, so is $f(v_r)$ (Lemma 1.17). This is a contradiction: s_2 is below s_1, as claimed. Similarly s_3 is above s_2, s_4 below s_3, and so on.

Claim: for $3 \le h \le m$, s_h is below s_1 and above s_2. To begin with, let e_2 and e_3 be the leftmost nondegenerate edges occurring in s_2 and s_3 (last and first, respectively). Since $f(s_1)$ contains the leftmost point $f(v_i)$, and the rightmost points in $f(s_1)$ and $f(s_2)$ are the same, $f(e_2)$ and $f(e_3)$ are contained within $f(s_1)$. Also, e_3 is above e_2. It follows that there exists a nondegenerate edge e_1 in s_1 and a vertical vertex-avoiding line V which intersects $f(e_1), f(e_2)$, and $f(e_3)$. Suppose that s_3 is above s_1.

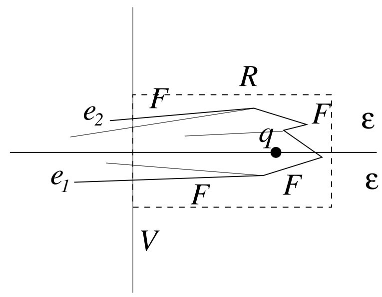

Figure 6: Why s_1 cannot be below s_2.

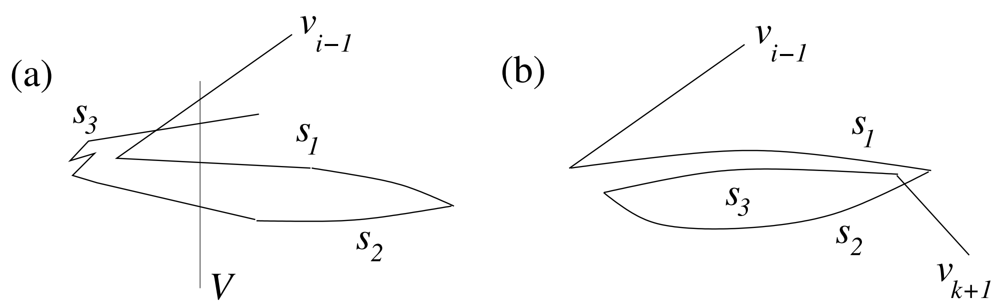

Figure 7: (a) s_3 is below s_1; (b) v_k is between s_1 and s_2.

Choose $\varepsilon > 0$ so that V is ε-vertex-avoiding. Let q_2 be the intersection of $f^\varepsilon(e_2)$ with V, and similarly q_3. By hypothesis (and Lemma 2.15), $f(e_1)$ crosses V between q_2 and q_3. There is a topological path $\pi \subseteq f^\varepsilon(s_2 \cup s_3)$ joining q_1 to q_3 which can be completed along q_3q_1 to a Jordan curve J which is crossed by $f^\varepsilon(e_1)$. The left endpoint p of $f^\varepsilon(s_1)$ is inside J. J, and p, can be made arbitrarily close to the x-axis, and $f^\varepsilon(v_{i-1})f^\varepsilon(v_i)$ connects p to a point bounded away from the x-axis, so if ε is small enough then $f^\varepsilon(v_{i-1})f^\varepsilon(v_i)$ crosses π, which is false. Therefore s_3 is between s_1 and s_2. See Figure 7.

If s_4 exists, then the right endpoints of $f(s_3)$ and $f(s_4)$ coincide, and it follows easily that s_4 is between s_1 and s_2. Generally speaking, if s_g exists, and g is odd (respectively, even), then the argument concerning s_3 (respectively, s_4) applies to show s_g is between s_1 and s_2. It follows that $f^\varepsilon(v_k)$ is between $f^\varepsilon(s_1)$ and $f^\varepsilon(s_2)$ for sufficiently small ε, and $f^\varepsilon(v_k)f^\varepsilon(v_{k+1})$ crosses $f^\varepsilon(s_1 \cup s_2)$, which is impossible. This contradiction shows that no reflex corner exists. **Q.E.D.**

(2.19) Corollary *If G is biconnected and F is a face of G then $f(\partial F)$ is either a point, or a line-segment, or a convex polygon.*

Proof. Let $S = f(\partial F)$ be described in the usual way as a union of line-segments p_ip_{i+1}, $1 \le i \le k$ (interpret p_{k+1} as p_1). Suppose that not all points p_i are collinear.

Claim that S is a simple polygon (though adjacent line-segments $p_{i-1}p_i$ and p_ip_{i+1} may be collinear). As usual, since S is the limit of simple polygons, it is connected, and edges do not cross though they may overlap.

The interiors of no two edge-images p_ip_{i+1} and p_jp_{j+1} can overlap. Otherwise one can extend them to two maximal collinear chains of edges which overlap. These chains contain no reflex corners (Lemma 2.18). Let I be their intersection. I is bounded by points p incident to the images of three or more edges, not all collinear, which is impossible (Lemma 2.13): this proves that edge images do not overlap.

Again, if a point p is incident to the images of more than two edges, then all these edge-images are collinear (Lemma 2.13), and edge images would overlap, which is false. Therefore S is a simple polygon (though successive edges could be collinear).

It remains to show that S is a convex polygon. Otherwise it has a concave corner $p_{\ell-1}p_\ell p_{\ell+1}$ in the sense that the interior of S is on the concave side of this broken line. In particular, p_ℓ is interior to the convex hull of S so p_ℓ is not a corner of the bounding polygon P.

By the argument showing that reflex corners do not exist, as illustrated in Figure 6, there would exist a vertex v such that $f(v) = p_\ell$, for all neighbours u of v, $f(u)$ is in the convex wedge containing $p_{\ell-1}, p_\ell$, and $p_{\ell+1}$, and for some neighbour u of v, $f(u) \neq f(v)$. Also, v is an internal vertex. This contradicts Lemma 1.17. **Q.E.D.**

(2.20) Lemma *If G is convex embeddable then the map f does not collapse faces onto non-degenerate line-segments.*

Proof. For f to collapse a face F into a nondegenerate line-segment means that $f(\partial F)$ is not a point and is contained in a line L. Suppose this is the case. F must be bounded. Let I be the maximal connected union of nondegenerate line-segments, including $f(\partial F)$, which are collinear and are the images of face-boundaries.

Let V be a vertex-avoiding directed line orthogonal to L which intersects the relative interior of $f(\partial F)$, and is directed into hull(P) (this only matters if $I \subseteq P$). Therefore V intersects at least one edge-image above L. Let e_1' be the highest edge (with respect to the relation 'e_1 is above e_2 on V' (2.14)) such that $f(e_1') \subseteq L$ and $V \cap f(e_1') \neq \emptyset$. Let e_1 be the lowest edge above e_1' along V.

Choose $\varepsilon > 0$ so that V is ε-vertex-avoiding. For all δ with $0 < \delta \leq \varepsilon$, $V \cap f^\delta(F)$ is nonempty.

Also, $V \cap f^\varepsilon(e_1)$ and $V \cap f^\varepsilon(e_1')$ are joined along V by a line-segment which meets no other edge-image. Therefore they are in the same face of $f^\varepsilon(G)$ and hence there exists a (bounded) face F_1 of G containing both e_1 and e_1'. Since $f(F_1)$ intersects $f(e_1)$ and $f(e_1')$, $f(\partial F_1)$ is a convex polygon S_1 joining points p_i, some of which may be collinear, but which are in cyclically monotone order around S_1 (Corollary 2.19). $S_1 \cap L$ is a line-segment I_1. Let $P_1 = u_1, \ldots, v_1$ be the maximal path such that $f(P_1) = I_1$. P_1 contains e_1' and its complementary path $Q_1 \subseteq \partial F_1$, joining u_1 to v_1, contains e_1.

There are two cases: (i) I intersects the interior of hull(P) and (ii) I is contained in a side of P.

In case (i), if we reverse the direction of V, we get corresponding data $e_2', e_2, F_2, S_2, P_2, u_2, v_2$, and Q_2. We shall see that $\partial F_1 \cap \partial F_2$ is disconnected, so G is not convex embeddable.

Without loss of generality, L is the x-axis, $f(u_1)$ is left of $f(v_1)$, and $f(u_2)$ is left of $f(v_2)$.

First, for all sufficiently small δ, $V \cap f^\delta(e_1') \neq V \cap f^\delta(e_2')$. This is because $f(e_1) \cap V$ and $f(e_2) \cap V$ are on opposite sides of L, so for all sufficiently small δ, $V \cap f^\delta(e_1)$ and $V \cap f^\delta(e_2)$ are on opposite sides of L and their distance from L is bounded below, whereas $f^\delta(F)$ can be made arbitrarily close to L. Therefore V intersects $f^\delta(F)$ between $f^\delta(e_1)$ and $f^\delta(e_2)$. By choice of e_1' and e_2', $V \cap f^\delta(F)$ separates $V \cap f^\delta(e_1')$ from $V \cap f^\delta(e_2')$. Hence the intersection-points differ.

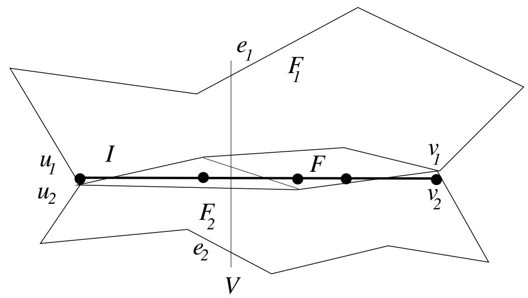

Figure 8: Illustrating F_1 and F_2 under f^δ (Lemma 2.20). Note: $u_1 = u_2$ and $v_1 = v_2$.

Since $f(\partial F_1)$ is a convex polygon (Corollary 2.19) and V is ε-vertex-avoiding and intersects $f(e_1)$ and $f(e_1')$, V intersects $f^\varepsilon(\partial F_1)$ in these edges alone. Hence $V \cap f^\varepsilon(P_1) = V \cap f^\varepsilon(e_1')$. Also $V \cap f^\varepsilon(P_2) = V \cap f^\varepsilon(e_2')$. Therefore $P_1 \neq P_2$.

Next, $f(u_1) = f(u_2)$. Otherwise, without loss of generality, $f(u_1)$ is in the relative interior of $f(u_2)f(v_2)$. Thus $f(u_1)$ is inside P and u_1 is an internal vertex. Since u_1 has a neighbour w in ∂F_1 where $f(w_1) \notin L$, and f is a convex combination map, u_1 has a neighbour y_1 such that $f(y_1)$ and $f(w_1)$ are on opposite sides of L. Then the line-segment $f(u_1)f(y_1)$ intersects the interior of S_2. Therefore, for sufficiently small $\delta > 0$, $f^\delta(u_1)f^\delta(y_1)$ intersects the interior of the face $f^\delta(F_2)$, which is impossible. From this contradiction, $f(u_1) = f(u_2)$, and also u_1 has neighbours w_1 and y_1 such that $f(w_1)$ and $f(y_1)$ are on opposite sides of L; similarly, u_2 has neighbours w_2 and y_2 with $f(w_2)$ and $f(y_2)$ on opposite sides of L. See Figure 8.

Next, $u_1 = u_2$. If $u_1 \neq u_2$ then there are two distinct paths $s_1 = w_1u_1y_1$ and $s_2 = w_2u_2y_2$ such that $f(s_1)$ crosses $f(s_2)$. For sufficiently small $\delta > 0$, $f^\delta(s_1)$ would cross $f^\delta(s_2)$, which is impossible. Hence $u_1 = u_2$. Similarly, $v_1 = v_2$.

Thus $\partial F_1 \cap \partial F_2$ contains u_1 and v_1. If $\partial F_1 \cap \partial F_2$ is connected then it contains a path Q joining u_1 to v_1 in both ∂F_1 and ∂F_2, $Q = P_1$ or $Q = Q_1$, and $Q = P_2$ or $Q = Q_2$. But Q_1 contains $e_1 \notin \partial F_2$, so $Q \neq Q_1$; also, $Q \neq Q_2$. Therefore $P_1 = P_2$ which has already been shown to be false, so $\partial F_1 \cap \partial F_2$ is disconnected. This concludes Case (i).

Case (ii): I is contained in a side of P. Let H be the closed half-plane containing P and bounded by L. We have the data $V, e_1', e_1, F_1, S_1, P_1, u_1, v_1$, and Q_1. First, $f(u_1)$ is a corner of P. Otherwise u_1 is an internal vertex, and since all vertices are mapped into H, and $f(v) \notin L$ where v is the neighbour of u_1 in Q_1, this contradicts Lemma 1.17. Since $f(u_1)$ is a corner, there is only one vertex mapped to $f(u_1)$ (Lemma 2.4), so u_1, and similarly v_1, is an external vertex. Let $e_2' = \{u_1, v_1\}$, so $f(e_2') = I$. $V \cap f(e_1)$ is bounded away from L and and $f(\partial F) \subseteq L$, so for all sufficiently small δ, $V \cap f^\delta(F)$ is between $V \cap f^\delta(e_1')$ and $V \cap f^\delta(e_2')$. Therefore e_2' is not incident to ∂F_1, whereas $u_1, v_1 \in \partial F_1$, and G has an inverted subgraph, which is false. **Q.E.D.**

(2.21) Corollary *If G is convex embeddable and $e \neq e'$ are edges then $f(e)$ and $f(e')$ don't overlap.*

Proof. Otherwise take a directed vertex-avoiding line V intersecting $f(e) \cap f(e')$ orthogonally. Without loss of generality, e is above e' along V. Let F be the face incident to e such that $f^\delta(F)$ is below $f^\delta(e)$ for all sufficiently small δ. $f^\delta(F) \cap V$ is between $f^\delta(e)$ and $f^\delta(e')$,

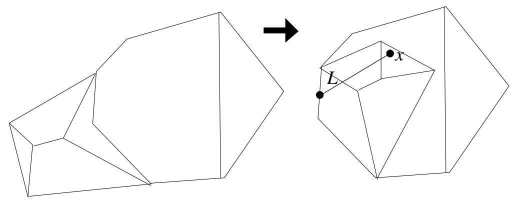

Figure 9: loosely illustrating G and a many-to-one map which is one-to-one on individual faces.

so in the limit $f(\partial F)$ is not a point nor a simple polygon, so it is a nontrivial line-segment (Corollary 2.19), which is impossible. **Q.E.D.**

(2.22) Lemma *If G is convex embeddable, then f does not collapse edges to points.*

Proof. (This is similar to Lemma 1.17.) Otherwise let H be a maximal connected subgraph of G such that $f(H)$ is a single point, p, say. For each $u \in H$, let N_u be the set of neighbours v of u such that $f(v) \neq p$. There must be more than one vertex u such that $N_u \neq \emptyset$, since otherwise G or some $G \backslash u$ would be disconnected.

Given $u_1 \neq u_2 \in H$, $v_i, w_i \in N_{u_i}$, $i = 1, 2$, the paths $f(v_1)f(u_1)f(w_1)$ and $f(v_2)f(u_2)f(w_2)$ cannot cross, since otherwise, for some $\delta > 0$, $f^\delta(v_1)f^\delta(u_1)f^\delta(w_1)$ and $f^\delta(v_2)f^\delta(u_2)f^\delta(w_2)$ would cross.

By Lemma 2.4, all vertices in H are internal. Let D be a closed disc centred at p such that for every vertex v, if $f(v) \neq p$, then $f(v) \notin D$. We can partition ∂D into minimal arcs A_u, one for each u in H such that $N_u \neq \emptyset$, where

$$A_u \supseteq \partial D \cap \{pf(v) : \; v \in N_u\}.$$

By Lemma 1.17, there are exactly two such arcs A_{u_1} and A_{u_2}, disjoint except perhaps at their endpoints, and for all $v \in A_{u_1} \cup A_{u_2}$, $pf(v)$ are collinear, and also u_1 has neighbours v_1 and w_1 in N_{u_1} such that $pf(v_1)$ and $pf(v_2)$ do not overlap. The same goes for u_2. It follows that there must be overlapping edges $pf(v_1)$ and $pf(v_2)$, say, contradicting Corollary 2.21. **Q.E.D.**

We have established that if G is convex embeddable then f maps face boundaries injectively to convex polygons. This is enough to prove that f is an embedding, by Tutte's arguments [16], which are as follows.

Provisionally, let us define $f(F)$ as $f(\partial F) \cup \text{interior}(f(\partial F))$ for every bounded face F.

For every point x inside the bounding (convex) polygon P, its *covering number* is the number of faces F such that $x \in f(F)$. See Figure 9.

This number is 1 on the bounding polygon, and if we take a vertex-avoiding line L from the boundary to x, the number can only change where an edge is crossed. However, to every internal edge e there are exactly two incident faces F_1 and F_2, and $f(F_1)$ and $f(F_2)$ are incident to $f(e)$ from opposite sides. Otherwise $f^\delta(F_1)$ and $f^\delta(F_2)$ would overlap for sufficiently small δ. It follows that the covering number does not change where L crosses edges, so it is 1 for all x, and f is injective. This completes the proof of our main theorem.

(2.23) Theorem *If G is convex embeddable then f is an embedding. Therefore G admits a convex embedding if and only if every convex combination map is an embedding.* ∎

3 Ambient isotopy

In [13], Stein considered plane embedded graphs in which every face boundary is a simple cycle and no two *bounded* faces have disconnected intersection (see also [15]). By our earlier results, all nodally 3-connected plane embedded graphs have this property. Stein showed that all such graphs admit convex embeddings, where the bounded faces map to convex polygons, so long as edges can be embedded piecewise linear rather than straight. Equivalently, one can allow new vertices (of degree 2) to be introduced. The existence of inverted subgraphs becomes irrelevant. Let us call such graphs *general convex embeddable*, or GCE for short. Stein also allowed them to have multiple edges.

Stein remarked in [13] that any two (convex) embeddings, with the same orientation, of a GCE graph are ambient isotopic, but does not include a proof.

(3.1) Definition *Given topological spaces X and Y, an* isotopy *is a continuous map $h : [0,1] \times X \to Y$ such that for each t, $0 \leq t \leq 1$, the map $h_t : X \to Y; \quad x \mapsto h(t,x)$ is a homeomorphism.*

This section gives an outline proof of the following isotopy theorem (Corollary 3.7). Let G^1 and G^2 be two plane embeddings of the same GCE graph G, such that their external boundaries are images of the same cycle C of G, with the same orientation. Then there exists an isotopy: $\mathbb{R}^2 \to \mathbb{R}^2$ taking the vertices, edges, and faces of G^1 to those of G^2.

(3.2) Proposition *Suppose G is a GCE plane embedded graph Then either G has just one bounded face or there exist two bounded faces F' and F'' such that $\partial F' \cap \partial F'' = Q$ is nonempty (and connected), and if $F = F' \cup \mathrm{interior}(Q) \cup F''$, then for every other face A of G, $\partial A \cap \partial F$ is connected.*

Furthermore, if G' is the embedded graph obtained by removing the edges and inner vertices on Q, hence merging F' and F'' into a single face F, then G' is also GCE, with the same external boundary as G. (The first part was proved in [13]*, and the rest follows immediately.)* ∎

(3.3) Definition *Let G^1 and G^2 be two plane embedded graphs. The embeddings are* ambient homeomorphic *(respectively,* ambient isotopic*) if there is a homeomorphism (respectively, an isotopy) from $\mathbb{R}^2$ to itself taking the vertices, edges, and faces of G^1 bijectively onto those of G^2.*

(3.4) Definition *A θ-*graph *is a plane embedded graph consisting of two nodes connected by three disjoint paths. It resembles the Greek letter θ.*

(3.5) Lemma *If G^1 and G^2 are plane embeddings of a θ-graph G, with the same orientation, then they are ambient homeomorphic. (Follows from the Schönflies theorem 1.9: proof omitted.)* ∎

(3.6) Corollary *If G^1 and G^2 be two GCE embeddings of the same graph G with the same orientation and the same boundary cycle, then they are ambient homeomorphic.*

Proof. This is a simple application of Stein's result (Lemma 3.2), and is by induction on the number of bounded faces. If G is a simple cycle then this is just the Schönflies Theorem (Proposition 1.9).

For the inductive step, choose faces F' and F'' of G^1 separated by a path Q such that $F = F' \cup \text{interior}(Q) \cup F''$ has the properties stated in Lemma 3.2. Let H be the subgraph of G obtained by removing the edges and inner vertices of Q, and let H^1 be the modified embedding where F' and F'' are merged into F. Then H^1 is a GCE embedding of H. Similarly a modified embedding H^2 is obtained from G^2. By induction, H^1 and H^2 are ambient homeomorphic through a homeomorphism h'. Let D^1 and D^2 be the images of $\overline{F}$ under the respective embeddings. $D^2 = h'(D^1)$. They contain images Q^1 and Q^2 of the path Q.

By Lemma 3.5, there exists a homeomorphism $h : D^1 \to D^2$ which agrees with h' on ∂D^1 and takes $(F')^1$ to $(F')^2$, $(F'')^1$ to $(F'')^2$, and Q^1 to Q^2, and also takes the vertices and edges in Q^1 to those in Q^2. Extend h to $\mathbb{R}^2$ by making it coincide with h' outside $(\partial F)^1$. Then h is an ambient homeomorphism between G^1 and G^2. **Q.E.D.**

(3.7) Corollary *If G^1 and G^2 are GCE embeddings of the same graph with the same external boundary in the same anticlockwise order C^1 and C^2, then the embeddings are connected by an isotopy.*

Sketch proof. There is an ambient homeomorphism h connecting them (Corollary 3.6). According to [14], h is isotopic to the identity or to reflection in the x-axis. Furthermore, if h preserves the orientation of any Jordan curve, as it does in this case, it is isotopic to the identity. This yields an isotopy carrying G^2 to G^1. ∎

Let G be a convex embeddable plane-embedded graph. We can let G^1 correspond to the identity map on $\mathbb{R}^2$, and G^2 correspond to an orientation-preserving convex combination map f. Then

(3.8) Corollary *Let G be an convex embeddable graph and suppose f is an orientation-preserving convex-combination map. Then there is an istotopy of $\mathbb{R}^2$ taking each vertex v, edge e, and face F of G to $f(v)$, $f(e)$, and $f(F)$, respectively.* ∎

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