A Reynolds– and Prandtl–uniform numerical method for Prandtl's boundary layer problem for flow past a wedge with heat transfer

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Abstract

In this paper we consider Prandtl's boundary layer problem for incompressible laminar flow past a wedge with heat transfer. When the Reynolds or Prandtl number is large the solution of this roblem has two parabolic boundary layers; one in the velocity components, the other in the temperature component. We construct a direct numerical method for computing approximations to the solution of this problem using a compound piecewise-uniform mesh appropriately fitted to the parabolic boundary layers. Using this numerical method we approximate the self–similar solution of Prandtl's problem in a finite rectangle excluding the leading edge of the wedge, which is the source of an additional singularity caused by incompatibility of the problem data. By means of extensive numerical experiments, for a range of values of the Reynolds number, Prandtl number and number of mesh points, we verify that the constructed numerical method is Reynolds and Prandtl uniform, in the sense that the computed errors for the velocity components, their derivatives and the temperature component, in the discrete maximum norm are Reynolds and Prandtl uniform. We use a special numerical method related to the Blasius technique to compute a semi–analytic reference solution, with required accuracy with respect to the Reynolds and Prandtl numbers, for use in the error analysis.

Keywords: Thermal boundary layer, Prandtl boundary layer, numerical method, Reynolds– and Prandtl–uniform.

1 Introduction

Incompressible laminar flow past a semi-infinite wedge W in the domain $D = \mathbf{R}^2 \backslash W$ is governed by the Navier-Stokes equations. Using Prandtl's approach the vertical momentum equation is omitted and the horizontal momentum equation is simplified, see [2]. The new momentum equation and energy equation are parabolic and singularly perturbed, which means that the highest order derivative in each equation is multiplied by a singular perturbation parameter. In the case of the momentum equation the parameter is the reciprocal of the Reynolds number Re. In the case of the energy equation the parameter is the reciprocal of the product of the Reynolds number and Prandtl number Pr. For convenience we use the

notation $\varepsilon = \frac{1}{Re}$ and $\varepsilon_{Pr} = \frac{1}{Pr}$.
It is well known that for flow problems with large Reynolds or Prandtl number boundary layers arise on the surface of the wedge. Also, when classical numerical methods are applied to these problems, large errors occur especially in approximations of the derivatives, which grow unboundedly as the Reynolds or Prandtl number increases. For this reason robust layer-resolving numerical methods, in which the error is independent of the singular perturbation parameters, are required. We want to solve Prandtl's problem in a region including the parabolic boundary layers. Since the solution of the problem has another singularity at the leading edge of the wedge, we take as the computational domain the finite rectangle $\Omega = (0.1, 1.1) \times (0, 1)$ on the upper side of the wedge, which is sufficiently far from the leading edge that the leading edge singularity does not cause problems for the numerical method. We denote the boundary of Ω by $\Gamma = \Gamma_L \bigcup \Gamma_T \bigcup \Gamma_B \bigcup \Gamma_R$ where Γ_L, Γ_T, Γ_B and Γ_R denote, respectively the left-hand, top, bottom and right-hand edges of Ω. Prandtl's boundary layer problem in Ω is then

$$(P_\varepsilon) \begin{cases} \text{Find } \mathbf{u}_\varepsilon = (u_\varepsilon, v_\varepsilon) \text{ and } t_\varepsilon \text{ such that for all } (x,y) \in \Omega \\ \mathbf{u}_\varepsilon \text{ and } t_\varepsilon \text{ satisfies the differential equations} \\ \\ -\frac{1}{Re}\frac{\partial^2 u_\varepsilon}{\partial^2 y} + u_\varepsilon \frac{\partial v_\varepsilon}{\partial x} + v_\varepsilon \frac{\partial u_\varepsilon}{\partial y} = U\frac{dU}{dx} \\ \\ -\frac{1}{Re}\frac{1}{Pr}\frac{\partial^2 t_\varepsilon}{\partial^2 y} + u_\varepsilon \frac{\partial t_\varepsilon}{\partial x} + v_\varepsilon \frac{\partial t_\varepsilon}{\partial y} = 0 \\ \\ \frac{\partial u_\varepsilon}{\partial x} + \frac{\partial v_\varepsilon}{\partial y} = 0 \\ \\ \text{with boundary conditions} \\ u_\varepsilon = 0, \quad t_\varepsilon = 1 \text{ and } v_\varepsilon = 0 \text{ on } \Gamma_B \\ \mathbf{u}_\varepsilon = \mathbf{u}_P \quad \Gamma_L \bigcup \Gamma_T \end{cases}$$

where $U(x) = x^m$, $m = \frac{\beta}{2-\beta}$ and $\beta\pi$ is the angle of the wedge in radians.
Our goal is to construct an (Re, Pr, β)–uniform numerical method for solving (P_ε), in the sense that the method has error bounds, for the solutions and their derivatives, independent of Re, Pr and β, for all $Re \in [1, \infty)$, $Pr \in [1, 10000]$ and all $\beta \in [0, 0.5]$.

2 Blasius similarity solution

Using the similarity transformation (see, for example, Reference [4])

$$\eta = y\sqrt{\frac{(m+1.0)Re}{2xU}}$$

the velocity components of the Blasius' solution $\mathbf{u}_B$ and temperature component of the Blasius' solution t_B of (P_ε) are given in terms of f and θ by

$$u_B(x,y) = Uf'(\eta),\, v_B(x,y) = -\sqrt{\tfrac{(m+1)\varepsilon U}{2x}}(f + \tfrac{m-1}{m+1}\eta f'(\eta)),\, t_B(x,y) = \theta(\eta)$$

and their scaled derivatives by similar expressions, for example

$$\frac{\partial u_B}{\partial y} = U\sqrt{\frac{(m+1.0)Re}{2xU}}f''(\eta)$$

where f and θ are the solutions of the coupled non–linear problem

$$(P_B)\begin{cases} \text{Find } f \in C^3([0,\infty)) \text{ such that for all } \eta \in [0,\infty) \\ f'''(\eta) + f(\eta)f''(\eta) + \beta(1 - f'^2(\eta)) = 0 \\ \theta''(\eta) + Prf(\eta)\theta'(\eta) = 0 \\ f(0) = 0, \quad \theta(0) = 1, \quad f'(0) = 0, \quad \lim_{\eta\to\infty} f'(\eta) = 1 \quad \lim_{\eta\to\infty} \theta(\eta) = 0. \end{cases}$$

To find the components u_B, v_B of $\mathbf{u_B}$, their derivatives and t_B, on the finite domain Ω for all $Re \in [1,\infty]$, $Pr \in [1,\infty]$ and $\beta \in [0, 0.5]$, we need to solve (P_B) numerically for f and θ and their derivatives on the semi-infinite domain $[0,\infty)$. Then we apply post-processing to determine numerical approximations to $\mathbf{u}_\varepsilon$ and t_ε. An analogous process is described in detail in [1] for flow past a two dimensional wedge.
Here, we make use of the Blasius similarity solution of Prandtl's problem in two ways. First, we use it to provide the required artificial boundary conditions on the boundary of Ω in the direct numerical method for Prandtl's problem discussed in the next section. Secondly, we use it as a reference solution for the unknown exact solution in the expression for the error. Since the Blasius solution is known to converge (Re, Pr, β)–uniformly to the solution of Prandtl's problem, we can compute (Re, Pr, β)–uniform error bounds. For this purpose we use the Blasius solution for (P_B) when N=8192, namely $\mathbf{U}_B^{8192}$ and T_B^{8192}, which provides the required accuracy for the velocity components U_B^{8192}, V_B^{8192}, their derivatives $D_xV_B^{8192}$, $D_yV_B^{8192}$, their scaled derivatives $\sqrt{\varepsilon}D_yU_B^{8192}$ and the temperature component T_B^{8192}.

3 Direct Numerical method for Prandtl's Problem

The aim of this section is to construct a direct numerical method to solve the Prandtl problem (P_ε) for all $Re \in [1,\infty)$, $Pr \in [1, 10000]$ and $\beta \in [0, 0.5]$. We require a compound piecewise–uniform fitted mesh $\Omega_\varepsilon^{\mathbf{N}}$ in the rectangle Ω, where $\mathbf{N}$=(N_x, N_y). We define the mesh as the tensor product $\Omega_\varepsilon^{\mathbf{N}} = \Omega_u^{N_x} \times \Omega_\varepsilon^{N_y}$, where the one–dimensional mesh in the x direction is the uniform mesh $\Omega_u^{N_x} = \{x_i : x_i = 0.1 + iN_x^{-1}, 0 \le i \le N_x\}$ and the mesh in the y-direction is the compound piecewise–uniform fitted mesh

$$\Omega_\varepsilon^{N_y} = \{y_j : y_j = \sigma_{Pr} j\tfrac{4}{N_y}, 0 \le j \le \tfrac{N_y}{4}; y_j = \sigma_{Pr} + (\sigma - \sigma_{Pr})(j - \tfrac{N_y}{4})\tfrac{4}{N_y}, \tfrac{N_y}{4} \le j \le \tfrac{N_y}{2};$$
$$y_j = \sigma + (1-\sigma)(j - \tfrac{N_y}{2})\tfrac{2}{N_y}, \tfrac{N_y}{2} \le j \le N_y\}.$$

It is important to note that the transition points σ and σ_{Pr} are chosen so that there are fine meshes in the boundary layers whenever they are required. The appropriate choices in this case are

$$\sigma = \min\{\tfrac{1}{2}, \sqrt{\varepsilon} ln N_y\}, \, \sigma_{Pr} = \min\{\tfrac{1}{2}\sigma, \sqrt{\varepsilon}\sqrt{\varepsilon_{Pr}} ln N_y\}.$$

The factors $\sqrt{\varepsilon}$ and $\sqrt{\varepsilon}\sqrt{\varepsilon_{Pr}}$ may be motivated from *a priori* estimates of the derivatives of the solution $\mathbf{u}_\varepsilon, t_\varepsilon$ or from asymptotic analysis. For simplicity we take $N_x = N_y = N$.
The problem (P_ε) is discretized by the following non-linear upwind finite difference method on the piecewise uniform fitted mesh $\Omega_\varepsilon^{\mathbf{N}}$

$$(P_\varepsilon^N) \begin{cases} \text{Find } \mathbf{U}_\varepsilon = (U_\varepsilon, V_\varepsilon) \text{ and } T_\varepsilon \text{ such that for all mesh points } (x_i, y_j) \in \Omega_\varepsilon^{\mathbf{N}} \\ \mathbf{U}_\varepsilon \text{ and } T_\varepsilon \text{ satisfies the finite mesh difference equations} \\ -\varepsilon\delta_y^2 U_\varepsilon(x_i, y_j) + U_\varepsilon D_x^- U_\varepsilon(x_i, y_j) + V_\varepsilon D_y^u U_\varepsilon(x_i, y_j) = U\frac{dU}{dx} \\ -\varepsilon\varepsilon_{Pr}\delta_y^2 T_\varepsilon(x_i, y_j) + U_\varepsilon D_x^- T_\varepsilon(x_i, y_j) + V_\varepsilon D_y^u T_\varepsilon(x_i, y_j) = 0 \\ D_x^- U_\varepsilon(x_i, y_j) + D_y^- V_\varepsilon(x_i, y_j) = 0 \\ \text{with boundary conditions} \\ U_\varepsilon = 0, \quad T_\varepsilon = 1 \text{ and } V_\varepsilon = 0 \text{ on } \Gamma_B \\ U_\varepsilon = U_B \text{ and } T_\varepsilon = T_B \quad \Gamma_L \bigcup \Gamma_T \end{cases}$$

where D_x^-, D_x^+ and D_y^-, D_y^+ are the standard first–order backward, respectively forward, finite difference operators in the x and y directions, the upwind finite difference operator D_y^u is defined by

$$V_\varepsilon(x_i, y_j) D_y^u U_\varepsilon(x_i, y_j) = \begin{cases} V_\varepsilon(x_i, y_j) D_y^- U_\varepsilon(x_i, y_j) \text{ if } V_\varepsilon(x_i, y_j) \geq 0 \\ V_\varepsilon(x_i, y_j) D_y^+ U_\varepsilon(x_i, y_j) \text{ if } V_\varepsilon(x_i, y_j) < 0 \end{cases}$$

and δ_y^2 is the standard second order centered finite difference operator in the y direction. Changes between forward and backward differences are required because, at angles $\beta > 0.1$, V_ε is initially negative and then becomes positive. Note that, without these changes, the tridiagonal system is not diagonally dominant and the continuation algorithm fails to converge.
Since $(P_\varepsilon^{\mathbf{N}})$ is a non–linear finite difference method an iterative method is required for its solution. This is obtained by replacing the system of non–linear equations by the following

sequence of systems of linear equations

$$
(A_\varepsilon^{\mathbf{N}})\begin{cases}
\text{With the boundary condition } \mathbf{U}_\varepsilon^{\mathbf{M}} = \mathbf{U}_{\mathrm{B}}^{\mathbf{8192}} \text{ on } \Gamma_{\mathrm{L}}, \text{for each } i, 1 \le i \le N, \\
\text{use the initial guess } \mathbf{U}_\varepsilon^0|_{X_i} = \mathbf{U}_\varepsilon^{M_{i-1}}|_{X_{i-1}} \text{and for } \; m = 1, \dots, M_i \; \text{ solve the following} \\
\text{two point boundary value problem for } U_\varepsilon^m(x_i, y_j) \\
-\varepsilon\delta_y^2 U_\varepsilon^m(x_i, y_j) + U_\varepsilon^{m-1} D_x^- U_\varepsilon^m(x_i, y_j) + V_\varepsilon^{m-1} D_y^u U_\varepsilon^m(x_i, y_j) = U\frac{dU}{dx} \quad 1 \le j \le N-1 \\
\text{with the boundary conditions } U_\varepsilon^m = U_{\mathrm{B}} \text{ on } \Gamma_{\mathrm{B}} \cup \Gamma_{\mathrm{T}}, \text{and the initial guess for } V_\varepsilon^0|_{X_1} = 0. \\
\text{Also solve the initial value problem for } V_\varepsilon^m(x_i, y_j) \\
D_x^- U_\varepsilon^m(x_i, y_j) + D_y^- V_\varepsilon^m(x_i, y_j) = 0 \\
\text{with initial condition } V_\varepsilon^m = 0 \text{ on } \Gamma_{\mathrm{B}}. \\
\text{Continue to iterate between the equations for } \mathbf{U}_\varepsilon^m \text{ until } m = M_i, \text{where } M_i \text{ is such that} \\
\max(|U_\varepsilon^{M_i} - U_\varepsilon^{M_i-1}|_{\overline{X}_i}, \frac{1}{V^*}|V_\varepsilon^{M_i} - V_\varepsilon^{M_i-1}|_{\overline{X}_i}) \le tol. \\
\text{Finally, solve the two point boundary value problem for } T_\varepsilon(x_i, y_j) \\
-\varepsilon\varepsilon_{Pr}\delta_y^2 T_\varepsilon(x_i, y_j) + U_\varepsilon^{M_i} D_x^- T_\varepsilon(x_i, y_j) + V_\varepsilon^{M_i} D_y^u T_\varepsilon(x_i, y_j) = 0, \quad 1 \le j \le N-1 \\
\text{with the boundary conditions } T_\varepsilon = T_{\mathrm{B}} \text{ on } \Gamma_{\mathrm{B}} \cup \Gamma_{\mathrm{T}} \cup \Gamma_L.
\end{cases}
$$

For notational simplicity, we suppress explicit mention of the iteration superscript M_i, and henceforth we write simply $\mathbf{U}_\varepsilon, T_\varepsilon$ for the solution generated by $(A_\varepsilon^{\mathbf{N}})$. We take $tol = 10^{-6}$ in the computations. We note that there are no known theoretical results concerning the convergence of the solutions $\mathbf{U}_\varepsilon$ and T_ε of $(P_\varepsilon^{\mathbf{N}})$ to the solutions $\mathbf{u}_\varepsilon$ and t_ε of (P_ε) and no theoretical estimate for the pointwise errors $(\mathbf{U}_\varepsilon - \mathbf{u}_\varepsilon)(x_i, y_j)$ and $(T_\varepsilon - t_\varepsilon)(x_i, y_j)$. It is for this reason that in the error analysis of the next section, we are forced to use controllable experimental techniques, which are adapted to the problem under consideration and are of crucial value to our understanding of these computational problems.

4 Error Analysis

In this section, we compute (Re, Pr, β)–uniform approximate maximum norm errors in the approximations generated by the direct numerical method described in the previous section. For the sake of brevity, we discuss here the approximate error in only the discrete temperature component and for only one value of the wedge angle and the Prandtl number, namely $\beta = 0.5$ and $Pr = 9000$.

We compare the approximations generated by the direct numerical method (A_ε^N) of the previous section with the corresponding values of T_B^{8192}. We use the following definition for the errors

$$E_\varepsilon^{\mathbf{N}}(T_\varepsilon) = ||T_\varepsilon - \overline{T_B}^{8192}||_{\overline{\Omega}_\varepsilon^N}$$

In Table 1 we display the computed maximum pointwise errors of the approximations to the temperature components. From these numerical experiments it follows that the method is Re-uniform for $\beta = 0.5$ and $Pr = 9000$. Further computations, not reported here, show that the method is (Re, P, β)–uniform for all scaled velocity components, their first order derivatives, and the temperature component for $Re \in [1, \infty)$, $Pr \in [1, 10000]$ and $\beta \in [0, 0.5]$. We define the computed local order of convergence $p^N_{\varepsilon,comp}$ for the temperature component T^N_ε and the ε–uniform order p^N_{comp} by

$$p^N_{\varepsilon,comp} = log_2 \frac{||T^N_\varepsilon - T^{8192}_B||_{\Omega^N_\varepsilon}}{||T^{2N}_\varepsilon - T^{8192}_B||_{\Omega^{2N}_\varepsilon}}\ p^N_{comp} = log_2 \frac{\max_\varepsilon ||T^N_\varepsilon - T^{8192}_B||_{\Omega^N_\varepsilon}}{\max_\varepsilon ||T^{2N}_\varepsilon - T^{8192}_B||_{\Omega^{2N}_\varepsilon}}.$$

In Table 2 we display the computed orders of convergence for the approximations of the temperature component T_ε obtained from the corresponding Table 1. We see that for each value of N, the orders of convergence stabilize as $\varepsilon \to 0$ for $\beta = 0.5$ and $Pr = 9000$. In additional computations, not reported here, similar behaviour is observed for all scaled velocity components, their first order derivatives and the temperature component for all $\beta \in [0, 0.5]$ and $Pr \in [1, 10000]$.

$\varepsilon \backslash N$	32	64	128	256	512
2^{-0}	2.65e-02	7.43e-03	5.53e-03	5.25e-03	5.11e-03
2^{-2}	5.85e-02	2.21e-02	8.97e-03	5.39e-03	5.13e-03
2^{-4}	9.21e-02	6.24e-02	3.10e-02	1.22e-02	5.72e-03
2^{-6}	1.47e-01	1.31e-01	8.18e-02	3.82e-02	1.38e-02
2^{-8}	1.47e-01	1.35e-01	1.01e-01	6.00e-02	2.81e-02
.	.	.	.	.	.
2^{-20}	1.47e-01	1.35e-01	1.01e-01	6.00e-02	2.81e-02
E^N	1.47e-01	1.35e-01	1.01e-01	6.00e-02	2.81e-02

Table 1: Computed maximum pointwise error $E^{\mathbf{N}}_\varepsilon(T_\varepsilon)$ where T_ε is generated by (A^N_ε) for various values of ε, N, $\beta = 0.5$ and $Pr = 9000$

$\varepsilon \backslash N$	32	64	128	256
2^{-0}	1.83	0.43	0.07	0.04
2^{-2}	1.41	1.30	0.73	0.07
2^{-4}	0.56	1.01	1.34	1.10
2^{-6}	0.17	0.68	1.10	1.47
2^{-8}	0.13	0.42	0.75	1.09
2^{-10}	0.13	0.42	0.75	1.09
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2^{-20}	0.13	0.42	0.75	1.09
p^N_{comp}	0.13	0.42	0.75	1.09

Table 2: Computed orders of convergence $p^N_{\varepsilon,comp}$, p^N_{comp} for $\sqrt{\varepsilon}(T_\varepsilon - \overline{T^{8192}_B})$ where T_ε is generated by (A^N_ε) for $\beta = 0.5$, $Pr = 9000$ and various values of ε, N.

5 Conclusion

We considered Prandtl's boundary layer equations for incompressible laminar flow past a wedge with angle $\beta\pi$, $\beta \in [0, 0.5]$ with heat transfer. When the Reynolds number and Prandtl number are large the solution of this problem has two parabolic boundary layers. We constructed a direct numerical method for computing approximations to the solution of this problem using a compound piecewise uniform fitted mesh technique appropriate to the parabolic boundary layers. We used the method to approximate the self–similar solution of Prandtl's problem in a finite rectangle excluding the leading edge of the wedge for various values of Re, Pr and β. We constructed and applied a special numerical method, related to the Blasius technique, to compute reference solutions to the problem. These were used to obtain approximate boundary conditions on the artificial boundaries of the computational domain and in the error analysis of the velocity components, their derivatives and the temperature component. Extensive numerical experiments indicated that the constructed direct numerical method is (Re, Pr, β)–uniform.

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