Grid Approximations of Multiscale Problems Arising in Singularly Perturbed Parabolic Reaction-Diffusion Equations in an Unbounded Domain *

J.J.H. Miller [1], R.E. O'Malley [2], and G.I. Shishkin [3]

[1] Department of Mathematics, Trinity College, Dublin 2, Ireland and

[1] Department of Computational Science, National University of Singapore, Singapore 117543

jmiller@tcd.ie jmiller@cz3.nus.edu.sg

[2] Department of Applied Mathematics, University of Washington,

Seatle, Washington 98195-2420, USA omalley@amath.washington.edu

[3] Institute of Mathematics and Mechanics, Ural Branch of Russian Academy of Sciences,

16 S. Kovalevskaya Street, Ekaterinburg 620219, Russia shishkin@imm.uran.ru and

Department of Mathematics, Trinity College, Dublin 2, Ireland shishkin@maths.tcd.ie

Abstract

We consider an initial boundary value problem for singularly perturbed parabolic reaction-diffusion equations on a semiaxis. The highest derivative in the equations is multiplied by a small parameter ε, $\varepsilon \in (0,1]$. The solution of such a problem exhibits multiple scales. Besides the usual (natural) scale, related to a variation of the problem data, one can observe a resolution scale, which is specified by the width of the domain on which the numerical solutions are being computed, and a boundary-layer scale controlled by the parameter ε. In this paper we solve the multiscale problem using the renormalization method, that is, we construct the following (normalized and renormalized) finite difference schemes:

(a) *formal* (nonconstructive) *schemes*, i.e., schemes on meshes with an *infinite number* of nodes, which lead to approximate solutions converging ε-uniformly at each node; and

(b) *constructive schemes*, i.e., schemes on meshes with a *finite number* of the nodes, which lead to approximate solutions converging for fixed values of the parameter ε at each node of arbitrarily chosen bounded subdomains whose widths increase as the number of nodes grows.

With standard constructive schemes, generally speaking, the accuracy of the approximate solutions deteriorates and the widths of the subdomains decrease when $\varepsilon \to 0$. Here, conditions are given under which the approximate solutions generated by the constructive schemes converge ε-uniformly, i.e. the accuracy of the numerical approximations and the widths of the subdomains, on which the schemes converge, are independent of the parameter ε.

To construct the schemes, we use classical finite difference approximations on piecewise uniform meshes which are refined in a neighbourhood of the boundary layer.

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1. Introduction

The use of grid methods to solve a regular initial boundary value problem in unbounded domains gives rise to difficulties arising from the following contradiction. To solve the problem, it is possible to use numerical algorithms on a grid set with a finite (however, large) number of nodes. On the other hand, for an effective approximation of the solution of the problem on the whole domain, in general, it is necessary to use a grid set with an infinite number of nodes. For this reason the following problem appears: to construct special numerical methods (finite difference schemes) on finite grid sets that allow us to approximate the solution of the problem on chosen bounded subdomains with diameters that grow as the number of nodes in the corresponding set of grids increases.

In the case of problems for singularly perturbed equations in unbounded domains, the problem of constructing such special numerical methods is complicated by additional singularities generated by the boundary (interior) layers that arise for small values of the perturbation parameter ε. Such difficulties in the numerical solution of singularly perturbed boundary and initial boundary value problems in bounded subdomains are quite well-known (see, e.g., [1–6]), [11]).

In this paper we construct finite difference schemes in the case of an initial boundary value problem for a singularly perturbed parabolic reaction-diffusion equation on a domain which is semi-infinite in the space variable. The highest derivative in the equation is multiplied by a small parameter ε^2; the perturbation parameter ε takes arbitrary values in the half-interval $(0, 1]$. In the case of grids with *an infinite number* of nodes we construct special *formal* schemes, i.e., schemes with *an infinite number* of nodes, which converge uniformly with respect to the parameter ε on the whole (unbounded) grid set (or, more concisely, converge ε-uniformly). To construct the schemes, we use the condensing grid method i.e. classical finite difference approximations of the initial boundary value problem on piecewise uniform grids [7, 8] which are refined in a neighbourhood of the boundary layer (for a description of this method for bounded domains see, e.g., [1, 4, 5, 11]. Such schemes on grids with an infinite number of nodes are nonconstructive, in the sense that they cannot, in general, be used for computation.

For the initial boundary value problem we also construct special constructive schemes, i.e. schemes on grids with a *finite number* of nodes, which lead to approximate solutions, converging for fixed values of the parameter ε, at all nodes of arbitrarily chosen bounded subdomains, whose widths increase as the number of nodes grows. The accuracy of the approximations and the widths of the subdomains, on which the grid solutions converge, depend essentially on the value of the parameter ε. Moreover, the accuracy of the approximate solutions deteriorates and the widths of the subdomains decrease when $\varepsilon \to 0$. In the case of grids condensing in a neighbourhood of the boundary layer, conditions are given under which the accuracy of the numerical approximations and the widths of the subdomains, on which the schemes converge, are independent of the parameter ε (we say that such schemes are *really ε-uniformly convergent*).

The construction of special really ε-uniformly convergent schemes for singularly perturbed equations in unbounded domains were not examined previously. We note the publications [4, 12] where, for singularly perturbed equations of parabolic reaction-diffusion type [4] and of elliptic convection-diffusion type [12] in unbounded domains, finite difference schemes convergent on bounded subdomains were constructed.

2. Problem Formulation

1. On the set $\overline{G}$, where

$$\overline{G} = G \cup S, \quad G = D \times (0, T], \quad D = (0, \infty), \tag{2.1}$$

we consider the Dirichlet problem for the parabolic equation [1]:

$$L_{(2.2)} u(x,t) \equiv \left\{ \varepsilon^2 a(x,t) \frac{\partial^2}{\partial x^2} - c(x,t) - p(x,t) \frac{\partial}{\partial t} \right\} u(x,t) = \\ = f(x,t), \quad (x,t) \in G, \tag{2.2a}$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S. \tag{2.2b}$$

The functions $a(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$, and $\varphi(x,t)$ are sufficiently smooth respectively on the sets $\overline{G}$ and S_0, $\overline{S}^L$. Here $S = S_0 \cup S^L$, where $S_0 = \overline{D} \times \{t = 0\}$ is the initial boundary and S^L is the lateral boundary of the set G. The function $\varphi(x,t)$ is continuous on the set S. Moreover, the following conditions are satisfied [2]

$$0 < a_0 \le a(x,t) \le a^0, \quad 0 \le c(x,t) \le c^0, \quad 0 < p_0 \le p(x,t) \le p^0, \tag{2.3}$$
$$|f(x,t)| \le M, \quad (x,t) \in \overline{G}; \quad |\varphi(x,t)| \le M, \quad (x,t) \in S; \quad T \le M.$$

[1] Throughout the paper, the notation $L_{(j.k)}$ ($M_{(j.k)}$, $G_{h(j.k)}$) means that these operators (constants, grids) are introduced in equation $(j.k)$.

[2] Here and below M, M_i (or m) denote sufficiently large (small) positive constants which do not depend on ε nor on the discretization parameters.

The parameter ε takes arbitrary values in the half-interval $(0, 1]$. We consider solutions of the boundary value problem which are bounded on the set $\overline{G}$.

When ε tends to zero, a boundary layer appears in a neighbourhood of the set S^L. This layer is parabolic.

Problems of this type arise in the modelling of a diffusion process in a reacting substance (heat and/or reactive components), if the diffusion coefficient is small and/or the flow velocity is high, and if the effective diameter of the domain is sufficiently large compared to sizes, with respect to x and t, of the domain on which it is possible to solve/resolve the discrete problem with the computational technique employed.

2. We now discuss specific properties of the problem (2.2) (2.1) and the aim of this research.

The initial boundary value problem (2.2) in the unbounded domain (2.1) belongs to the class of singular problems, even for finite values of the parameter ε. For example, a problem for a regular equation in a bounded domain, the width of which may take arbitrarily large values, can be transformed, by a change of the space variable, into a problem for a singularly perturbed equation in a domain of unit width. Thus, problem (2.2) (2.1), in addition to a singularity of boundary layer type, has a singularity generated by the unboundedness of the domain.

In the numerical solution of problem (2.2), even in the finite domain

$$\overline{G}_0 = \overline{D}_0 \times [0, T], \quad D_0 = (0, d_0), \quad d_0 \leq M, \tag{2.4}$$

difficulties arise when classical methods are applied; the errors arising from the use of such methods depend on the parameter ε and become large (comparable with the solution itself) for small values of the parameter ε. For problem (2.2), (2.4), such schemes converge only under the condition

$$N^{*-1} \ll \varepsilon,$$

where $N^* + 1$ is the number of nodes corresponding to the variable x (on the set $\overline{G}_{0(2.4)}$, it is the number of nodes on the segment $\overline{D}_0$). For the problem (2.2), (2.4) in the case of the scheme (4.2), (6.14) see e.g. Remark 4 in Section 6.

For boundary and initial boundary value problems in unbounded domains, the construction of numerical methods is essentially complicated. For such problems,in general, even for regular equations, there do not exist numerical methods on grids with a finite number of nodes, for which the numerical solutions converge on the whole domain (see, for example, the statement of Lemma 6.1 in Section 6). Thus, the following computational problem seems to be natural: for the initial boundary value problem (2.2), (2.1) construct a numerical method that approximates the solution of the problem on a preliminary chosen bounded set $\overline{G}^0$ from $\overline{G}$ where

$$\overline{G}^0 = G^0 \cup S^0, \quad G^0 = D^0 \times (0, T], \tag{2.5}$$

D^0 is a simply connected set with a width d^0; $D^0 = (x^0, x^0 + d^0)$; x^0 is an arbitrary point from $(0, \infty)$. The width d^0 (the diameter with respect to x) of the set $\overline{G}^0$, on which convergence of the solutions is achievable using a classical scheme on a grid set with a finite number of nodes, essentially depends on the value of the parameter ε. This width becomes, in general, arbitrarily small under the condition

$$N^* = \mathcal{O}(\varepsilon^{-1}),$$

where $N^* + 1$ is the number of nodes in the grid on the x-axis (see, e.g., the estimates (6.9) and Remark 3 from Section 6).

Thus, for classic approximations of problem (2.2), (2.1) on grids with a finite number of nodes, (we refer to these as *constructive methods*), the width of the set on which convergence of the schemes is achievable, and the error of the discrete solution on this set, depend essentially on the value of the parameter ε. It seems attractive to develop constructive grid methods for which both the error in their discrete solutions and the diameter d^0 of the subdomain $\overline{G}^0$, where their discrete solutions are defined, are independent of the parameter ε. Furthermore, the diameter d^0 and the error are to depend only on the number of nodes in the corresponding grids. We refer to such methods as ε-uniformly convergent methods.

Our aim is to develop a constructive difference scheme for the initial boundary value problem (2.2), (2.1) that converges ε-uniformly on finite subdomains of the set $\overline{G}$. Note that, such an approach was not previously applied to the construction of special numerical methods for singularly perturbed equations in unbounded domains.

3. A-priori estimates

We now give *a-priori* estimates of the solutions of problem (2.2), (2.1) used in our constructions; the technique from [4] is applied to derive estimates of the solutions and their derivatives.

We assume that the problem data satisfy compatibility conditions on the set $S^c = S_0 \cap \overline{S}^L$, (i.e. at the set of corner points), so that the solution of the problem is smooth on the set $\overline{G}$ for each fixed value of the parameter ε (see, e.g., [10]).

By means of a comparison principle we establish the ε-uniform boundedness of the solution of problem (2.2), (2.1)

$$|u(x,t)| \le M, \quad (x,t) \in \overline{G}. \tag{3.1}$$

In the variables $\xi = \varepsilon^{-1}x$, t the equation (2.2a) becomes regular; in order to estimate the solution, and its derivatives, for the regular problem (which has a bounded solution), we use interior *a-priori* estimates and estimates up to the boundary [9, 10]. Returning to the variables x, t, we then obtain the estimate

$$\left| \frac{\partial^{k+k_0}}{\partial x^k\, \partial t^{k_0}} u(x,t) \right| \le M\, \varepsilon^{-k}, \quad (x,t) \in \overline{G}, \quad k + 2k_0 \le K, \tag{3.2}$$

where K is a sufficiently large number depending on the smoothness of the problem data.

We now describe more accurate estimates obtained from asymptotic representations. We represent the solution of the problem as a sum of functions

$$u(x,t) = U(x,t) + V(x,t), \quad (x,t) \in \overline{G}, \tag{3.3}$$

where $U(x,t)$ and $V(x,t)$ are the regular and singular parts of the problem solution. The function $U(x,t)$, $(x,t) \in \overline{G}$ is the restriction to $\overline{G}$ of the function $U^0(x,t)$, $(x,t) \in \overline{G}^0$, $\overline{G}^0 = R \times [0,T]$. The function $U^0(x,t)$ is the solution of the Cauchy problem

$$L^0 U^0(x,t) = f^0(x,t), \quad (x,t) \in G^0, \quad U^0(x,t) = \varphi^0(x,t), \quad (x,t) \in S^0.$$

Here L^0 and $f^0(x,t)$ are smooth extensions of the operator $L_{(2.2)}$ and the function $f(x,t)$ to the set $\overline{G}^0$, which preserve conditions (2.3); $\varphi^0(x,t)$, $x \in R$ is a smooth extension of the function $\varphi(x,t)$, $(x,t) \in S_0$. The function $V(x,t)$ is the solution of the problem

$$L_{(2.2)} V(x,t) = 0, \quad (x,t) \in G,$$

$$V(x,t) = \begin{cases} \varphi(x,t) - U(x,t), & (x,t) \in S^L, \\ 0, & (x,t) \in S_0. \end{cases}$$

The function $U(x,t)$ can be represented as the following sum of functions

$$U(x,t) = \sum_{k=0}^{n} \varepsilon^{2k} U_k(x,t) + v_U^{[n]}(x,t) \equiv U^{[n]}(x,t) + v_U^{[n]}(x,t), \tag{3.4}$$

$$(x,t) \in \overline{G}, \quad n \ge 0,$$

where $U_k(x,t)$ are the components of the regular part in the representation of the solution of the problem and $v_U^{[n]}(x,t)$ is the remainder term. The functions $U_k(x,t)$, $(x,t)\in\overline{G}$ are the solutions of the problems

$$L^1 U_0(x,t) \equiv \left\{-c(x,t) - p(x,t)\frac{\partial}{\partial t}\right\} U_0(x,t) = f(x,t), \quad (x,t)\in\overline{G}\setminus S_0,$$

$$U^0(x,t) = \varphi(x,t), \quad (x,t)\in S_0;$$

$$L^1 U_k(x,t) = -a(x,t)\frac{\partial^2}{\partial x^2}U_{k-1}(x,t), \quad (x,t)\in\overline{G}\setminus S_0,$$

$$U_k(x,t) = 0, \quad (x,t)\in S_0, \quad k>0.$$

Let the following condition be satisfied for the function $u(x,t)$:

$$u\in C^{l_1+\alpha,l_0+\alpha/2}(\overline{G}), \quad l_1\ge 4, \quad l_0\ge 2, \quad \alpha>0. \tag{3.5}$$

Then, for the components in the representations (3.3), (3.4) we obtain the estimates

$$\left|\frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}}U(x,t)\right| \le M\,[1+\varepsilon^{2n+2-k}], \tag{3.6}$$

$$\left|\frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}}V(x,t)\right| \le M\,\varepsilon^{-k}\exp(-m\,\varepsilon^{-1}\,x\,),$$

$$\left|\frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}}U^{[n]}(x,t)\right| \le M,$$

$$\left|\frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}}v_U^{[n]}(x,t)\right| \le M\,\varepsilon^{2n+2-k},$$

$$(x,t)\in\overline{G}, \quad k\le K_1, \quad k_0\le K_0,$$

where m is a constant, $K_1 = l_{1(3.5)}$, $K_0 = l_{0(3.5)}$.

These results are stated formally in the following

Theorem 3.1. *Assume that in equation* (2.2) a, c, p, $f\in C^{l_1+\alpha,l_0+\alpha/2}(\overline{G})$, $\varphi\in C^{l_1+\alpha}(S_0)\cap C^{l_0+1+\alpha/2}(\overline{S}^L)\cap C(S)$, $l_1\ge 2n+l_{1(3.5)}$, $l_0 = l_{0(3.5)}-1$, $n\ge 0$, $\alpha>0$, *and let condition* (3.5) *be fulfilled. Then, for the solution of problem* (2.2), (2.1) *and for its components in the representations* (3.3), (3.4), *the estimates* (3.1), (3.2), (3.6) *are valid.*

4. Classical formal difference schemes

On the set $\overline{G}$ we introduce the grid

$$\overline{G}_h = \overline{\omega}_1 \times \overline{\omega}_0, \tag{4.1}$$

where $\overline{\omega}_1$ and $\overline{\omega}_0$ are grids on $\overline{D}$ and $[0,T]$ respectively; the grids $\overline{\omega}_1$, $\overline{\omega}_0$ are, in general, *non-uniform*. Define $h^i = x^{i+1} - x^i$, $x^i, x^{i+1} \in \overline{\omega}_1$, $h_t^j = t^{j+1} - t^j$, $t^j, t^{j+1} \in \overline{\omega}_0$, $h = \max_i h^i$, $h_t = \max_j h_t^j$. We denote by $N+1$ the maximal number of nodes in the grid $\overline{\omega}_1$ on any unit interval of $\overline{D}$ and by N_0+1 the number of nodes in the grid $\overline{\omega}_0$. We suppose that the condition $h \le M\,N^{-1}$, $h_t \le M\,N_0^{-1}$ is satisfied.

On the grid $\overline{G}_h$, we associate with (2.2) the finite difference scheme

$$\Lambda_{(4.2)} z(x,t) \equiv \left\{ \varepsilon^2 a(x,t)\delta_{\overline{x}\widehat{x}} - c(x,t) - p(x,t)\delta_{\overline{t}} \right\} z(x,t) = f(x,t), \tag{4.2}$$

$$(x,t) \in G_h,$$

$$z(x,t) = \varphi(x,t), \quad (x,t) \in S_h.$$

Here $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$ and $\delta_{\overline{x}\widehat{x}} z(x,t)$, $\delta_{\overline{t}} z(x,t)$ are respectively the second (central), first (backward) differences on the *non-uniform* grids, for example,
$\delta_{\overline{x}\widehat{x}} z(x,t) = 2(h^{i-1} + h^i)^{-1}(\delta_x z(x,t) - \delta_{\overline{x}} z(x,t))$, $x = x^i$.

The difference scheme (4.2), (4.1) is monotone (see [8]) on grids with an arbitrary distribution of nodes.

2. We now consider the scheme (4.2), (4.1). Taking into account the *a-priori* estimates of the solution of problem (2.2), (2.1), we find the following bounds

$$|u(x,t) - z(x,t)| \le M\,[(\varepsilon + N^{-1})^{-1}\,N^{-1} + N_0^{-1}], \quad (x,t) \in \overline{G}_h. \tag{4.3}$$

In the case of the mesh

$$\overline{G}_h, \tag{4.4}$$

which is *uniform* w.r.t. x, we have the estimate

$$|u(x,t) - z(x,t)| \le M\,[(\varepsilon + N^{-1})^{-2}\,N^{-2} + N_0^{-1}], \quad (x,t) \in \overline{G}_h. \tag{4.5}$$

Definition. Let the function $z(x,t)$, $(x,t) \in \overline{G}_h$, be the solution of some difference scheme. An estimate of the following form

$$|u(x,t) - z(x,t)| \le M\,\mu(N^{-1}, N_0^{-1}; \varepsilon), \quad (x,t) \in \overline{G}_h$$

is said to be *unimprovable* with respect to the values of N, N_0, ε if the estimate

$$|u(x,t) - z(x,t)| \le M\,\mu_0(N^{-1}, N_0^{-1}; \varepsilon), \quad (x,t) \in \overline{G}_h$$

fails for some values of N, N_0 and ε, where N, $N_0 \ge M$, $\varepsilon \in (0,1]$.

The estimate (4.5) is unimprovable with respect to the values of N, N_0, ε. The condition

$$N^{-1} = o(\varepsilon) \tag{4.6}$$

is necessary and sufficient for the convergence (for $N, N_0 \to \infty$) of the solutions of the difference scheme (4.2), (4.4).

Definition. We say that the solutions $z(x,t)$, $(x,t) \in \overline{G}_h$ of a difference scheme converge with respect to the parameter ε with a defect ν (or, shortly, *converges with the defect* ν) if, for a constant $\nu > 0$, there exists a function $\mu_1(N^{-1}, N_0^{-1})$, $\mu_1(N^{-1}, N_0^{-1}) \to 0$ ε-uniformly for $N, N_0 \to \infty$, such that the following estimate holds:

$$|u(x,t) - z(x,t)| \le M\, \mu_1(\varepsilon^{-\nu} N^{-1}, N_0^{-1}), \quad (x,t) \in \overline{G}_h.$$

If $\nu = 0$ the scheme converges ε-uniformly.

The convergence defect of both of the schemes (4.2), (4.1) and (4.2), (4.4) is equal to one (this defect is unimprovable).

Theorem 4.1. *Suppose that, for the components of the representations* (3.3), (3.4) *of the solution of the initial boundary value problem* (2.2), (2.1), *the a-priori estimates* (3.6) *with* $K_1 = 4$, $K_0 = 2$ *hold. For the difference schemes* (4.2), (4.1) *and* (4.2), (4.4) *the condition* (4.6) *is necessary and sufficient for the convergence of the discrete solutions, for* $N, N_0 \to \infty$ *and* $\varepsilon \in (0,1]$, *to the solution of problem* (2.2), (2.1). *For the grid solutions the estimates* (4.3), (4.5) *hold; the estimate* (4.5) *is unimprovable with respect to the values of* N, N_0, ε.

5. Special formal difference scheme

Note that a maximum of the error in the solution of the difference scheme (4.2) on the grids (4.1), (4.4) is achievable in a neighbourhood of a boundary layer. In the construction of special schemes, convergent ε-uniformly, we use a grid condensing in the boundary layer region (similar to the case of singularly perturbed problems in bounded domains; see, e.g., [4, 5, 11]).

We consider the difference scheme (4.2) on the following special grid condensing in the boundary layer

$$\overline{G}_h = \overline{\omega}_1^S \times \overline{\omega}_2 = \overline{G}^S_{h(5.1)}. \tag{5.1a}$$

Here $\overline{G}_{h(5.1)} = \overline{G}_{h(4.1)}$, where $\overline{\omega}_1 = \overline{\omega}_1^S(\sigma)$ is a *piecewise uniform* grid, σ is a parameter depending on ε and N. The grid $\overline{\omega}_1^s$ is constructed as follows. The set $\overline{D}$ is divided in two subsets $[0,\sigma]$ and $[\sigma,\infty)$. Then, in each subset, the step-size of the grid is taken to be constant and equal to $h_{(1)} = 2\sigma N^{-1}$ and $h_{(2)} = 2(1-\sigma)N^{-1}$ respectively, $\sigma \le 2^{-1}$. The value of σ is chosen to satisfy the relation

$$\sigma = \sigma(\varepsilon, N) = \min\,[\,2^{-1}, \varepsilon\, m^{-1}\,\ln N\,], \tag{5.1b}$$

where $m = m_{(3.6)}$.

On the grid (5.1) we have the estimate

$$|u(x,t) - z(x,t)| \le M\,\{\min{}^2\,[\,\varepsilon^{-1}, \ln N\,]\,N^{-2} + N_0^{-1}\}, \quad (x,t) \in \overline{G}_h, \tag{5.2}$$

and also the following ε-uniform estimate

$$|u(x,t) - z(x,t)| \le M\,[\,N^{-2}\,\ln^2 N + N_0^{-1}\,], \quad (x,t) \in \overline{G}_h. \tag{5.3}$$

Estimates (5.2) and (5.3) are unimprovable with respect to the values of N, N_0, ε and N, N_0 respectively.

A formal statement of these results is contained in

Theorem 5.1. *Let the condition of Theorem* 4.1 *be fulfilled. Then the solutions of the scheme* (4.2), (5.1) *converge to the solution of the boundary value problem* (2.2), (2.1) *ε-uniformly. For the discrete solutions the estimates* (5.2) *and* (5.3) *are valid. These estimates are unimprovable with respect to the values of N, N_0, ε and N, N_0 respectively.*

Remark 1. Taking into account the *a-priori* estimates of the solutions of problem (2.2), (2.1), we obtain the following estimates for the solutions of the difference scheme (4.2), (5.1)

$$|u(x,t) - \overline{z}(x,t)| \le M\,\{\min{}^2\,[\,\varepsilon^{-1}, \ln N\,]\,N^{-2} + N_0^{-1}\}, \quad (x,t) \in \overline{G},$$

$$|u(x,t) - \overline{z}(x,t)| \le M\,[\,N^{-2}\,\ln^2 N + N_0^{-1}\,], \quad (x,t) \in \overline{G},$$

where $\overline{z}(x,t)$, $(x,t) \in \overline{G}$ is the bilinear interpolant, which is constructed from the values of the function $z(x,t)$, $(x,t) \in \overline{G}_h$. Thus, the difference scheme (4.2), (5.1) allows us to construct approximations of the solution on the whole set $\overline{G}$.

6. Constructive difference schemes

The difference schemes (4.2) on the grids (4.1), (5.1) are formal; they are unfit for numerical computation, since the number of nodes in such grids is infinite. To solve problem (2.2), (2.1), it is of interest to develop constructive difference schemes, i.e., schemes on grids with a finite number of the nodes. We now discuss such schemes.

The following lemma holds

Lemma 6.1. *For the initial boundary value problem* (2.2), (2.1), *there do not exist constructive difference schemes, that allow us to approximate the solution of the problem on the set* $\overline{G}$, *even for fixed values of the parameter* ε.

1. Now we consider constructive schemes, which allow us to approximate the solution of problem (2.2), (2.1) in bounded subdomains of $\overline{G}$.

Suppose that, for the solution of the initial boundary value problem (2.2), (2.1), it is required to construct a grid approximation in the bounded subdomain $\overline{G}^0_{(2.5)}$ of $\overline{G}$. Let $G^1 = D^1 \times (0,T]$ where the set $D^1 = D^1(\eta) = (x^0 - \eta, x^0 + d^0 + \eta)$ contains $\overline{D}^0$ together with its η-neighbourhood. Assume that

$$G^{[0]} = G^{[0]}(\eta) = G^1(\eta) \cap G; \quad D^{[0]} = D^1 \cap D. \tag{6.1}$$

To find the solution of (2.2) on the set $\overline{G}^0$, we can use the solution of an auxiliary problem, which is the "restriction" to $\overline{G}^{[0]}$ of problem (2.2), (2.1). Let $u^{[0]}(x,t) = u^{[0]}(x,t;\eta)$, $(x,t) \in \overline{G}^{[0]}(\eta)$ be the solution of the problem

$$L_{(2.2)} u^{[0]}(x,t) = f(x,t), \quad (x,t) \in G^{[0]}, \tag{6.2a}$$

$$u^{[0]}(x,t) = \varphi(x,t), \quad (x,t) \in S^{[0]} \cap S,$$

$$u^{[0]}(x,t) = 0, \qquad (x,t) \in S^{[0]} \setminus S, \tag{6.2b}$$

where $S^{[0]} = \overline{G}^{[0]} \setminus G^{[0]}$. Note that, if in (6.2b) $u^{[0]}(x,t) = u(x,t)$, $(x,t) \in S^{[0]} \setminus S$, then $u^{[0]}(x,t) = u(x,t)$, $(x,t) \in \overline{G}^{[0]}$ where $u(x,t)$ is the solution of problem (2.2), (2.1).

The solution of problem (6.2), (6.1) satisfies the estimate

$$|u(x,t) - u^{[0]}(x,t)| \le \lambda(\eta), \quad (x,t) \in \overline{G}^0,$$

where $\lambda(\eta) \to 0$ ε-uniformly for $\eta \to \infty$. Thus, the function $u^{[0]}(x,t)$, which is defined on $\overline{G}^{[0]}$, approximates the solution of problem (2.1), (2.2) on the set $\overline{G}^0$ ε-uniformly as $\eta \to \infty$.

Using a comparison principle, we obtain

$$|u(x,t) - u^{[0]}(x,t)| \le M \exp\left(-m\,\varepsilon^{-1}\,\eta\right), \quad (x,t) \in \overline{G}^0, \tag{6.3}$$

where $m = m_{(3.6)}$.

Lemma 6.2. *For* $\eta \to \infty$ *the solution of the initial boundary value problem* (6.2), (6.1) *converges on* $\overline{G}^0$ *to the solution of the original initial boundary value problem* (2.2), (2.1) *ε-uniformly; for the solution of* (6.2), (6.1) *the estimate* (6.3) *holds.*

Definition. When, for a chosen set $\overline{G}^0$ and an arbitrary value $\beta \in (0, M]$, there exists a set $\overline{G}^{[0]}$ such that, for the solution of problem (2.2), (2.1), the following estimate holds

$$|u(x,t) - u^{[0]}(x,t)| \le M\,\beta, \quad (x,t) \in \overline{G}^0,$$

we say that the set $\overline{G}^{[0]}$ is the *domain of dependence* of the data on the solution of the initial boundary value problem on the set $\overline{G}^0$ with a threshold of disturbance β (or, for short, the *domain of dependence with a threshold* β).

From the estimate (6.3) it follows that for problem (2.2), (2.1) the domain of dependence with a fixed threshold β is ε-uniformly bounded; in general, this domain increases as $\beta \to 0$ (as the threshold β decreases).

To approximate the problem (6.2), (6.1), we introduce on $\overline{G}^{[0]}$ the grid

$$\overline{G}_h^{[0]} = \overline{G}^{[0]} \cap \overline{G}_h, \quad \overline{G}^{[0]} = \overline{G}^{[0]}(\eta), \tag{6.4}$$

where $\overline{G}_h$ is the basic grid on $\overline{G}$ (we can use the grid (4.1) or the grid (4.4) or another grid as the basic grid $\overline{G}_h$). Let $\eta \ge 1$. We denote by N^*+1 the number of nodes in the grid on the segment $\overline{D}^{[0]}$. The value N^* can be estimated from $N^* \le M(d^0 + \eta) N_{(4.1)}$, where d^0 is the diameter of the set $\overline{D}^0$ (the number of nodes in the grid $\overline{G}_h^{[0]}$ is finite).

On the grid $\overline{G}_h^{[0]}$ we consider the discrete problem

$$\Lambda_{(4.2)} z^{[0]}(x,t) = f(x,t), \quad (x,t) \in G_h^{[0]}, \tag{6.5}$$

$$z^{[0]}(x,t) = \varphi(x,t), \quad (x,t) \in S_h^{[0]} \cap S, \quad z^{[0]}(x,t) = 0, \quad (x,t) \in S_h^{[0]} \setminus S.$$

The difference scheme (6.5), (6.4) is constructive.

2. We now deduce estimates for the solutions of the scheme (6.5), (6.4).

The following estimate holds for solutions of the scheme (6.5), (6.4) in the case of the basic grid (4.1):

$$|u(x,t) - z^{[0]}(x,t)| \le M\left[\exp(-m\,\eta) + (\varepsilon + h^*)^{-1} h^* + N_0^{-1}\right], \tag{6.6a}$$

$$(x,t) \in \overline{G}_h^0,$$

where $h^* = (d^0 + \eta + \min[x^0, \eta])\, N^{*-1}$, $m = m_{(6.3)}$, $\overline{G}_h^0 = \overline{G}^0 \cap \overline{G}_h$.

In the case of the basic grid (4.4) we find the following estimate

$$|u(x,t) - z^{[0]}(x,t)| \le M\left[\exp(-m\,\eta) + (\varepsilon + h^*)^{-2} h^{*2} + N_0^{-1}\right], \tag{6.6b}$$

$$(x,t) \in \overline{G}_h^0,$$

where $h^* = h^*_{(6.6a)}$; this estimate is unimprovable with respect to the value of h^*, N_0, ε and the value of η up to a constant factor.

The conditions

$$\eta = \eta(N^*) \to \infty, \quad \eta\, N^{*-1} \to 0 \quad \text{for} \quad N^* \to \infty; \tag{6.7a}$$

$$d^0 = d^0(N^*) \to \infty, \quad d^0\, N^{*-1} \to 0 \quad \text{for} \quad N^* \to \infty;$$

$$h^* = o(\varepsilon), h^* = h^*_{(6.6)}(d^0, \eta, N^*) \tag{6.7b}$$

are necessary and sufficient for the convergence of the solutions of the difference scheme (6.5), (6.4) on the grids (4.1), (4.4) for N^*, $N_0 \to \infty$ and $\varepsilon \in (0, 1]$.

Under the condition

$$\eta = l\, m^{-1} \ln N^*, \quad m = m_{(6.3)} \tag{6.8}$$

where $l = 1$ for the grid (4.1) and $l = 2$ for the grid (4.4), we have the following estimate for the grid (4.1)

$$|u(x,t) - z^{[0]}(x,t)| \le M \left[\varepsilon^{-1} d^0\, N^{*-1} + \varepsilon^{-1} N^{*-1} \ln N^* + N_0^{-1}\right], \tag{6.9a}$$

$$(x,t) \in \overline{G}^0_h;$$

and the following estimate for the grid (4.4)

$$|u(x,t) - z^{[0]}(x,t)| \le M \left[(\varepsilon^{-1} d^0\, N^{*-1})^2 + (\varepsilon^{-1} N^{*-1} \ln N^*)^2 + N_0^{-1}\right], \tag{6.9b}$$

$$(x,t) \in \overline{G}^0_h;$$

However, if the set $\overline{G}^{[0]}$ is separated from the boundary $\overline{S}^L$, then under the condition

$$r(\overline{G}^{[0]}, \overline{S}^L) \ge \eta \tag{6.10}$$

we obtain, for the grid (4.1), the estimate

$$|u(x,t) - z^{[0]}(x,t)| \le M \left[\exp(-m\,\eta) + h^* + N_0^{-1}\right], \quad (x,t) \in \overline{G}^0_h, \tag{6.11a}$$

and, for the grid (4.4), the estimate

$$|u(x,t) - z^{[0]}(x,t)| \le M \left[\exp(-m\,\eta) + h^{*2} + N_0^{-1}\right], \quad (x,t) \in \overline{G}^0_h. \tag{6.11b}$$

The estimate (6.11b) is unimprovable with respect to the value of h^*, N_0.

Under condition (6.10) and the additional condition (6.7a) the constructive schemes converge ε-uniformly. On the grid (4.1), under condition (6.8), we have the estimate

$$|u(x,t) - z^{[0]}(x,t)| \le M \left[\, d^0\, N^{*-1} + N^{*-1} \ln N^* + N_0^{-1}\right], \tag{6.12a}$$

$$(x,t) \in \overline{G}^0_h$$

and, for the grid (4.4), the estimate

$$|u(x,t) - z^{[0]}(x,t)| \le M \left[(d^{0\,2}\, N^{*-2}) + N^{*-2} \ln^2 N^* + N_0^{-1} \right], \quad (6.12b)$$

$$(x,t) \in \overline{G}_h^0;$$

Theorem 6.1. *Let the condition of Theorem* 4.1 *be fulfilled and let the boundary value problem* (2.2), (2.1) *be approximated by the difference scheme* (6.5), (6.4) *on the grids* (4.1), (4.4). *The condition* (6.7) *is necessary and sufficient for the convergence of the grid solutions to the solution of the problem* (2.2), (2.1) *on the set* $\overline{G}^0_{(2.5)}$ *for* N^*, $N_0 \to \infty$ *and* $\varepsilon \to 0$; *for the discrete solutions the estimates* (6.6) *and also the estimates* (6.9) *under the condition* (6.8) *are valid. However, if the condition* (6.10) *holds, the estimates* (6.11) *are satisfied, and under the additional condition* (6.7a) *the schemes converge* ε*-uniformly and the estimates* (6.12) *are satisfied.*

3. We now deduce some properties of the solutions of the scheme (6.5), (6.4).

Remark 1. In the case of the nonconstructive schemes (4.2) on the grids (4.1), (4.4), by virtue of condition (4.6), the convergence defect is equal to one. However, for the constructive schemes (6.5), (6.4) on the grids (4.1), (4.4), even for $d^0 \approx m$, by virtue of condition (6.7), a more restrictive condition, compared to the condition $\varepsilon^{-1} N^{*-1} \to 0$ as $N^* \to \infty$, is required for their convergence. This condition is given in the following lemma.

Lemma 6.3. *Under the condition* $d^0 \ge m$, *in the class of constructive schemes* (6.5), (6.4) *on the basic grids* (4.1), (4.4), *there do not exist schemes with convergence defect (on* $\overline{G}^0$*) equal to one.*

Note that, by virtue of the estimates (6.9), under the condition (6.8) and the condition

$$m \le d^0 \le M \ln N^*,$$

the convergence defect of the schemes (6.5), (6.4) on the grids (4.1), (4.4) is equal to one up to the logarithmic factor $M \ln N^*$, which could be considered as an error constant. Here the error constant means a constant M_0 in the case when the error estimate has the following form

$$|u(x,t) - z^{[0]}(x,t)| \le M_0 \left[(\varepsilon^{-\nu} N^{*-1})^p + N^{*-p_0} \right], \quad (x,t) \in \overline{G}_h^0,$$

(see, e.g., [11], p.157). The convergence defect grows without bound as d^0 tends to infinity.

Remark 2. For constructive schemes it is attractive, for a fixed number of nodes in the grid domains where the problem is to be solved, to find a solution on a domain with a possibly larger diameter and a discrete solution with a possibly smaller error. However, for the schemes (6.5), (6.4) such a task is *contradictory*. This follows from the unimprovability of the estimate (6.6b) of the error δ_1 on $\overline{G}^0$, where δ_1 denotes the error due to the discretization of the problem with respect to x. In the case of condition (6.8) on the grid (4.4) for $d^0 \geq m \ln N^*$, we have the estimate

$$\delta_1^{1/2}(d^0)^{-1} \leq M \min[\varepsilon^{-1} N^{*-1}, 1]; \tag{6.13}$$

this estimate is unimprovable.

In the case of the grid (4.1) the estimate, in general, is no better.

Lemma 6.4. *For the difference schemes* (6.5), (6.4) *on the grid* (4.4) *under the condition* η, $d^0 \geq M \ln N^*$, *for the value* δ_1, *i.e. the component of the error of the grid solution which is due to the discretization of the problem* (6.2), (6.1) *with respect to* x *on the set* $\overline{G}^0$, *the estimate* (6.13) *is valid. This estimate is unimprovable with respect to the values of* N^*, ε, d^0.

Remark 3. We denote by $\widehat{d}^0$ the largest size of the set $\overline{G}^0$ w.r.t x on which convergence of the scheme (6.5), (6.4) on the grids (4.1), (4.4) for $N^*, N_0 \to \infty$ is achievable. Under the condition $\overline{G}^0 \cap \overline{S^L} \neq \emptyset$, we derive the condition $\widehat{d}^0 = o(\varepsilon N^*)$. But if (6.10) holds, then we get $\widehat{d}^0 = o(N^*)$, moreover, in this case the scheme converges ε-uniformly.

Remark 4. We give an estimate for the solutions of the classic difference scheme for problem (2.2), (2.4). The basic grids (4.1), (4.4) generate the grid on $\overline{G}_0$

$$\overline{G}_{0h} = \overline{G}_0 \cap \overline{G}_h, \tag{6.14}$$

where $\overline{G}_h$ is either the grid (4.1) or the grid (4.4); N^*+1 is the number of nodes in the grid on the segment $\overline{D}_0$. For the solutions of the difference scheme (4.2), (6.14) in the case of the grid (4.1), we obtain the estimate

$$|u_0(x,t) - z_0(x,t)| \leq M \left[\varepsilon^{-1} N^{*-1} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_{0h}, \tag{6.15a}$$

and for the grid (4.4) the estimate

$$|u_0(x,t) - z_0(x,t)| \leq M \left[(\varepsilon^{-1} N^{*-1})^2 + N_0^{-1}\right], \quad (x,t) \in \overline{G}_{0h}. \tag{6.15b}$$

Here $u_0(x,t)$ and $z_0(x,t)$ are the solutions of the problem (2.2), (2.4) and the scheme (4.2), (6.14) respectively. From the estimates (6.9) and (6.15), it follows that, under the condition $d^0 = d_0$, the convergence rate of the scheme (4.2), (6.14) is the same as that for the scheme (6.5), (6.4) on the set $\overline{G}^0$ up to a logarithmic factor.

7. Special constructive difference schemes

In this section we consider approximations of problem (6.2), (6.1) on constructive grids condensing in the boundary layer region. We assume that the following condition holds

$$d^0 \geq M, \quad M \geq 1. \tag{7.1}$$

On the set $\overline{G}^{[0]}$, using the grid (5.1) as a basic grid, we introduce the grid

$$\overline{G}_h^{[0]} = \overline{G}_h^{[0]}(\overline{G}_{h(5.1)}) = \overline{G}^{[0]} \cap \overline{G}_{h(5.1)}; \tag{7.2}$$

$N^* + 1$ is the number of nodes in the grid $\overline{\omega}_1^{[0]}$ on the set $\overline{D}^{[0]}$,
$m\,(d^0 + \eta)\,N_{(5.1)} \leq N^* \leq M\,(d^0 + \eta)\,N_{(5.1)}$.

For the solution of the difference scheme (6.5), (7.2), taking into account the *a-priori* estimates, we obtain the estimate

$$|u(x,t) - z^{[0]}(x,t)| \leq \tag{7.3}$$

$$\leq M\,[\exp(-m\,\eta) + (\varepsilon + h^*_{(1)})^{-2}\,(h^*_{(1)})^2 + (h^*_{(2)})^2 + N_0^{-1}], \quad (x,t) \in \overline{G}_h^0,$$

where $h^*_{(i)} = h_{(i)(5.1)}$, $i = 1, 2$, $m = m_{(6.3)}$;

$$m\,(d^0 + \eta)\,N^{*-1} \leq h^*_{(2)} \leq M\,(d^0 + \eta)\,N^{*-1},$$

$$m\,\{\min[\varepsilon\,\ln((d^0 + \eta)^{-1}\,N^*), 1]\,(d^0 + \eta)\,N^{*-1}\} \leq h^*_{(1)} \leq$$

$$\leq M\,\{\min[\varepsilon\,\ln((d^0 + \eta)^{-1}\,N^*), 1]\,(d^0 + \eta)\,N^{*-1}\};$$

estimate (7.3) is unimprovable with respect to the values of $h^*_{(1)}$, $h^*_{(2)}$, N_0, ε and it is also unimprovable up to a constant-factor with respect to the value of η.

From the estimate (7.3) it follows that the condition (6.7a) is necessary and sufficient for the ε-uniform convergence on $\overline{G}^0$ of the solutions of the difference scheme (6.5), (7.2) for $N^*, N_0 \to \infty$.

Under the condition

$$\eta = 2\,m^{-1}\,\ln N^*, \quad m = m_{(6.3)} \tag{7.4}$$

we have the estimate

$$|u(x,t) - z^{[0]}(x,t)| \leq \tag{7.5a}$$

$$\leq M\,\left\{[\min[\varepsilon^{-1}, \ln((d^0 + \ln N^*)^{-1}\,N^*)]\,(d^0 + \ln N^*)\,N^{*-1}]^2 + N_0^{-1}\right\},$$

$$(x,t) \in \overline{G}_h^0,$$

and also the ε-uniform estimate

$$|u(x,t) - z^{[0]}(x,t)| \le \tag{7.5b}$$
$$\le M \ \left\{[\ln((d^0 + \ln N^*)^{-1} N^*) \, (d^0 + \ln N^*) \, N^{*-1})]^2 + N_0{}^{-1}\right\}, \quad (x,t) \in \overline{G}{}^0_h.$$

Assume that the condition (7.4) is satisfied. Then, under the condition

$$d^0 \le M \ \ln N^* \tag{7.6a}$$

we have the estimate

$$|u(x,t) - z^{[0]}(x,t)| \le M \ \left[N^{*-2} \ln^4 N^* + N_0{}^{-1}\right], \quad (x,t) \in \overline{G}{}^0_h, \tag{7.7a}$$

and under the condition

$$d^0 \ge m \ \ln N^* \tag{7.6b}$$

we have the estimate

$$|u(x,t) - z^{[0]}(x,t)| \le M \ \left\{[\, d^0 \, N^{*-1} \ln((d^0)^{-1} N^*)]^2 + N_0{}^{-1}\right\}, \tag{7.7b}$$
$$(x,t) \in \overline{G}{}^0_h.$$

A formal statement of these results is contained in

Theorem 7.1. *Let the condition of Theorem* 4.1 *and also the condition* (7.1) *be fulfilled. For the difference scheme* (6.5), (7.2) *the condition* (6.7a) *is necessary and sufficient for the ε-uniform convergence of the grid solutions of the solution of the problem* (2.2), (2.1) *on the set* $\overline{G}{}^0_{(2.5)}$ *for* N^*, $N_0 \to \infty$. *For the solutions of the difference scheme the estimate* (7.3) *holds, and also estimates* (7.5) *and* (7.7), *respectively, are valid under the conditions* (7.4) *and* (7.4), (7.6).

Remark 1. Under condition (7.4) and the additional condition (7.6a), the ε-uniform convergence rate of the scheme (6.5), (7.2) with respect to the variable x (i.e., the value δ_1, the component of the error caused by the discretization of the problem with respect to x) is equal to 2, up to a logarithmic factor. However, under condition (7.6b), the convergence order decreases as the value of d^0 grows exponentially ; for the value δ_1 we have the unimprovable estimate $\delta_1^{1/2} \le M(d^0/N^*) \ln(N^*/d^0)$. The largest width $\widehat{d}^0$ of the set $\overline{G}{}^0$, on which the scheme converges ε-uniformly, satisfies the condition $\widehat{d}^0 = o(N^*)$.

Remark 2. If condition (6.10) holds, when solving a grid problem, it is convenient to use the scheme (6.5), (6.4), (6.8) on the uniform grid (4.4). However, if the condition (6.10) is not satisfied, then we use the scheme (6.5), (7.2), (7.4), i.e., the scheme on the piecewise uniform grid with respect to x. For the grid solutions, under condition (6.10), the estimate (6.12b) holds, and the estimate (7.6b) is valid when the condition (6.10) is not fulfilled.

Remark 3. Taking into account the *a-priori* estimates of the solutions for the initial boundary value problem, and the estimates (7.7) for the solutions of the difference scheme (6.5), (7.2), we find the following estimate under condition (7.6a)

$$|u(x,t) - \overline{z}^{[0]}(x,t)| \le M \, \left[N^{*-2} \ln^4 N^* + {N_0}^{-1} \right], \quad (x,t) \in \overline{G}^0,$$

and, under condition (7.6b), the estimate

$$|u(x,t) - \overline{z}^{[0]}(x,t)| \le M \, \left\{ [\, d^0 \, N^{*-1} \ln((d^0)^{-1} N^*)]^2 + {N_0}^{-1} \right\}, \quad (x,t) \in \overline{G}^0,$$

where $\overline{z}^{[0]}(x,t)$, $x \in \overline{G}$ is the bilinear interpolant, which is constructed from the values of the function $z^{[0]}(x,t)$, $(x,t) \in \overline{G}_h$. Under conditions (6.10), (6.7a) for the solutions of the difference scheme (6.5), (6.4) on the grid (4.4), the following estimate holds

$$|u(x,t) - \overline{z}^{[0]}(x,t)| \le M \, \left[(d^0)^2 \, N^{*-2} + N^{*-2} \ln^2 N^* + {N_0}^{-1} \right], \quad (x,t) \in \overline{G}^0.$$

Thus, the schemes (6.5), (7.2) and (6.5), (6.4), (4.4) allow us to construct approximations of the solution of the problem (2.1), (2.1) on the set $\overline{G}^0$ with exactly the same estimate as on the mesh $\overline{G}_h^0$.

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