High-Order Accurate Decomposition Methods Based on Richardson's Extrapolation for a Singularly Perturbed Elliptic Reaction-Diffusion Equation on a Strip *

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Abstract

We consider the Dirichlet problem on a strip for a singularly perturbed elliptic equation of reaction-diffusion type. For such a problem well-known finite difference schemes converge ε-uniformly with the order of accuracy not higher than second. This can imply some restrictions for practical use of these schemes in applications. Based on special schemes on *piecewise uniform* meshes and using the Richardson extrapolation technique on a sequence of *embedded* meshes, we construct a difference scheme that converges ε-uniformly with the third-order accuracy up to a logarithmic factor and with the fourth-order accuracy with respect to the variables which are respectively orthogonal and tangent to the boundary. For the above Richardson scheme we construct a decomposition method on overlapping subdomains for which the ε-uniform accuracy of the Richardson scheme is preserved.

1. Introduction

At present special numerical methods for sufficiently wide classes of singularly perturbed boundary value problems have been constructed and studied. These methods, as opposed to methods developed for regular boundary value problems (see, e.g., [1], [2]), allow us to obtain solutions which converge independently of the perturbation parameter ε (in other words, convergent ε-uniformly). In the case of boundary value problems for elliptic reaction-diffusion equations,

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the order of ε-uniform convergence for known regular methods on special condensing meshes does not exceed two, and for convection-diffusion equations it is not higher than one even if the problem data are smooth (see, e.g., [3]–[6] and the references therein; fitted operator methods on uniform meshes see in [7], [8]). That is why it is interesting to construct special schemes for reaction-diffusion and convection-diffusion problems for which the order of ε-uniform convergence is respectively higher than two and one.

In the case of regular boundary value problems, defect-correction and Richardson methods allow one to improve the order of accuracy for discrete solutions (see, e.g., [1], [9], [10] and the references therein). These methods are applied also in order to improve the order of time-accuracy for singularly perturbed problems (see, e.g., [11]–[13]). Note that the time mesh is uniform in these papers, which essentially simplifies the construction and investigation of high-order time-accurate schemes. For wide classes of singularly perturbed boundary value problems, the use of meshes that are condensing in a boundary layer region (with respect to the space variable in the direction across the boundary layer) is a necessary requirement for ε-uniform convergence of schemes (see, e.g., [4], [14]). Thus, we are interested in developing ε-uniform schemes of high-order accuracy with respect to the variables for which the step size changes sharply at the point of transition from a fine to a coarse mesh.

In this paper we consider a boundary value problem on a vertical strip for a singularly perturbed elliptic equation of reaction-diffusion type. For this problem, using the Richardson method we construct a special scheme that converges ε-uniformly at a rate of $\mathcal{O}(N_1^{-3} \ln^3 N_1 + N_2^{-4})$, where the values N_1 and N_2 define the number of mesh points across the strip, i.e. in the direction across the layer, and the number of mesh points on a unit interval along the strip. When constructing the scheme we use meshes condensing in a neighbourhood of the boundary layer. For this Richardson scheme we construct an overlapping domain decomposition method for which the ε-uniform accuracy of the Richardson scheme is preserved. The technique developed in this paper allows to construct parallel Richardson schemes, i.e. high-order accurate scheme for parallel computations.

Note that a similar technique, based on Richardson's extrapolation, applied in order to improve the solution accuracy was used in [15] for elliptic convection-diffusion equations to construct schemes convergent ε-uniformly at a rate of $\mathcal{O}(N_1^{-2} \ln^3 N_1 + N_2^{-2})$.

2. Problem Formulation

On the vertical strip $\overline{D}$, where

$$D = \{x : \quad x_1 \in (0, d), \quad x_2 \in R\}, \tag{2.1}$$

we consider the boundary value problem for a singularly perturbed elliptic equation

$$\begin{aligned} Lu(x) &\equiv \left\{ \varepsilon^2 \sum_{s=1,2} a_s(x) \frac{\partial^2}{\partial x_s^2} - c(x) \right\} u(x) = f(x), \quad x \in D, \\ u(x) &= \varphi(x), \quad x \in \Gamma. \end{aligned} \tag{2.2}$$

Here $\Gamma = \overline{D} \backslash D$, the functions $a_s(x)$, $c(x)$, $f(x)$ and $\varphi(x)$ are sufficiently smooth on the sets $\overline{D}$ and Γ respectively, and also satisfying the conditions:

$$\begin{gathered} a_0 \le a_s(x) \le a^0, \quad s = 1, 2, \quad c_0 \le c(x) \le c^0, \qquad a_0,\, c_0 > 0; \\ |f(x)| \le M, \quad x \in \overline{D}; \quad |\varphi(x)| \le M, \quad x \in \Gamma; \end{gathered} \tag{2.3}$$

the parameter ε takes arbitrary values from the half-open interval (0,1]. Here and below M, M_i (or m) denote sufficiently large (small) positive constants which do not depend on ε and on the discretization parameters. Throughout the paper, the notation $L_{(j.k)}$ ($M_{(j.k)}$, $G_{h(j.k)}$) means that these operators (constants, grids) are introduced in equation $(j.k)$.

As the parameter ε tends to zero, a regular boundary layer appears in a neighbourhood of the boundary Γ.

For the boundary value problem (2.2), (2.1) it is required to construct a difference scheme convergent ε-uniformly with the order of accuracy higher than second for each variable.

3. Base Scheme for Problem (2.2), (2.1)

In this section we give a classical finite difference scheme and also the special (base) scheme convergent ε-uniformly with the order $\mathcal{O}(N_1^{-2} \ln^2 N_1 + N_2^{-2})$, i.e., a scheme of the second-order accuracy with respect to x_2.

On the set $\overline{D}$ we introduce the rectangular mesh

$$\overline{D}_h = \overline{\omega}_1 \times \omega_2, \tag{3.1}$$

where $\overline{\omega}_1$ and ω_2 are arbitrary, generally speaking, non-iniform meshes on the interval $[0,d]$ and the axis x_2, respectively. We define $h_s^i = x_s^{i+1} - x_s^i$, x_s^i, $x_s^{i+1} \in \overline{\omega}_1$ for $s=1$ and x_s^i, $x_s^{i+1} \in \omega_2$ for $s=2$; let $h_s = \max_i h_s^i$, $h = \max_s h_s$. We assume the following condition to be satisfied: $h \le MN^{-1}$, where $N = \min[N_1, N_2]$; N_1+1 is the number of nodes in the mesh $\overline{\omega}_1$, and N_2+1 is the minimal number of nodes in the mesh ω_2 on an interval of unit length.

Problem (2.2), (2.1) is approximated by the difference scheme

$$\Lambda z(x) \equiv \left\{ \varepsilon^2 \sum_{s=1,2} a_s(x)\delta_{\overline{xs}\,\widehat{xs}} - c(x) \right\} z(x) = f(x), \quad x \in D_h,$$
$$z(x) = \varphi(x), \quad x \in \Gamma_h. \tag{3.2}$$

Here $D_h = D \cap \overline{D}_h$, $\Gamma_h = \Gamma \cap \overline{D}_h$, $\delta_{\overline{xs}\,\widehat{xs}} z(x) = z_{\overline{xs}\,\widehat{xs}}(x)$ is the second (central) difference derivative on a non-uniform mesh [2], for example,

$$\delta_{\overline{x1}\,\widehat{x1}} z(x) = 2\left(h_1^i + h_1^{i-1}\right)^{-1}\left[\delta_{x1} z(x) - \delta_{\overline{x1}} z(x)\right], \quad x = (x_1^i, x_2) \in D_h \quad .$$

Scheme (3.2), (3.1) is monotone [2] ε-uniformly.

For solutions of the difference scheme we derive the estimate

$$|u(x) - z(x)| \le M \left[\left(\varepsilon + N_1^{-1}\right)^{-1} N_1^{-1} + N_2^{-1}\right], \quad x \in \overline{D}_h. \tag{3.3}$$

On the meshes

$$\overline{D}_h^u = \overline{\omega}_1 \times \omega_2, \tag{3.4}$$

which are uniform with respect to both of the variables (with the step-sizes $h_1 = dN_1^{-1}$ and $h_2 = N_2^{-1}$), we have

$$|u(x) - z(x)| \le M \left[\left(\varepsilon + N_1^{-1}\right)^{-2} N_1^{-2} + N_2^{-2}\right], \quad x \in \overline{D}_h^u. \tag{3.5}$$

Now we construct a *piecewise uniform* mesh on which scheme (3.2) converges ε-uniformly [4], [5]. On the set $\overline{D}$ we introduce the special mesh condensing in a neighbourhood of the boundary layer:

$$\overline{D}_h = \overline{\omega}_1^* \times \omega_2, \tag{3.6a}$$

where $\omega_2 = \omega_{2(3.4)}$, $\overline{\omega}_1^*$ is a *piecewise uniform* mesh constructed as follows. Let σ be a parameter of the mesh, which depends on ε and N_1, $\sigma \le 4^{-1}d$. The interval $[0,d]$ is divided into three parts $[0,\sigma]$, $[\sigma, d-\sigma]$, $[d-\sigma, d]$. In each part the mesh size is constant and equal to $h_1^{(1)} = 4\sigma N_1^{-1}$ on the intervals $[0,\sigma]$, $[d-\sigma, d]$ and $h_1^{(2)} = 2(d-2\sigma)N_1^{-1}$ on the interval $[\sigma, d-\sigma]$. The parameter σ is defined by the relation

$$\sigma = \sigma(\varepsilon,\, N_1,\, l) = \min\left[4^{-1}d,\, l\, m^{-1}\, \varepsilon \ln N_1\right], \tag{3.6b}$$

where $m = m_{(7.2)}$. Here $l = 2$; in other meshes (below) this parameter will be chosen. The mesh $\overline{\omega}_1^*$ and hence the mesh $\overline{D}_h = \overline{D}_h(l = 2)$ are constructed.

For the solutions of scheme (3.2), (3.6) we obtain the estimate

$$|u(x) - z(x)| \le M \, \left\{ N_1^{-2} \min \left[\varepsilon^{-2}, \, \ln^2 N_1 \right] + N_2^{-2} \right\}, \quad x \in \overline{D}_h, \qquad (3.7)$$

and also the ε-uniform estimate

$$|u(x) - z(x)| \le M \, \left[N_1^{-2} \ln^2 N_1 + N_2^{-2} \right], \quad x \in \overline{D}_h. \qquad (3.8)$$

Theorem 3.1. *Let the solution $u(x)$ of problem (2.2), (2.1) satisfy the a priori estimates of Theorem 7.1 for $K = 4$. Then the difference scheme (3.2), (3.6) (schemes (3.2), (3.1) and (3.2), (3.4)) converges ε-uniformly (for fixed values of the parameter ε). For the discrete solutions the estimates (3.3), (3.5), (3.7) and (3.8) are valid.*

4. Richardson Scheme for Problem (2.2), (2.1)

We describe the Richardson method used in this paper to improve the accuracy of solutions of the special difference scheme.

1. On the set $\overline{D}$ we construct the meshes

$$\overline{D}_h^i = \overline{\omega}_1^{*i} \times \omega_2^i, \quad i = 1, 2, \qquad (4.1a)$$

where $\overline{D}_h^1 = \overline{D}_{h(3.6)}(l)$ for $l = 4$, $\overline{\omega}_1^{*2}$ is a *piecewise uniform* grid whose step-size on the intervals $[0, \sigma]$, $[\sigma, d - \sigma]$, $[d - \sigma, d]$, where $\sigma = \sigma_{(3.6b)}(\varepsilon, N_1 = N_{1(3.6)}, l = 4)$ is k times smaller than the step-size of the grid $\overline{\omega}_1^{*1}$; the step-size of the grid ω_2^2 is k times smaller than the step-size of ω_2^1; $kN_1 + 1$ and $kN_2 + 1$ are respectively the number of nodes in the mesh $\overline{\omega}_1^{*2}$ and on a unit interval in the mesh ω_2^2.

We denote

$$\overline{D}_h^0 = \overline{D}_h^1 \cap \overline{D}_h^2. \qquad (4.1b)$$

Here, $\overline{D}_h^0 = \overline{D}_h^1$ for each integer k ($k \ge 2$), and $\overline{D}_h^0 \ne \overline{D}_h^1$ for a noninteger k.

Let $z^i(x)$, $x \in \overline{D}_h^i$, $i = 1, 2$ be solutions of the finite difference schemes

$$\Lambda_{(3.2)} \, z^i(x) = f(x), \quad x \in D_h^i, \quad z^i(x) = \varphi(x), \quad x \in \Gamma_h^i, \quad i = 1, 2. \qquad (4.2a)$$

Suppose

$$z^0(x) = \gamma \, z^1(x) + (1 - \gamma) \, z^2(x), \quad x \in \overline{D}_h^0, \qquad (4.2b)$$

where $\gamma = \gamma(k) = -(k^2-1)^{-1}$. We call the function $z^0(x)$, $x \in \overline{D}_h^0$, the solution of difference scheme (4.2), (4.1), i.e., the scheme based on the Richardson method on two embedded meshes.

2. To justify the convergence of scheme (4.2), (4.1), it is convenient to decompose the functions $z^i(x)$, $x \in \overline{D}_h^i$, $i = 1, 2$ with respect to N_1^{-1} and N_2^{-1}

$$z^i(x) = u(x) + k^{-2(i-1)} \left[N_1^{-2} u_1(x) + N_2^{-2} u_2(x)\right] + v^i(x), \quad x \in \overline{D}_h^i, \quad i = 1, 2, \tag{4.3}$$

where $v^i(x)$ is the remainder term. The function $u_2(x)$ is the solution of the problem

$$L_{(2.2)} u_2(x) = -12^{-1} \varepsilon^2 a_2(x) \frac{\partial^4}{\partial x_2^4} u(x), \quad x \in D, \quad u_2(x) = 0, \quad x \in \Gamma.$$

The function $u_1(x)$ can be represented as a sum of two functions

$$u_1(x) = u_{11}(x) + u_{12}(x), \quad x \in \overline{D}, \tag{4.4}$$

where $u_{1j}(x)$, $x \in \overline{D}$, $j = 1, 2$ are the solutions of the problems

$$L_{(2.2)} u_{11}(x) = -12^{-1} \varepsilon^2 N_1^2 a_1(x) \frac{\partial^4}{\partial x_1^4} u(x) \begin{Bmatrix} \left(h_1^{(1)}\right)^2, & x_1 \in (0, d) \setminus [\sigma, d-\sigma] \\ \left(h_1^{(2)}\right)^2, & x_1 \in (\sigma, d-\sigma) \end{Bmatrix},$$

$$x \in D, \quad x_1 \neq \sigma,\ d-\sigma,$$

$$\begin{aligned}
[u_{11}(x)] &= \left[\frac{\partial}{\partial x_1} u_{11}(x)\right] = 0, && x \in D,\ x_1 = \sigma,\ d-\sigma, \\
u_{11}(x) &= 0, && x \in \Gamma; \\
L_{(2.2)} u_{12}(x) &= 0, && x \in D,\ x_1 \neq \sigma,\ d-\sigma, \\
u_{12}(x) &= \psi(x), && x \in D,\ x_1 = \sigma,\ d-\sigma, \\
u_{12}(x) &= 0, && x \in \Gamma.
\end{aligned}$$

Here $\psi(x) = w(x)$, $x \in D$, $x_1 = \sigma, d-\sigma$, where $w(x)$ is the solution of the following problem for the method of lines with respect to x_2:

$$L^h w(x) \equiv \left\{\varepsilon^2 \left(a_1(x) \delta_{\overline{x1}\widehat{x1}} + a_2(x) \frac{\partial^2}{\partial x_2^2}\right) - c(x)\right\} w(x) =$$

$$= \begin{Bmatrix} 0, & x_1 \neq \sigma,\ d-\sigma, \\ -\dfrac{\varepsilon^2}{3} N_1^2 a_1(x) \dfrac{\partial^3}{\partial x_1^3} U(x) \begin{cases} h_1^{(2)} - h_1^{(1)}, & x_1 = \sigma, \\ -h_1^{(2)} + h_1^{(1)}, & x_1 = d-\sigma \end{cases} \end{Bmatrix}, \quad x \in D,\ x_1 = \omega_1^{*1},$$

$$w(x) = 0, \quad x \in \Gamma, \quad h_1^{(1)} = h_{1(3.6)}^{(1)}, \quad h_1^{(2)} = h_{1(3.6)}^{(2)}, \quad U(x) = U_{(7.1)}(x).$$

The functions $u_{11}(x)$, $u_{12}(x)$, $u_2(x)$, $x \in \overline{D}$ satisfy the estimates

$$|u_2(x)| \le M\,\varepsilon^2, \quad |u_{11}(x)| \le M \min\left[\varepsilon^{-2}, \ln^2 N_1\right], \quad |u_{12}(x)| \le M\,\varepsilon, \quad x \in \overline{D}.$$

The component $u_1(x)$ is sufficiently smooth on $\overline{D}$, and the component $u_2(x)$ is also sufficiently smooth on $\overline{D}$ on the strips $x_1 \le \sigma$, $\sigma \le x_1 \le d - \sigma$, $x_1 \ge d - \sigma$. Taking into consideration the estimates for the components $u_1(x)$, $u_2(x)$, we estimate the function $v^i(x)$ as follows

$$|v^i(x)| \le M \left\{N_1^{-3} \min\left[\varepsilon^{-3}, \ln^3 N_1\right] + N_2^{-4}\right\}, \quad x \in \overline{D}_h^i, \quad i = 1, 2. \tag{4.5}$$

Taking (4.5) into account, we find the estimate

$$|u(x) - z^0(x)| \le M \left\{N_1^{-3} \min\left[\varepsilon^{-3}, \ln^3 N_1\right] + N_2^{-4}\right\}, \quad x \in \overline{D}_h^0, \tag{4.6}$$

and the ε-uniform estimate

$$|u(x) - z^0(x)| \le M\left[N_1^{-3} \ln^3 N_1 + N_2^{-4}\right], \quad x \in \overline{D}_h^0. \tag{4.7}$$

Theorem 4.1. *Let the data of problem* (2.2), (2.1) *satisfy conditions* (2.3) *and, moreover,* a_s, c, $f \in C^{6+\alpha}(\overline{D})$, $\varphi \in C^{6+\alpha}(\Gamma)$, $\alpha > 0$, *and let a priori estimates* (7.2) *with* $K = 6$ *be fulfilled for the components of the problem solution from representation* (7.1). *Then the solution of scheme* (4.2), (4.1) *converges to the solution of the boundary value problem* ε*-uniformly at a rate of*

$$\mathcal{O}(N_1^{-3} \ln^3 N_1 + N_2^{-4}).$$

For the discrete solutions the estimate (4.6), (4.7) *are valid.*

5. Decomposition of Scheme (3.2), (3.1)

Let us give a domain decomposition scheme based on the difference scheme (3.2), (3.1).

1. Let the domain D be covered by overlapping subdomains $D^{(k)}$

$$D = D^{(1)} \cup D^{(2)}, \quad D^{(1)} = (0, d_1) \times R, \quad D^{(2)} = (d_2, d) \times R, \tag{5.1a}$$
$$0 < d_2 < d_1 < d,$$

that have overlap $\delta = d_1 - d_2 > 0$. On the sets $\overline{D}^{(k)}$, $k = 1, 2$ we introduce the meshes

$$\overline{D}_h^{(k)} = \overline{D}^{(k)} \cap \overline{D}_h, \quad k = 1, 2, \tag{5.1b}$$

where $\overline{D}_h = \overline{D}_{h(3.1)}$; we suppose that boundaries of the subdomains $D^{(1)}$, $D^{(2)}$ pass through the nodes of the mesh $\overline{\omega}_1$.

Let an arbitrary bounded function $z_0(x)$ be given on $\overline{D}_h$, and assume that the functions $z_1(x), \ldots, z_{n-1}(x)$, $x \in \overline{D}_h$ are already known, where $z_i(x) = \varphi(x)$, $x \in \Gamma_h$. We find the function $z_n(x)$ by solving the following problems

$$\Lambda z_n^{(1)}(x) = f(x), \ \ x \in D_h^{(1)}, \quad z_n^{(1)}(x) = z_{n-1}(x), \qquad x \in \Gamma_h^{(1)}; \tag{5.2a}$$

$$\Lambda z_n^{(2)}(x) = f(x), \ \ x \in D_h^{(2)}, \quad z_n^{(2)}(x) = \begin{cases} z_n^{(1)}(x), & x \in \Gamma_h^{(2)} \setminus \Gamma, \\ z_{n-1}(x), & x \in \Gamma_h^{(2)} \cap \Gamma. \end{cases}$$

Suppose

$$z_n(x) = \left\{ \begin{array}{ll} z_n^{(1)}(x), & x \in D_h^{(1)} \setminus D^{(2)} \\ z_n^{(2)}(x), & x \in D_h^{(2)} \end{array} \right\}, \quad x \in \overline{D}_h. \tag{5.2b}$$

We call the function $z_{n(5.2)}(x)$, $x \in \overline{D}_h$, $n = 1, 2, \ldots$, the solution of difference scheme (5.2), (5.1), i.e., the domain decomposition scheme with overlapping subdomains. Difference scheme (5.2), (5.1) is an approximation of the following continual scheme of the Schwartz method.

Let $u_0(x)$, $x \in \overline{D}$ be an arbitrary bounded function, and let functions $u_1(x), \ldots, u_{n-1}(x)$, $x \in \overline{D}$ have been already constructed; $u_i(x) = \varphi(x)$, $x \in \Gamma$. We wish to construct the function $u_n(x)$. Beforehand, we solve problems

$$L u_n^{(1)}(x) = f(x), \ \ x \in D^{(1)}, \quad u_n^{(1)}(x) = u_{n-1}(x), \qquad x \in \Gamma^{(1)}; \tag{5.3a}$$

$$L u_n^{(2)}(x) = f(x), \ \ x \in D^{(2)}, \quad u_n^{(2)}(x) = \begin{cases} u_n^{(1)}(x), & x \in \Gamma^{(2)} \setminus \Gamma, \\ u_{n-1}(x), & x \in \Gamma^{(2)} \cap \Gamma. \end{cases}$$

Further we suppose

$$u_n(x) = \left\{ \begin{array}{ll} u_n^{(1)}(x), & x \in D^{(1)} \setminus D^{(2)} \\ u_n^{(2)}(x), & x \in D^{(2)} \end{array} \right\}, \quad x \in \overline{D}. \tag{5.3b}$$

We call the function $u_n(x)$, $x \in \overline{D}$, $n = 1, 2, \ldots$ the solution of the continual scheme of the Schwartz method (on overlapping subdomains).

For $z_0(x) = u_0(x)$, $x \in \overline{D}_h$ scheme (5.2), (5.1) is in fact an approximation of problem (5.3), (5.1a).

2. Note that the width of overlap, i.e., the value of δ can depend on the parameter ε; $\delta = \delta(\varepsilon)$.

In the case if

$$\inf_{\varepsilon}(\varepsilon^{-1}\delta) > 0 \tag{5.4}$$

for the solutions of schemes (3.2), (3.1) and (5.2), (5.1), (3.1) we have the estimate [4]

$$|z(x) - z_n(x)| \le M\, q^n, \quad x \in \overline{D}_{h(3.1)}, \tag{5.5}$$

moreover, $q \le 1 - m$. On the special mesh (3.6) we have the estimate [4]

$$|u(x) - z_n(x)| \le M\{N_1^{-2} \min\left[\varepsilon^{-2}, \ln^2 N_1\right] + N_2^{-2} + q^n\}, \quad x \in \overline{D}_{h(3.6)} \tag{5.6}$$

and the ε-uniform estimate

$$|u(x) - z_n(x)| \le M[N_1^{-2} \ln^2 N_1 + N_2^{-2} + q^n], \quad x \in \overline{D}_{h(3.6)}. \tag{5.7}$$

We say that the scheme (5.2), (5.1) for $n = n_*$ is consistent with respect to both the accuracy of the limit (for $n = \infty$) solution $z_\infty(x)$ (i.e., a solution of the base scheme) and the number of iterations n (or, shortly, consistent), if the following estimate is fulfilled:

$$\max_{\overline{D}_h} |z_\infty(x) - z_n(x)| \le M \max_{\overline{D}_h} |u(x) - z(x)|, \quad n \ge n^*,$$

where $z_\infty(x) = z(x)$ is the solution of scheme (3.2) on the mesh $\overline{D}_h$.

For the consistent scheme (5.2), (5.1), (3.6) we have

$$|u(x) - z_n(x)| \le M\left\{N_1^{-2} \min\left[\varepsilon^{-2}, \ln^2 N_1\right] + N_2^{-2}\right\} \le$$

$$\le M\left[N_1^{-2} \ln^2 N_1 + N_2^{-2}\right], \quad x \in \overline{D}_{h(3.6)}, \quad n \ge n^*, \tag{5.8a}$$

moreover, the value $n^* = n^*_{(5.8)}$ satisfies the estimate

$$n^* \le M \ln\left(\min[N_1, N_2]\right) \le M \ln N. \tag{5.8b}$$

Theorem 5.1. *Let the hypotheses of Theorem* 3.1 *be fulfilled. Then the solution of scheme* (5.2), (5.1), (3.6) *for* N_1, N_2, $n \to \infty$ *converges to the solution of the boundary value problem* (2.2), (2.1) ε*-uniformly. For the discrete solutions the estimates* (5.5) *and* (5.8) *are valid.*

6. Decomposition of Scheme (4.2), (4.1)

In this section we give a decomposition scheme based on difference scheme (4.2), (4.1), which is the Richardson scheme of the fourth-order accuracy with respect to x_2.

1. On the subdomains $\overline{D}^{(1)}$, $\overline{D}^{(2)}$ we construct meshes

$$\overline{D}_h^{(k)i} = \overline{D}^{(k)} \cap \overline{D}_h^i, \quad k, i = 1, 2, \tag{6.1}$$

where $\overline{D}_h^i = \overline{D}^i_{h(4.1)}$, $\overline{D}^{(k)} = \overline{D}^{(k)}_{(5.1)}$. We consider that the boundaries of the sets $\overline{D}^{(k)}$ pass through the nodes of the mesh $\overline{D}^0_{h(4.1)}$.

Let the function $z_0^i(x)$, $x \in \overline{D}_h^i$, $i = 1, 2$, be given, and also $z_0^i(x) = \varphi(x)$, $x \in \Gamma_h^i$, $z_0^1(x) = z_0^2(x)$, $x \in \overline{D}_h^0$. We suppose that the functions $z_1^i(x), \ldots, z_{n-1}^i(x)$, $x \in \overline{D}_h^i$ are known. We solve the problems

$$\Lambda z_n^{(1)i}(x) = f(x),\ x \in D_h^{(1)i},\ \ z_n^{(1)i}(x) = z_{n-1}^i(x), \qquad x \in \Gamma_h^{(1)i}; \tag{6.2a}$$

$$\Lambda z_n^{(2)i}(x) = f(x),\ x \in D_h^{(2)i},\ \ z_n^{(2)i}(x) = \begin{cases} z_n^{(1)i}(x), & x \in \Gamma_h^{(2)i} \setminus \Gamma, \\ z_{n-1}^i(x), & x \in \Gamma_h^{(2)i} \cap \Gamma; \quad i = 1, 2. \end{cases}$$

Then we suppose

$$z_n^i(x) = \left\{ \begin{array}{ll} z_n^{(1)i}(x), & x \in \overline{D}_h^{(1)i} \setminus \overline{D}^{(2)}, \\ z_n^{(2)i}(x), & x \in \overline{D}_h^{(2)i} \end{array} \right\}, \quad x \in \overline{D}_h^i, \quad i = 1, 2. \tag{6.2b}$$

The function $z_n^i(x)$, $x \in \overline{D}_h^i$, $i = 1, 2$ is the solution of problem (6.2a), (6.1). This iterative process converges for $n \to \infty$. From the values of the functions $z_n^i(x)$, $x \in \overline{D}_h^i$, $i = 1, 2$ we find the function

$$z_n^0(x) = \gamma z_n^1(x) + (1 - \gamma) z_n^2(x), \quad x \in \overline{D}_h^0, \quad n = 1, 2, \ldots; \tag{6.2c}$$

we call $z_n^0(x)$ the solution of scheme (6.2), (6.1), i.e., decomposition of the Richardson scheme (4.2), (4.1).

2. In virtue of estimate (5.5) we have

$$|z^i(x) - z_n^i(x)| \le M q^n, \quad x \in \overline{D}_h^i, \quad i = 1, 2. \tag{6.3}$$

Taking into account representation (4.3), and also estimates (4.6), (4.7) and (6.3), for the solutions of difference schemes 6.2), (6.1) we obtain the estimate

$$|u(x) - z_n^0(x)| \le M \left\{ N_1^{-3} \min\left[\varepsilon^{-3}, \ln^3 N_1\right] + N_2^{-4} + q^n \right\}, \quad x \in \overline{D}_h^0, \tag{6.4}$$

and also the ε-uniform estimate

$$|u(x) - z_n^0(x)| \leq M \, \left[N_1^{-3} \ln^3 N_1 + N_2^{-4} + q^n\right], \quad x \in \overline{D}_h^0; \tag{6.5}$$

in (6.4), (6.5) $q \leq 1 - m$.

For the consistent scheme (6.2), (6.1) we have

$$|u(x) - z_n^0(x)| \leq M \, \left\{N_1^{-3} \min\left[\varepsilon^{-3}, \ln^3 N_1\right] + N_2^{-4}\right\} \leq$$
$$\leq M \, \left[N_1^{-3} \ln^3 N_1 + N_2^{-4}\right], \quad x \in \overline{D}_h^0, \quad n \geq n^*, \tag{6.6a}$$

where the value of $n^* = n^*_{(6.6)}$ satisfies the estimate similar to (5.8b):

$$n^* \leq M \, \ln\left(\min\left[N_1, \, N_2\right]\right) \leq M \, \ln N. \tag{6.6b}$$

Theorem 6.1. *Let the hypotheses of Theorem* 4.1 *be fulfilled. Then the solution of difference scheme* (6.2), (6.1) *under condition* (5.4) *converges, as* $N_1, N_2, n \to \infty$, *to the solution of boundary value problem* (2.2), (2.1) *ε-uniformly. For the discrete solutions the estimates* (6.5) *and* (6.6) *are valid.*

Remark. By using the given technique of constructing scheme (6.2), (6.1) and also a technique from [16] – [18], we can similarly construct parallel Richardson schemes, that is, decomposition schemes which allow parallel computations.

7. Appendix

Let us give *a priori* estimates of solutions to boundary value problem (2.2), (2.1), which we use in our constructions (see, e.g.,[4]). These estimates are established with regard to interior *a priori* estimates and estimates up to a boundary [19].

We represent the solution of problem (2.2), (2.1) as a sum of functions

$$u(x) = U(x) + V(x), \quad x \in \overline{D}, \tag{7.1}$$

where $U(x)$ and $V(x)$ are the regular and singular parts of the problem solution. For the components from (7.1) the following estimates are fulfilled:

$$\begin{aligned} \left| \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2}} U(x) \right| &\leq M, \\ \left| \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2}} V(x) \right| &\leq M \varepsilon^{-k_1} \exp(-m \varepsilon^{-1} r(x, \Gamma)), \quad x \in \overline{D}, \quad k = k_1 + k_2 \leq K, \end{aligned} \tag{7.2}$$

where $r(x,\mathit{\Gamma})$ is the distance from the point x to the set $\mathit{\Gamma}$, m is an arbitrary number from the interval $(0, m_0)$, $m_0 = \min_{\overline{D}}[a_1^{-1}(x)c(x)]^{1/2}$. The value of K depends on smoothness of the problem data.

The function $u(x)$ satisfies the following estimate

$$\left|\frac{\partial^k}{\partial x_1^{k_1}\partial x_2^{k_2}}u(x)\right| \le M\varepsilon^{-k}, \quad x\in\overline{D}, \quad k\le K. \tag{7.3}$$

Theorem 7.1. *Let the data of boundary value problem* (2.2), (2.1) *satisfy conditions* (2.3), *and let* a_s, c, $f \in C^{K+\alpha}(\overline{D})$, $\varphi \in C^{K+\alpha}(\mathit{\Gamma})$, $s = 1, 2$, $K \ge 0$, $\alpha > 0$. *Then, for the components from representation* (7.1), *the estimates* (7.2) *are valid.*

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