Limitations of Adaptive Refinement Mesh Methods for a Singularly Perturbed Parabolic Equation in a Composed Domain with a Concentrated Source on the Moving Interface Boundary

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Abstract

In a composed domain on an axis $I\!R$ with the moving interface boundary between two subdomains we consider an initial value problem for a singularly perturbed parabolic reaction-diffusion equation in the presence of a concentrated source on the interface boundary. Monotone classical difference schemes for problems from this class converge only when $\varepsilon \gg N^{-1} + N_0^{-1}$, where ε is the perturbation parameter, N and N_0 define the number of mesh points with respect to x (on segments of unit length) and t. Therefore, in the case of such problems with moving interior layers, it is necessary to develop special numerical methods whose errors depend rather weakly on the parameter ε and, in particular, are independent of ε (i.e. ε-uniformly convergent methods).

In this paper we study schemes on adaptive meshes which are locally condensing in a neighbourhood of the set γ^*, that is, the trajectory of the moving source. It turns out that in the class of difference schemes consisting of a standard finite difference operator on rectangular meshes which are (*a priori* or *a posteriori*) locally condensing in x and t there are no schemes which converge ε-uniformly and, in particular, even under the condition $\varepsilon \approx N^{-2} + N_0^{-2}$ if the total number of the mesh points between the cross-sections x_0 and $x_0 + 1$ for any $x_0 \in I\!R$ has order of NN_0. Thus, the adaptive mesh refinement techniques used directly do not allow us to widen essentially the convergence range of classical numerical methods. On the other hand, the use of condensing meshes but in a local coordinate system fitted to the set γ^* makes it possible to construct schemes which converge ε-uniformly for N, $N_0 \to \infty$; such a scheme converges with the rate $\mathcal{O}\left(N^{-1}\ln N + N_0^{-1}\right)$.

1. Introduction

Numerical analysis of heat and mass transfer with fixed concentrated sources in media characterized by small coefficients of heat conductivity/diffusion often bring us to diffraction boundary value problems for singularly perturbed partial differential equations. Here the singular perturbation parameter ε is a coefficient multiplying the highest derivatives of the equations. The solutions of such problems for small values of the parameter ε typically exhibit boundary and transition (interior) layers, moreover, for fixed finite values of ε their derivatives are discontinuous at the points where the concentrated sources act. The singular behaviour of the solutions is complicated in the case of moving concentrated sources. So, the solutions of the reduced (for $\varepsilon = 0$) problems have discontinuities of the first kind on the trajectories of the moving sources.

In this paper we consider an initial value problem on an axis $I\!R$ for a singularly perturbed parabolic reaction-diffusion equation in a composed domain with a moving interface boundary between two subdomains; the concentrated source acts on the interface boundary. Note that the solution of such a problem, in contrast to boundary value problems, has no boundary-layer singularities. However, singularities generated by the moving concentrated source still occur that give rise to difficulties in the numerical solution (see, e.g., Theorem 1). Namely, classical finite difference schemes for this problem converge only when $\varepsilon \gg N^{-1} + N_0^{-1}$, where N and N_0 define the number of nodes in the grids with respect to x (on segments of unit length) and t.

Therefore, in the case of problems with moving transition layers it it necessary to develop special numerical methods whose errors depend rather weakly on the parameter ε and, in particular, are independent of ε (i.e. ε-uniformly convergent methods). To this end, it seems expedient to apply the techniques based on locally condensing meshes that have been earlier proposed (see, for example, [1–4] and the bibliography therein for several singularly perturbed problems with stationary boundary or transition layers; for the case of a regular boundary value problem with singularities in its solution see also [5]). So, in the case of regular problems whose solutions have singularities, the effect of improving the accuracy of a numerical solution can be achieved by *a priori* or *a posteriori* local grid refinement in those subregions where the errors in the approximate solution are large (see, e.g., [6, 7, 5]). However, the direct use of this approach in the case of singularly perturbed problems with moving concentrated sources is not sufficiently effective.

We study the class of difference schemes consisting of a standard finite difference operator on adaptive meshes which are locally refined in a neighbourhood of the set γ^*, that is, the trajectory of the moving source. At first sight, such adaptive grid refinement can resolve the layer phenomena numerically in a completely satisfactory manner. Nevertheless, it turns out that in the case of rectangular meshes which are (*a priori* or *a posteriori*) locally condensing in x and t there are no schemes of the above class which converge ε-uniformly and, in particular, even for $\varepsilon^{1/2} \approx N^{-1} + N_0^{-1}$ (see, e.g., the conclusion of Theorem 2) if the total number of the mesh points between the cross-sections x_0 and $x_0 + 1$ for any $x_0 \in I\!R$ has order of NN_0. Thus, the adaptive mesh techniques used directly do not allow us to widen essentially the convergence range of classical numerical methods. On the other hand, the use of condensing (in the nearest vicinity of the singularity) meshes in a local coordinate system fitted to the set γ^* makes it possible to construct schemes that converge ε-uniformly for $N, N_0 \to \infty$ (see, e.g., Remark 3 in Section 4).

2. Problem formulation. The objective of research

1. In an infinite one-dimensional composed domain with the moving interface boundary between its subdomains we consider an initial value problem for a singularly perturbed parabolic equation in the presence of a concentrated source acting on the interface boundary.

Let the domain $\overline{G}$ with boundary $S = \overline{G} \setminus G$, where $G = R \times (0, T]$, be decomposed into non-overlapping subdomains

$$\overline{G} = \overline{G}^1 \bigcup \overline{G}^2, \quad G^1 \bigcap G^2 = \emptyset, \tag{2.1}$$

in each of which we consider the equation

$$L\,u(x,t) \equiv \left\{ \varepsilon a(x,t)\frac{\partial^2}{\partial x^2} - c(x,t) - p(x,t)\frac{\partial}{\partial t} \right\} u(x,t) = f(x,t),$$

$$(x,t) \in G, \quad k = 1, 2, \tag{2.2a}$$

where $a(x,t) = a^k(x,t),\ \ldots,\ f(x,t) = f^k(x,t),\quad (x,t) \in \overline{G}^{\,k}$,

$$G^1 = \{(x,t):\quad x < \beta(t),\quad t \in (0,T]\,\},$$
$$G^2 = \{(x,t):\quad x > \beta(t),\quad t \in (0,T]\,\}, \tag{2.3}$$

the interface boundary between the subdomains $\gamma^* = \{(x,t)\ :\ x = \beta(t),\ t \in (0,T]\}$ is sufficiently smooth. On the set S the function $u(x,t)$ satisfies the initial condition

$$u(x,t) = \varphi(x), \qquad (x,t) \in S, \tag{2.2b}$$

and on the interface boundary γ^* it obeys the conjugation condition

$$[\,u(x,t)\,] = 0, \quad l\,u(x,t) \equiv \varepsilon\left[a(x,t)\,\frac{\partial}{\partial x}\,u(x,t)\right] = -q(t), \qquad (x,t) \in \gamma^*. \tag{2.2c}$$

Here ε is a parameter taking arbitrary values from the half-interval (0,1]; $a^k(x,t)$, $c^k(x,t)$, $p^k(x,t)$, $f^k(x,t)$, $(x,t) \in \overline{G}^{\,k}$, $k = 1,2$, $\varphi(x)$, $x \in I\!R$, $\beta(t)$ and $q(t)$, $t \in [0,T]$ are sufficiently smooth functions, and also [1]

$$\begin{aligned}
&a_0 \le a^k(x,t) \le a^0, \quad 0 \le c^k(x,t) \le c_0, \quad p_0 \le p^k(x,t) \le p^0, \quad (x,t) \in \overline{G}^{\,k},\\
&v_0 \le (\,d/dt\,)\beta(t) \equiv v(t) \le v^0, \quad t \in [0,T], \quad a_0,\, p_0,\, v_0 > 0;\\
&|\,f^k(x,t)\,| \le M, \quad (x,t) \in \overline{G}^{\,k}, \qquad |\,\varphi(x)\,| \le M, \quad x \in I\!R,\\
&|\,q(t)\,| \le M, \quad t \in [0,T]; \qquad k = 1,2;
\end{aligned} \tag{2.4}$$

The function $\beta(t)$ specifies the velocity of motion of the interface boundary, and $q(t)$ defines the power of the concentrated source. The symbol $[\,v(x,t)\,]$ denotes the jump of the function $v(x,t)$ when passing through γ^* from the set G^1 into the set G^2, for example,

$$[\,v(x^*,t)\,] \;=\; \lim_{x\to x^*+0} v(x,t) - \lim_{x\to x^*-0} v(x,t), \qquad (x^*,t) \in \gamma^*.$$

For simplicity, we assume that the compatibility conditions are fulfilled at the point $\gamma^0 = (\beta(0),0)$ to ensure sufficient smoothness of the solution of problem (2.2) on each of the subsets $\overline{G}^{\,k}$ (for fixed values of the parameter ε); suppose $S^k = \overline{G}^{\,k} \setminus G^k$, $k = 1,2$.

As $\varepsilon \to 0$, in a neighbourhood of the set γ^* (on the right from it) there appears a transition layer decreasing exponentially when the point (x,t) recedes away from γ^* to the right. The solution of the reduced problem is a function being sufficiently smooth outside the set γ^* and having a discontinuity of the first kind at γ^*.

2. The errors in the solutions of finite difference schemes based on classical difference approximations to problem (2.2), (2.1) depend on the parameter ε and become small only for those values of ε that essentially exceed the "effective" mesh widths with respect to x and t. So, by virtue of estimate (3.7), the classical difference scheme (3.4), (3.6) (see Section 3) converges under the condition

$$\varepsilon \gg N^{-1} + N_0^{-1} \tag{2.5}$$

where the values $N+1$ and N_0+1 is the number of mesh points with respect to x (on a unit interval) and t respectively. If this condition is violated, the solutions of the difference scheme do not converge to the solution of problem (2.2), (2.1).

[1] Here and below M, M_i (or m) denote sufficiently large (small) positive constants which do not depend on ε and on the discretization parameters. Throughout the paper, the notation $L_{(j.k)}$ ($M_{(j.k)}$, $G_{h(j.k)}$) means that these operators (constants, grids) are introduced in equation $(j.k)$.

By this argument, we are interested in constructing special difference schemes whose errors do not depend on the value of the parameter ε. In particular, it is of interest to develop such schemes that converge under a weaker condition than condition (2.5).

For the initial value problem (2.2), (2.1), by using the condensing mesh method we are thus to construct ε-uniformly convergent schemes and also nearly such schemes, namely, schemes convergent for the values of ε much less than in (2.5), which is our purpose in this paper.

3. Classical difference schemes

Let us give a classical difference scheme for problem (2.2), (2.1) and show some difficulties arising in the numerical solution of the problem for small values of the parameter ε.

1. We consider a difference scheme based on "direct" approximation of the conjugation condition (2.2c). For this we need meshes which contain nodes on the set γ^* at each time level $t = t^j$ of the difference scheme. Let us construct such meshes.

On the domain $\overline{G}$, we introduce rectangular base meshes, on the basis of which we will construct the required grid sets. Let

$$\overline{G}_h = \omega_1 \times \overline{\omega}_0, \tag{3.1}$$

where ω_1 and $\overline{\omega}_0$ are grids on the axis x and the segment $[0,T]$ respectively; ω_1 and $\overline{\omega}_0$ are grids with any distribution of the nodes satisfying only the condition $h \le MN^{-1}$, $h_t \le MN_0^{-1}$, where $h = \max_i h^i$, $h^i = x^{i+1} - x^i$, $x^i, x^{i+1} \in \omega_1$, $h_t = \max_j h_t^j$, $h_t^j = t^{j+1} - t^j$, $t^j, t^{j+1} \in \overline{\omega}_0$. Here $N+1$ and N_0+1 are the maximal number of nodes on a segment of unit length on the axis x and the number of nodes in the grid $\overline{\omega}_0$ respectively. It is of great interest to consider also difference schemes on the simplest meshes, which are uniform with respect to both x and t:

$$\overline{G}_h = \overline{G}_{h(3.1)}, \tag{3.2}$$

where ω_1 and $\overline{\omega}_0$ are uniform grids with step-sizes $h = N^{-1}$ and $h_t = TN_0^{-1}$.

On the set $\overline{G}$ we construct the mesh $\overline{G}_h^* = \overline{G}_h^*(\overline{G}_{h(3.1)})$ generated by the base mesh $\overline{G}_{h(3.1)}$. On the time level $t = t^n \in \overline{\omega}_0$ we introduce the grid set

$$\overline{G}_h^{*n} = G_h^{*n} \cup S_{0h}^{*n}, \quad G_h^{*n} = G_h^{(*)n} \cup \gamma^{*n}$$

. Here $\gamma_h^{*n} = \{x = \beta(t^n), t^n\}$; the set $G_h^{(*)n}$ is formed by those nodes $(x^i, t^n) \in \overline{G}_{h(3.1)}$, $(x^i, t^n) \notin \gamma^{*n}$, for which the segments $x^i \times [t^{n-1}, t^n]$ entirely belongs to either $\overline{G}^1$ or $\overline{G}^2$; the set S_{0h}^{*n} consists of the nodes $(x^i, t^{n-1}) \in \overline{G}_{h(3.1)}$ for which $(x^i, t^n) \in \overline{G}_h^{(*)n}$. We define the mesh $\overline{G}_h^*$ by

$$\overline{G}_h^* = \bigcup_{n=1}^{N_0} \overline{G}_h^{*n}. \tag{3.3}$$

We approximate problem (2.2), (2.1) by the implicit difference scheme [8]

$$\Lambda\, z(x,t) \equiv \left\{\varepsilon a(x,t)\delta_{\overline{x}\widehat{x}} - c(x,t) - p(x,t)\delta_{\overline{t}}\right\} z(x,t) = f(x,t), \quad (x,t) \in G_h^{(*)n}, \tag{3.4a}$$

$$l^h\, z(x,t) \equiv \varepsilon\left\{a^2(x,t)\,\delta_x\, z(x,t) - a^1(x,t)\,\delta_{\overline{x}}\, z(x,t)\right\} = -q(t), \quad (x,t) \in \gamma_h^{*n}, \tag{3.4b}$$

$$z(x,t) = \begin{cases} \overline{z}^{n-1}(x,t), & t^{n-1} > 0, \\ \varphi(x), & t^{n-1} = 0, \quad (x,t) \in S_{0h}^{*n}; \end{cases} \tag{3.4c}$$

$$(x,t) \in G_h^{*n}, \quad n = 1, \ldots, N_0. \tag{3.4d}$$

Here $z^n(x,t)=z(x,t)$ for $(x,t)\in G_h^{*n}$; $\overline{z}^n(x,t)$, $x\in R$, $t=t^n\in\overline{\omega}_0$ is the linear, in x, interpolant constructed from the values of $z^n(x,t)$, $(x,t)\in G_h^{*n}$; $\delta_{\overline{x}\widehat{x}}\,z(x,t)$, $\delta_x\,z(x,t)$, $\delta_{\overline{x}}\,z(x,t)$, $\delta_{\overline{t}}\,z(x,t)$ are the second and first difference derivatives; $\delta_{\overline{x}\widehat{x}}\,z(x,t)\;=\;2\,(h^i+h^{i-1})^{-1}\{\,\delta_x-\delta_{\overline{x}}\,\}\,z(x,t)$, $x\;=\;x^i$, h^{i-1} and h^i are the left and right "arms" of the three-point stencil on G_h^* (for the operator $\delta_{\overline{x}\widehat{x}}$) with center at the node $(x^i,t^j)\in G_h^{(*)}$. The function

$$z(x,t)=\begin{cases} z^n(x,t), & (x,t)\in G_h^{*n},\\ \overline{z}^{n-1}(x,t), & (x,t)\in S_{0h}^{*n};\quad (x,t)\in\overline{G}_h^{*n},\quad n=1,...,N_0;\quad (x,t)\in\overline{G}_h^*\end{cases}$$

will be called the solution of scheme (3.4), (3.3).

For the difference scheme (3.4), (3.3) the maximum principle is valid [8].

By using the majorant function technique we find the estimate

$$|\,z(x,t)\,|\;\le\;M\left[\,1+\varepsilon^{-1}\,\right],\qquad (x,t)\in\overline{G}_h^*. \tag{3.5}$$

In the case of the mesh

$$\overline{G}_h^*=\overline{G}_h^*(\overline{G}_{h(3.2)}) \tag{3.6}$$

the solution of problem (3.4) is bounded under the (unimprovable) condition $N^{-1},\;N_0^{-1}\;=\;\mathcal{O}\,(\varepsilon)$. Under this condition we obtain the (unimprovable) estimate

$$|\,u(x,t)-z(x,t)\,|\;\le\;M\varepsilon^{-1}\left[\,N^{-1}+N_0^{-1}\,\right],\qquad (x,t)\in\overline{G}^*_{h(3.6)}; \tag{3.7}$$

thus, scheme (3.4), (3.6) converges under the (unimprovable) condition

$$N^{-1},\;N_0^{-1}\;=\;o(\varepsilon). \tag{3.8}$$

Theorem 1. *Let* $a,c,p,f\;\in\;C^{4+\alpha}(\overline{G}^k)$, $\varphi\;\in\;C^{4+\alpha}(S_0^k)$, $q\;\in\;C^{2+\alpha/2}([0,T])$, $\beta\;\in\;C^{3+\alpha/2}([0,T])$, *and also* $u\in C^{4+\alpha,\,2+\alpha/2}(\overline{G}^k)$, $\alpha>0$, $k=1,2$, *and let the condition* (2.4) *hold. Then the condition* (3.8) *is necessary and sufficient for the convergence of the difference scheme* (3.4), (3.6) *as* N, $N_0\to\infty$. *For the discrete solutions the estimates* (3.5) *and* (3.7) *are valid.*

4. On the construction of ε-uniformly convergent schemes on locally condensing meshes

Note that the singularity inherent in the initial value problem (2.2), (2.1) does not extend to the set $\overline{G}^1$ and exponentially decreases on $\overline{G}^2$ when the point (x,t) recedes away from the set γ^*. The singular component $W(x,t)$ for $x\ge\beta(t)+\sigma$ does not exceed the value $M\delta$, where δ is a sufficiently small number, when $\sigma=m_1^{-1}\varepsilon\ln\delta^{-1}$, with $0<m_1<m_0$,

$$m_0=\min_{\overline{G}^2}\left[\left(a^2(x,t)\right)^{-1}\,p^2(x,t)(d/dt)\beta(t)\right]$$

. The residual of the difference scheme on the solution of the original problem is large but only in this neighbourhood, which is sufficiently narrow for small values of ε.

1. Bearing in mind the possible use of schemes on sufficiently arbitrary locally condensing meshes for solving the initial value problem, it would be convenient to introduce into consideration *balanced* meshes, that is, meshes with any distribution of their nodes (in x and t) but having the total number of the mesh points of order $\mathcal{O}\,(NN_0)$ in a unit vicinity of the set γ^*, which is the same order as that in the case of uniform meshes with respect to x and t. Thus, the amount of computational work (proportional to the number on the mesh points in which

it is necessary to find the solution of the grid problem) for balanced meshes is of the same order just as for uniform meshes. Balanced meshes are not in general the tensor product of one-dimensional meshes with respect to x and t.

2. We consider a class of difference schemes composed of classical approximations of the initial value problem (2.2), (2.1) and *"piecewise uniform"* locally condensing meshes, i.e., meshes which are uniform both in the nearest neighbourhood of the curve γ^* and outside its somewhat greater neighbourhood.

2.1. For simplicity, assume that $\beta(t) = t$. Let the following mesh have been constructed in some way:

$$\overline{G}_h^* = \overline{G}_h^*(\rho_1), \tag{4.1}$$

where $\rho_1 > 0$ is a parameter chosen below, which defines the distribution of the mesh points. This mesh is uniform on each of the sets $G_1^2 = G_1^2(\rho_1)$ and $G_2^2 = G^2 \setminus \overline{G}_1^2(M\rho_1)$, where

$$G_1^2(\rho_1) = \{(x,t):\ x \in (\beta(t), \beta(t)+\rho_1),\ t \in (0,T]\}$$

is the right ρ_1–neighbourhood of the set γ^*. The meshes $G_{ih}^2 = G_i^2 \bigcap \overline{G}_{h(4.1)}^*$ have step-sizes h_i and h_{it} in x and t respectively, $i = 1, 2$. We consider for simplicity that the stencils of four-point implicit schemes having, as a center, the nodes from G_{ih}^2 are regular, i.e., their left, right and "lower" arms equal h_i and h_{it} respectively.

Let us consider fragments of the grid problem from the class of difference schemes on the meshes (4.1), namely, the fragments on the sets $\overline{G}_{1h}^2$ and $\overline{G}_{2h}^2$. Let $z_i^2(x,t),\ (x,t) \in \overline{G}_{ih}^2$ be the solution of the grid problem

$$\begin{aligned} \Lambda_{(3.4)}\, z_i^2(x,t) &= f(x,t), \quad && (x,t) \in G_{ih}^2, \\ z_i^2(x,t) &= u(x,t), \quad && (x,t) \in S_{ih}^2, \qquad i = 1,2, \end{aligned} \tag{4.2}$$

where $u(x,t),\ (x,t) \in \overline{G}$ is the solution of problem (2.2), (2.1).

For the functions $z_i^2(x,t),\ (x,t) \in \overline{G}_{ih}^2$ we have the estimates

$$|u(x,t) - z_1^2(x,t)| \le M\left[(\varepsilon+h_1)^{-2}h_1^2 + (\varepsilon+h_{1t})^{-1}h_{1t}\right],\quad (x,t)\in\overline{G}_{1h}^2; \tag{4.3a}$$

$$|u(x,t) - z_2^2(x,t)| \le M\left\{\left[(\varepsilon+h_2)^{-2}h_2^2 + (\varepsilon+h_{2t})^{-1}h_{2t}\right] \times \right.$$
$$\left. \times \max_{S_{2h}^2} |W(x,t)| + h_2^2 + h_{2t}\right\},\quad (x,t)\in\overline{G}_{2h}^2, \tag{4.3b}$$

where $W(x,t)$ is the singular component of the solution $u(x,t)$; estimates (4.3a) and (4.3b) are unimprovable with respect to the entering values of h_1, h_{1t}, ε and h_2, h_{2t}, ε respectively.

In order that the function $z_2^2(x,t)$ converges ε-uniformly, it is necessary that the value ρ_1 satisfies the condition $\rho_1 \gg \varepsilon$ or

$$\varepsilon = o(\rho_1). \tag{4.4}$$

The following estimate for the function $z_1^2(x,t)$, which is the same with respect to the convergence order as the optimal estimate relatively to h_1, h_{1t} for the fixed, equal to MNN_0, number of nodes of the mesh $\overline{G}_{1h}^2$, can be obtained under the condition $(\varepsilon+h_{1t})^{-1}h_{1t} = (\varepsilon+h_1)^{-2}h_1^2$:

$$|u(x,t) - z_1^2(x,t)| \le M\varepsilon^{-4/3}\rho_1^{2/3}(NN_0)^{-2/3}\left[1+\varepsilon^{-4/3}\rho_1^{2/3}(NN_0)^{-2/3}\right]^{-1},$$
$$(x,t) \in \overline{G}_{1h}^2; \tag{4.5}$$

this estimate is unimprovable with respect to the entering values of ρ_1, $(N\,N_0)^{-1}$, ε. It follows from estimate (4.5) under condition (4.4) that the function $z_1^2(x,t)$ does not converge ε-uniformly for N, $N_0 \to \infty$.

Thus, the error analysis demonstrates that there are no piecewise uniform meshes $\overline{G}^*_{h(4.1)}$ on which the solutions of problems (4.2) for $i = 1, 2$ converge ε-uniformly to the solution of problem (2.2), (2.1). In the case of the auxiliary problems (4.2) on meshes (4.1), similarly to the above considerations we make sure of the fact that there exist no meshes on which the solutions of these problems converge even under the condition

$$N^{-1} + N_0^{-1} \geq \varepsilon^{1/2}. \tag{4.6}$$

2.2. From here it follows that this non-existence result remains valid also in the case of classical difference approximations of the problem and the family of meshes (4.1) as well as the family of meshes

$$\overline{G}^*_h \tag{4.7}$$

which are uniform in the right ρ-neighbourhood of the set γ^*, where $m\,\varepsilon \leq \rho \leq M\,\varepsilon$.

Theorem 2. *For the initial value problem* (2.2), (2.1), *in the class of balanced difference schemes composed of standard finite difference operators on locally condensing grids* (4.7) *there are no schemes convergent under condition* (4.6).

Remark 1. If we use the grid equations (3.4b) in order to approximate the conjugation conditions (2.2c), in the case of meshes (4.1) and (4.7) there exist no balanced schemes convergent under the condition $N^{-1} + N_0^{-1} \geq \varepsilon^{2/3}$.

Remark 2. It follows from the given considerations that the use of locally condensing meshes for problem (2.2), (2.1) does not allow us to weaken essentially the convergence condition (3.8) for classical difference schemes; it is impossible to reduce the order of the parameter ε in condition (2.5) more than twice on a class of sufficiently common locally condensing meshes, unless the stencil used is non-rectangular.

Remark 3 (Schemes on a stencil fitted to the transition layer). To construct schemes with an improved condition of convergence in comparison with (3.8), one can reformulate the initial value problem (2.2), (2.1) by transforming to variables connected with the moving source, in which the source already become fixed. For the problem in these new variables one can construct a difference scheme on rectangular meshes (in particular, a scheme convergent ε-uniformly) and then return to the old variables. The resulting meshes (i.e. meshes moving in agreement with the source) is no longer rectangular in the original variables. This, generally speaking, implies certain inconveniences for the construction of grid domains and the numerical solution of the problem under consideration. However, it is possible to construct a similar scheme only in a sufficiently small neighbourhood of the source; outside this neighbourhood one can use standard (e.g., uniform) meshes and finite difference operators. This approach leads to a scheme which converges ε-uniformly with the rate $\mathcal{O}\left(N^{-1}\ln N + N_0^{-1}\right)$.

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