The exponential growth in the number of active mathematicians in the present era is sometimes illustrated by the remark that there are as many mathematicians alive today as have lived - and died - since classical times. A less picturesque but more interesting indicator of mathematical activity is the rapidity with which well known conjectures and problems, sometimes of long standing, are being resolved. A recent article in the Newsletter (No. 11) by David Lewis on the Merkuryev-Suslin Theorem illustrates this point, and the present article (also expository, also concerned with Russian work) provides another example.

INTRODUCTION

Many readers will be familiar with, or at least aware of, the Burnside Problem in group theory, namely: must a group be finite if it is finitely generated and has exponent k? Having exponent k means that the group elements all satisfy the law $x^k = 1$ and some element has period precisely k. The problem was stated in 1902 [1], and answered negatively in 1968; an outline of developments and a bibliography, may be found in [4] and [3]. The story is by no means complete, and many problems remain open concerning these groups, but one problem concerning solvability has been settled completely by the work of Ju. P. Razmyslov in Moscow.

Let $B_k$ denote the Burnside Variety of all groups satisfying the law $x^k = 1$; let $B_k^n$ represent the free group of rank n in $B_k$ (then the n-generator groups of exponent k are just the quotient-groups of $B_k^n$). It has been known for many years (> 25) that:
\( B_{2,n} \) is finite and abelian
\( B_{3,n} \) is finite and metabelian
\( B_{6,n} \) is finite and solvable, of derived length 3
\( B_{4,n} \) is finite.

Of course \( B_{4,n} \) is a finite 2-group, and therefore is solvable—
but what is its derived length? What Razmyslov [3] calls
the Problem of Hall and Higgins, under attack since the 1950s,
could be put thus: Is the derived length of \( B_{4,n} \) independent
of \( n \)?

If this were so, then \( B_{4} \) would join the varieties \( B_{2}, B_{3}, \)
and \( B_{6} \) in being known to be "solvable" in the sense that all
groups in these three varieties are solvable, with bounded
derived lengths.

A great deal of work on \( B_{4,n} \) culminated in the proof by
Razmyslov that \( B_{4} \) is not solvable— and this, due to previous
work of Gupta and Newman, determined the precise nilpotency
class of \( B_{4,n} \), which in turn enabled Vaughan-Lee to decide the
precise derived length of \( B_{4,n} \).

There is, however, much more: Bachmuth and Mochizuki a
little earlier had shown that \( B_{5} \) is not solvable, but Razmy-
slov has constructed non-solvable groups of exponent \( p \) for
all primes \( p > 3 \) and also of exponent 9. A consequence of
all this is the following result which we might call

**Razmyslov’s Theorem:** The Burnside Variety \( B_{k} \) is solvable only
when \( k = 2, 3 \) or \( 6 \).

This is a satisfyingly complete result, although certainly
unexpected. The work has been announced and has appeared in
Russian sources during the past decade; some of the details
have only recently appeared in an English translation by J.
 Wiegold [3]. As an introduction to the ideas involved we

will explain here a relatively easy way of producing groups
of exponent \( p^{k} \) which are non-solvable when \( p > 2 \). This is
given in [3] as a concession to the readers really, to encou-
rage them to persevere with the far more complex details of
exponent \( p \).

The justification for presenting here what could be read
in [3] is that hopefully our account is less Delphic in style
than the original — and may perhaps achieve the aim of the
original if it encourages study of the entire paper. Fur-
thermore, the use of Lie-ring-theoretic methods has proved
to be a most important tool in certain problems of combinatori-
al group theory; the construction given here is a nice (if
not very deep) illustration of its power.

**FIRST STAGE — A GROUP OF EXPONENT \( p^{k} \)**

Let \( A_{0} \) be an associative algebra with identity 1, over
a field \( K \) and let \( A_{0} \) be generated by an infinite set of non-
commuting elements \( x_{1}, x_{2}, x_{3}, \ldots \). This means that an
element of \( A_{0} \) has the form \( \prod k_{j}d_{j} \) where the sum is finite,
\( k_{1} \in K \) and \( d_{j} \) is a product of generators \( x_{j} \).

In \( A_{0} \) we introduce the relations

\[ x_{1}w x_{1} = 0 \]

for every word \( w \) in \( A_{0} \); note that \( w \) may be the empty word.
We consider the quotient algebra \( A \) say, and we will continue
to use the symbols \( x_{1} \) for the images in \( A \) of the original
generators \( x_{1} \) of \( A_{0} \).

Now \( (1 + x_{1})(1 - x_{1}) = 1 \) in \( A \) so the elements

\[ q_{1} = (1 + x_{1}) \quad \text{and} \quad q_{1}^{-1} = (1 - x_{1}), \quad 1 \leq 1 \]
generate a group \( G \) embedded in \( A \).
We notice now that if \( c(y_1, y_2, \ldots, y_k) \) is any group commutator in elements \( y_1, y_2, \ldots, y_k \) we have
\[
c(g_1, \ldots, g_k) = 1 + c^k(x_1, x_2, \ldots, x_k)
\]
where \( c^k(x_1, x_2, \ldots, x_k) \) is the corresponding Lie commutator in \( A \).

For example, if \( c(g_1, g_2) = g_1^{-1}g_1^{-1}g_2g_2 \) then
\[
c^k(x_1, x_2) = x_1x_2 - x_2x_1.
\]

Notice also that the group \( G \) is locally nilpotent; thus for instance in the subgroup \( G(n) \) say generated by \( g_1, g_2, \ldots, g_n \) if \( c \) is any single commutator in the elements \( g_i \) which is of weight \( n + 1 \), the corresponding \( c^k \) will be a homogeneous polynomial of weight \( n + 1 \) in \( x_1, x_2, \ldots, x_n \) and each monomial term in \( c^k \) will have a repeated \( x_i \) and be \( 0 \) because of the relations in \( A \); that is \( c = 1 \) and so \( G(n) \) has nilpotency class \( n \). Of course any finitely generated subgroup of \( G \) lies in \( G(n) \) for a suitable \( n \).

If we now stipulate that \( K \) be a field of characteristic \( p \) we have \( g_i^p = 1 \) for every \( i \), and an element of \( G \) may be written as a product of positive powers of the generators \( g_i \).

These generators all have period \( p \), and indeed \( G \) now has the property that every element must have period a power of \( p \) - but we wish to do more than that, we want every element to satisfy the law \( g_i^{pk} = 1 \); in characteristic \( p \) this means that \( (g - 1)^{pk} = 0 \). We note that if \( g \in G \) then
\[
g = (1 + x_{i_1})(1 + x_{i_2}) \ldots (1 + x_{i_t})
\]
and we begin by considering
\[
g = (1 + x_1)(1 + x_2) \ldots (1 + x_t), \text{ where } t \geq 1.
\]

Let \( D = \{(1 + x_1)(1 + x_2) \ldots (1 + x_t) - 1\}^{pk} \) and let \( \delta(x_1, x_2, \ldots, x_t) \) be the homogeneous component of maximum weight in the expansion of \( D \); this weight must be \( t \), since terms of higher weight are killed because of repetitions. Of course \( D = 0 \) when \( t < p^k \).

We note that the homogeneous component of weight \( t - 1 \) in \( D \) must be the sum of \( t \) separate components, namely:
\[
\delta(x_1, x_2, \ldots, x_{t-1}) + \delta(x_1, x_2, \ldots, x_{t-2}, x_t) + \ldots +
\delta(x_2, x_3, \ldots, x_t).
\]

Similarly for the homogeneous components of lower weight in \( D \).

Thus finally if we let \( J \) be the ideal of \( A \) generated by \( \delta(x_{i_1}, x_{i_2}, \ldots, x_{i_s}) \) for all \( s \geq 1 \) and for all possible choices of \( i_1, i_2, \ldots, i_s \) we have an ideal which must contain \( (g - 1)^{pk} \) for all \( g \in G \).

Now if we take the quotient algebra \( A_2 = A/J \) the image of \( G \) in \( A_2 \) is a (locally nilpotent and finite) group of exponent \( p^k \).

We might remark by the way that the approach so far is not novel, and similar ideas were used in some earlier papers on groups with exponent \( a \).

However, we will see that in the particular algebra \( A \) which we are going to produce below, this last step is unnecessary; in other words \( J \) will be \( (0) \) already in \( A \) and so \( G \) will automatically have exponent \( p^k \).

**AN IDENTITY**

We digress now to consider an identity which holds in any associative algebra \( B \) of dimension \( s \) over a field of prime characteristic \( p \).
Consider the symmetric function

$$S_t(y_1, y_2, \ldots, y_t) = \sum \frac{1}{t!} Y_{010} y_2 \cdots y_t$$

where \(Y_{ij} \in B(1 \leq i \leq t)\) and \(o\) runs over all permutations of the set \(\{1, 2, \ldots, t\}\).

\(S_t\) is multilinear in all variables \(y_i\), so if \(b_1, b_2, \ldots, b_s\) is a basis for \(B\) and

$$y_i = \sum_{j=1}^{s} a_{ij} b_j$$

we get

$$S_t(y_1, y_2, \ldots, y_t) = \sum_{j_1=1}^{s} \sum_{j_2=1}^{s} \cdots \sum_{j_t=1}^{s} a_{1j_1} a_{2j_2} \cdots a_{tj_t} S_t(b_{j_1}, b_{j_2}, \ldots, b_{j_t})$$

Suppose now that we take \(t = s(p - 1) + 1\). Then in any \(S_t(b_{j_1}, b_{j_2}, \ldots, b_{j_t})\) some basis element must occur at least \(p\) times in the entries \(b_{j_i}\). Suppose, for example, that \(b_{j_1}\) occurs \((p + a)\) times. Then \(S_t(b_{j_1}, \ldots, b_{j_t})\) breaks into a sum of blocks each consisting of \((a + p)!\) identical products: since \((a + p)! \equiv 0 \mod p\) this means that every \(S_t(b_{j_1}, \ldots, b_{j_t}) = 0\) and so we see that

$$S_t(y_1, y_2, \ldots, y_t) = 0$$

is an identity in \(B\).

(We remark that \(S_t = 0\) implies \(S_{t+m} = 0\), all \(m \geq 0\)).

We wish to use this result where \(B\) is the algebra \(M\) of all \(2 \times 2\) matrices over an infinite field \(K\) of characteristic \(p\): then for \(t = 4(p - 1) + 1\) we have \(S_t(y_1, y_2, \ldots, y_t) = 0\) in \(M\).

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**FINAL STAGE**

Let us now return to the construction of a non-solvable group. For the algebra \(A_0\) we choose the free algebra, on free generators \(x_1, x_2, x_3, \ldots\), in the variety of algebras generated by the matrix algebra \(M\) referred to above. \(A_0\) comes furnished with characteristic \(p\); we construct the quotient algebra \(A\) containing the group \(G\) as before, \(A = A_0/R\) where \(R\) is the ideal in \(A_0\) generated by all expressions \(x_1 w x_1\), \(w\) being any (possibly empty) word in \(A_0\).

Now in the group \(G\) (generated by all \(q_{ij} = 1 + x_1, 1 \leq i \in A\)) let \((u, v)\) denote the group commutator \(u^{-1} v^{-1} u v\). Let \(\delta_1 = (q_{11}, q_{22}), \delta_2 = ((q_{11}, q_{22}), (q_{44}, q_{44})), \delta_3 = (\delta_2, ((q_{55}, q_{66}), (q_{77}, q_{88})))\) and so on; then \(\delta_0\) involves \(2^k\) generators and lies in the \(k\)th derived subgroup of \(G\). For every \(k \geq 1\) there is a corresponding \(\delta_k^z\) where \(\delta_k^z = 1 + \delta_k^z\); clearly \(\delta_k^z\) is a homogeneous polynomial in \(x_1, \ldots, x_{2k}\) of degree \(2^k\) where no term has a repeated factor \(x_1\). There is a preimage of \(\delta_k^z\) in \(A_0\) having exactly the same form, call it \(\delta_k^x\); the \(x_1\) which appear in \(\delta_k^x\) are the free generators of \(A_0\).

Since \(M\) contains the Lie algebra \(sl(2, K)\) which is simple when \(p > 2\) there is a Lie commutator \(\gamma(a_1, \ldots, a_{2^k}) \neq 0\) in \(M\), where \(\gamma\) has the same form as \(\delta_k^z\). For every \(k\); the mapping \(x_j \mapsto a_j (1 \leq j \leq 2^k)\) induces a homomorphism of \(A_0\) into \(M\) (we might map all other \(x_j\) onto \(0\)) which shows that \(\delta_k^x \neq 0\) for any \(k\). The form of \(\delta_k^x\) now shows that it does not lie in the ideal \(R\) in \(A_0\) so the image \(\delta_k^z\) is not 0 in \(A\). Thus finally \(\delta_k^z \neq 1\) in \(G\) for any \(k\) and so \(G\) is non-solvable.

Notice here that we need \(p \geq 3\); also that we have yet to show (as we promised) that \(G\) has exponent \(p^2\) (where we are now fixing \(k = 2\)).

What makes this work is the observation that the expression \(\delta(x_1, x_2, \ldots, x_1)\) is a sum of terms \(S_m(u_1, u_2, \ldots, u_m)\) where \(m = p^k\) and the \(u_i\) are certain monomials in the elements

- 62 -
when \( t \geq p^k \); when \( t < p^k \) we have \( \Delta(x_1, \ldots, x_t) = 0 \).

This is easy to see - an example will suffice - if, for example, \( p^4 = 3 \) and \( t = 5 \) we would have

\[
\Delta(x_1, x_2, x_3, x_4, x_5) = \sum_{i<j<k} S_{3}(x_i^1, x_j^1, x_k^1, x_i^2, x_j^2, x_k^2)
\]

where \( \{1, i, j, k, r, s\} = \{1, 2, 3, 4, 5\} \).

Now since \( A \) above is in the variety generated by \( M \) we have the identity \( S_{r}(y_1, \ldots, y_r) = 0 \) in \( A \) whenever \( t \leq 4(p-1) + 1 \).

But \( p^2 - (4p-3) - (p-2)^2 = 1 \geq 0 \) if \( p \geq 3 \). This means that already in \( A \) the relation \((g-1)g^2 = 0\) is satisfied for all \( g \) in \( G \).

Thus finally we have arrived at a non-solvable group \( G \) of exponent \( p^2 \) which is also locally a finite \( p \)-group.

FOCAL SCUIR

In consonance with the didactic tendency of this journal, we end with an exercise for the reader: taking 3x3 matrices and applying techniques similar to those used above, construct a non-solvable group of exponent \( 8 \).

REFERENCES

1. **BURNSIDE, W.**

2. **NEWMAN, M.F.**

3. **RAZMYSLOV, Yu. P.**

4. **TIBBIN, S.J.**