SUBDIVISION OF SIMPLEXES - IS BISECTION BEST?

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Problem 1

A closed and bounded interval in \( \mathbb{R} \) is to be subdivided into 2 intervals by insertion of a single point. These 2 intervals in turn are to be subdivided into 4 intervals by insertion of a point in each. Continue this process. Let \( d_n \) be the length of the longest interval at the \( n \)th stage. How should the insertion points be chosen so as to minimize \( d_n \)?

This is not very difficult! Obviously points should be inserted at the midpoints of intervals, i.e., the optimal policy is to bisect intervals at each stage.

Why is Problem 1 of interest? (No doubt for many readers this is a much harder question than Problem 1 itself.) Well, if we wish to solve \( f(x) = 0 \), \( f: [a,b] \to \mathbb{R} \) with \( f(a)f(b) < 0 \), and we want after some fixed number of function evaluations to find an interval of minimum length that is guaranteed to contain a root of \( f \), then Problem 1 shows that the classical bisection method is best. (For any other algorithm there is some function \( f \) for which the interval found is longer.) In this article we shall examine the relevance of bisection to an \( n \)-dimensional generalization of Problem 1 which we'll call Problem \( n \). This problem has as yet no complete solution. It arises in the comparison of methods used to solve \( f(x) = 0 \) for \( f: \mathbb{R}^n \to \mathbb{R}^n \), but we shall discuss it purely from a geometric viewpoint.

Generalizing Problem 1 to \( \mathbb{R}^n \)

We first replace closed and bounded intervals by \( n \)-simplexes (triangles when \( n=2 \), tetrahedra when \( n=3 \)). The reason that we generalize intervals to triangles and not rectangles is that if \( g: \mathbb{R}^2 \to \mathbb{R} \), then on each triangle in \( \mathbb{R}^2 \) there is a unique affine function which interpolates \( g \) at the vertices.

Definitions: For \( n \geq 1 \), an \( n \)-simplex \( S^n = (a_0a_1...a_n) \) is the closed convex hull of \( n+1 \) points \( a_0, a_1, ..., a_n \) in \( \mathbb{R}^q \), \( q \geq n \), such that the vectors \( a_1-a_0, a_2-a_0, ..., a_n-a_0 \) are linearly independent. The points \( a_0, a_1, ..., a_n \) are called the vertices of \( S^n \). Any \( m \)-simplex \((1 \leq m \leq n)\) formed by taking the closed convex hull of any \( m+1 \) vertices of \( S^n \) is called a face of \( S^n \). The one-dimensional faces \((a_1a_j)\), \( 0 \leq i < j \leq n \), are called the edges of \( S^n \). The diameter of \( S^n \), \( d(S^n) \), is the length of the longest edge of \( S^n \) in the Euclidean norm.

We subdivide any \( n \)-simplex \( S^n = (a_0a_1...a_n) \) as follows. Choose a point \( y \in S^n \). Form all \( n \)-simplexes \((a_0a_1...a_{i-1}y a_{i+1}...a_n)\). Note that if \( m \) is the minimum dimension of a face of \( S^n \) containing \( y \), then the subdivision yields \( m+1 \) \( n \)-simplexes.

Problem \( n \)

When an \( n \)-simplex \( S^n \) subdivide it as just described. This is the first stage. Similarly subdivide the resulting \( n \)-simplexes by inserting a point in each. This is the second stage. Continue thus. Let \( A_k \) denote the set of all \( n \)-simplexes \( T^n \) obtained at the \( k \)th stage. Define

\[
d_k = \max \{ d(T^n) | T^n \in A_k \}
\]

We find an algorithm for inserting points which will yield

\[
d_{kn} \leq C \varepsilon^k \quad \text{for} \quad k = 1, 2, 3, ...
\]

where \( \varepsilon > 0 \) (independent of \( S^n \)) is as small as possible and \( C \) depends on \( S^n \) only. (We consider \( d_{kn} \) rather than \( d_k \) as experience shows it's a more natural measure).
Example: For the $n=1$ case with $S^1 = [a, b]$ the bisection method yields an equality:

$$d_k = (b-a)(1/2)^k, \quad k = 1, 2, 3, \ldots$$

Only partial results have been obtained for Problem $n$. The principal reference is [7]. There it is shown (essentially) that for any $n$ and any algorithm one must have $r \geq \frac{1}{2}$, but it is also conjectured that in fact one must have $r \geq \frac{1}{4}$. An algorithm for which $r = \frac{1}{2}$ is exhibited in [7]; it is based on Whitney's simplicial subdivision [8, pp 358-360]. This algorithm may be fairly described as a generalization of the one-dimensional bisection method. Nevertheless a different generalization has become established as the "n-dimensional bisection method" [1,2,3,4,5,8]. We shall concentrate on this latter algorithm as it is simple to describe, it is clearly a generalization of the one-dimensional method, and yet it has not been satisfactorily analyzed up to now.

The n-Dimensional Bisection Method

For $n > 1$, given an $n$-simplex $T^n$ choose any edge $(a_1a_j)$ of $T^n$ whose length is $d(T^n)$. Let $b$ be the midpoint of this edge. Bisect $T^n$ into $(a_1, a_{j+1}, b, a_{j+2}, a_n)$ and $(a_2, a_{j+1}, b, a_{j+1}, a_n)$. That is, $b$ is the point inserted into $T^n$ to subdivide it.

In attacking Problem $n$ this method is intuitively attractive. To decrease the diameter of an $n$-simplex one must divide edges, and the bisection method bisects the longest edge. It's intuitively reasonable that the method will yield $n$-simplexes of diameters shrinking to zero, and this fact was implicitly assumed in [4]; however a proof did not appear until later [1].

To demonstrate the elementary nature of the arguments which can be used in relation to Problem $n$, we shall give a new proof of (a slightly stronger result than) the main theorem of [1].

Lemma. Given a triangle (2-simplex) of diameter $d$, the length of the median obtained by joining the midpoint of the longest edge to the opposite vertex is at most $\sqrt{3} \frac{d}{2}$.

![Figure 1](image-url)
Theorem. Let $S^n$ be an $n$-simplex having exactly $m+1$ vertices as endpoints of edges of length greater than $\sqrt{3d(S^n)}/2$. Then after $m$ iterations of the bisection method the diameter of any resulting $n$-simplex is at most $\sqrt{3d(S^n)/2}$.

Proof. Bisected edges have length at most $d(S^n)/2$. New edges have length at most $\sqrt{3d(S^n)/2}$ by the Lemma. So we need only show that after $m$ iterations any edge of $S^n$ whose length exceeds $\sqrt{3d(S^n)/2}$ has been bisected.

Let $S^n = (a_0, a_1, ..., a_m, a_n)$ where without loss of generality we assume that among all the $a_k$ only $a_0, a_1, ..., a_m$ are endpoints of edges whose lengths exceed $\sqrt{3d(S^n)/2}$. At the first bisection $S^n$ becomes

$$S^n_1 = (a_0, a_1, b, a_m, a_n)$$

and

$$S^n_2 = (a_0, b, a_1, ..., a_m, a_n)$$

where $b$ is the midpoint of $(a_1, a_2)$ and $0 < i < j < m$. Consider $S^n_1$. By the first paragraph of the proof only the vertices $a_0, a_1, ..., a_j, a_{j+1}, ..., a_m$ can be endpoints of edges exceeding $\sqrt{3d(S^n)/2}$ in length. That is, $S^n_1$ has at most $m$ vertices with this property. Similarly for $S^n_2$.

At the next iteration we will obtain 4 $n$-simplexes each having at most $m+1$ vertices which are endpoints of edges exceeding $\sqrt{3d(S^n)/2}$ in length. Repeating this argument $n$ times in all proves the theorem.

Corollary. (Notation as in Problem n). For the $n$-dimensional bisection method we have

$$d_{kn} \leq C(\sqrt{3}/2)^k, \quad k = 1, 2, 3,...$$

Proof. An $n$-simplex has $n+1$ vertices so in the Theorem we have $m = n$ at most. Thus the Theorem implies that

$$d_n \leq d(S^n)/\sqrt{3}/2.$$
This contradicts (*). Hence \( r < \frac{1}{4} \) is impossible.

References


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