IRISH MATHEMATICAL SOCIETY

NEWSLETTER

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IRISH MATHEMATICAL SOCIETY NEWSLETTER NUMBER 7

THE IRISH MATHEMATICAL SOCIETY

The aim of the *Newsletter* is to inform Society members about the activities of the Society and also about items of general mathematical interest.

The *Newsletter* also seeks articles of mathematical interest written in an expository manner. All parts of mathematics are welcome, pure and applied, old and new.

Manuscripts should be typewritten and double-spaced on A4 paper. Authors should send two copies and keep one copy as protection against possible loss. Prepare illustrations carefully on separate sheets of paper in black ink, the original without lettering and a copy with lettering added.

Correspondence relating to the *Newsletter* should be sent to:

Irish Mathematical Society Newsletter
Department of Mathematics,
University College,
Cork.

CONTENTS

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Editorial

At its recent meeting, the Committee of the Irish Mathematical Society decided that the Newsletter will be published three times each year; at the end of the months of March, September and December. The March and December issues will be available at the Mathematical Symposia in the Dublin Institute for Advanced Studies. It is hoped that the regular appearance and good quality printing will encourage members to write articles for the various sections of the Newsletter.

In the next issue there will be a new section on the History of Mathematics. I believe that there is a particular need to record the contributions of Mathematicians who were Irish or who worked in Ireland, so articles detailing these activities will be particularly welcome.

I would like to thank Ms. Una Sheehan for the typing of recent issues of the Newsletter.

D. Hurley

OBITUARY

Dr. Robert Charles Geary

Dr. Roy Geary, who died on February 8th, 1983, at the age of 86, was one of Ireland's most distinguished statisticians. He was the first director of the Economic Research Institute, later the Economic and Social Research Institute. As a civil servant he founded the Central Statistics Office and he also held senior appointments in the United Nations in New York.

For a time he was also visiting lecturer at Cambridge University. He addressed numerous economic and statistical conferences all over the world and published a great number of papers. Dr. Geary was an honorary fellow of the Royal Statistical Society, the American Statistical Association and the Institute of Mathematical Statistics. He was also elected president of the International Statistical Institute and chairman of the International Association for Research in Income and Wealth.

He had been engaged in the collection of Irish official statistics from 1923 and was responsible for inaugurating many new inquiries which have since become standard. When he retired as director of the Central Statistics Office in 1957 he became chief of the national accounts branch of the United Nations Statistical Office.

In 1960 he returned to become the first director of what was to become the Economic and Social Research Institute, and on his retirement as director, the institute honoured him by inaugurating the annual Geary Lecture which has attracted many of the most distinguished social scientists in the world to lecture in Ireland.

He held doctorates from the National University, Dublin University and Queen's University in Belfast. In 1981 the Royal Dublin Society awarded him the prestigious Boyle Medal for Science.

D. Hurley

IRISH MATHEMATICAL SOCIETY

Notice of Ordinary Meeting

An Ordinary Meeting of the Society will be held on Thursday 31st March, 1983, at 12.15 pm. The venue is the Dublin Institute For Advanced Studies, 10 Burlington Road, Dublin 4.

The agenda will include a discussion and vote on the following motion:

> *"That the Irish Mathematical Society abandon the idea of giving a prize for outstanding papers in the Proceedings of the Royal Irish Academy".*

If this motion is not passed, ~~it will~~ be proposed that the prize be awarded in the manner suggested by the Subcommittee. The report of the Subcommittee is given below.

Report on Proposed Prize Based on Papers in the Proceedings of the Royal Irish Academy

At Christmas, 1981, the Society set up a committee to consider ways in which a prize could be awarded. The committee sought information from the London Mathematical Society and other sources. The advice that came back, couched in the strongest terms, was that the amount of trouble and effort is out of proportion to the benefits. An interim report along these lines at Easter, 1982, was rejected by the Society, at an Ordinary Meeting, and the committee was charged to consider procedures. At Christmas, 1982, the committee proposed that, if a prize is to be established, then a system along the following lines be used:

1. That at most one prize be awarded per annum, and then

only to exceptional papers.

2. The prize should be restricted to mathematicians under 35 years of age who are Irish or based in Ireland.

3. A committee should be established to look at papers at the end of each year, and make a recommendation. The committee could arrange to get referees reports and analyses of papers.

At the next Ordinary Meeting, at Easter, the Society will discuss whether or not to proceed.

Anthony G. O'Farrell

Minutes of the Ordinary Meeting Held at 12.15 on December 21st 1982, in DIAS

1. There were 22 members present. Since the President, J.J.H. Miller was absent due to illness, it was agreed that D.J. Simms would chair the meeting. The minutes of the last Ordinary Meeting on April 7th, 1982, were read and signed.

2. The Secretary's and Treasurer's reports, which were reproduced in the Newsletter, were taken as read.

3. A motion proposed by the committee to change the constitution to increase the size of the committee by two members, was passed. (It must be passed again at the next ordinary meeting before it takes effect)

4. The following officers and committee members were nominated and elected without a vote:

President	A.G. O'Farrell	(2 years, to Dec. 1984)
Vice-President	F. Holland	(2 years, 2nd term, to Dec. 1984)
Secretary	R.M. Timoney	(1 year, to Dec. 1983)
Treasurer	G. Enright	(1 year, to Dec. 1983)
Committee	M. Clancy) J. Hannah) P. Boland)	(2 years, to Dec. 1984)

R. Bates, D. Hurley and M. Stynes continue on the committee until December 1983.

5. There was a discussion on the notice of the election given to members. The committee agreed to consider this and the possibility of a postal ballot.

6. T. Laffey reported on behalf of the sub-committee on the awarding of a prize that

(i) On the basis of advice from other mathematical societies the sub-committee was not in favour of having a prize at all.

(ii) If a prize is to be founded it should be for the best paper by an Irish Mathematician or Mathematician based in Ireland in the Proceedings of the Royal Irish Academy.

(iii) The prize should be restricted to Mathematicians under 35 and should only be awarded for an exceptionally good paper. The committee of the IMS should set up a jury to award the prize who would, in turn, solicit referees' reports if necessary.

A motion that

"(a) the report of the sub-committee should be published in the Newsletter, and

(b) the next ordinary meeting of the Irish Mathematical Society would decide whether or not to proceed with the prize".

was passed after a vote. Another motion to go ahead with awarding the prize was defeated.

7. The meeting discussed the cases of Mathematicians J.L. Massera (Uruguay) and Victor Kipnis (USSR). The Secretary reported that he had, on the instructions of the committee, written to the organisers of the campaign for Massera to add the IMS to their list of supporters. This action was endorsed by the meeting. A collection was taken for the Massera campaign. The meeting passed the following motion:

> "The IMS is concerned about J.L. Massera and V. Kipnis and supports the committee in whatever steps it considers necessary such as writing to the appropriate embassies and to relevant organisations".

8. R. Bates mentioned that the IMS might be able to take over the functions of the Irish Mechanics Group.

R. Timoney (Secretary)

Membership

Subscriptions for 1983 are now due from individual members. Fees, which are still just £3.50 per person, should be forwarded to the Treasurer as soon as possible.

Members, who are not already paying by Bankers' Order, might consider using this convenient method. Forms are available from the Treasurer.

Overseas members, who wish to pay in foreign currency, are asked to send the equivalent of IR£4.00 to allow for conversion costs.

All members are asked to encourage colleagues to join the Society in order to support Irish mathematical activities. You might also encourage your Department to become an Institutional Member.

Arrears are still being taken from members who forgot to renew in 1982. Your co-operation in getting fees collected quickly this year will be appreciated.

Remember: your £3.50 includes a year's supply of our new improved Newsletter.

G.M. Enright (Treasurer)

Institutional Members of the Irish Mathematical Society

National Board for Science and Technology

National Institute for Higher Education, Limerick

New University of Ulster, Coleraine

St. Patrick's College, Maynooth

University College, Dublin

PERSONAL ITEMS

Dr. P.D. Bourke of the Statistics Department, U.C.C. has been promoted to Statutory Lecturer.

Dr. R. Harte of the Mathematics Department, U.C.C. has been promoted to Associate Professor.

Dr. P. Hogan of the Mathematical Physics Department, U.C.D. is visiting the Institute of Theoretical Physics at the University of Warsaw, Poland, for two weeks during March.

Dr. D. Lewis of the Mathematics Department, U.C.D. is visiting the Department of Mathematical Sciences at McMaster University, Ontario, Canada, during the month of March.

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SUMMARY OF RESULTS OF 1982 IRISH NATIONAL MATHEMATICS CONTEST

The Fourth Irish National Mathematics Contest was held on Tuesday, March 9, 1982 and attracted 1,539 entries from 77 schools. The total represents a decrease of about 150 contestants from 1981. This is possibly accounted for by the increase in fee and the difficulty of the 1980 and 1981 Contests which may have deterred some schools from participating this year.

The results to hand indicate that the 1982 Contest was more difficult than the 1981 edition. The number of those who scored 80 or more was 26 - and includes 5 girls - as against 45 in 1981; the average mark for the top scorers was 84.9 versus 85.4 in 1981. These statistics are in line with those that were compiled from the results obtained by 418,009 contestants from 6,623 schools in the United States, Canada and American schools abroad: only 221 of these scored 100 marks or more as against 269 in 1981, when 422,231 students competed; of the 221, two actually achieved 145, the highest score recorded.

The highest score achieved so far by an Irish student was returned this year by our winner David Donnelly, who scored 115. David and the next four top scorers were each presented with prizes, for instance, David was given a ZX81 computer, at a reception held in U.C.D. on December 3, 1982.

The top five contestants were

Name	School	Score
David A. Donnelly	St. Michael's College, Omeath, Co. Louth.	115
Barry E. Ambrose	Presentation College, Western Road, Cork.	92

Name	School	Score
Chris J. Stenson	O'Connell Schools, Dublin 1.	89
Peter J. Johnston	Coleraine Academical Institution, Coleraine, Co. Londonderry.	88
Patrick J. Gaffney	Christian Brothers College, St. Patrick's Place, Cork	88

This year the Contest was sponsored by the Bank of Ireland and the National Board of Science and Technology as well as the Irish Mathematical Society and the Irish Mathematics Teachers Association, both of whom have supported the Contest from the start. It is a pleasure to record our thanks to these bodies, without whose help and interest the project would fail.

F. Holland

DUBLIN REPORTS

The National Institute for Higher Education (Dublin)

Introduction

The NIHE (D), after its initial planning stage (begun in 1975), was brought into existence through THE NATIONAL INSTITUTE FOR HIGHER EDUCATION (DUBLIN) ACT, 1980. The institute sees itself as having two main roles:

1. To train well qualified graduates with a breadth of practical knowledge attuned to a rapidly changing technologically-based society.

3. To engage itself in research and associated activities especially as they pertain to Irish industry.

The Academic Organization is structured along faculty lines, at present there being six faculties: Mathematical Sciences; Engineering and Design; Science and Paramedical Studies; Communications and Human Studies; and Education studies. Through the schools in these faculties, the institute offers degree/diploma programmes up to the Masters and Ph.D. level, which are accredited by the NATIONAL COUNCIL FOR EDUCATIONAL AWARDS. Normally a first degree programme is of four years duration and consultation with industry and business is a vital component in course planning. One special feature of the undergraduate training is the INTRA (Industrial Training) element which places the student in industry for a period of six months usually in the third year.

School of Mathematical Sciences

Among third level institutions in Ireland, NIHE (D) is unique in having a complete faculty devoted to Mathematics in its broadest sense (constituent schools: School of Mathematical Sciences; School of Computing and Quantitative Methods). The grouping of these schools in one faculty is designed to facil-

itate the integrated development of computing, statistics and mathematics.

At present, the school of Mathematical Sciences provides service teaching, up to final year, across the entire spectrum of undergraduate courses offered by the institute. A proposal for a degree in Mathematical Sciences itself is currently being considered by the NCEA and it is hoped that this course will have its first intake of students in October 1983. Besides teaching activities the school has active research interests which are fostered by weekly seminars as well as colloquia for visiting speakers. In 1982 the latter have included Prof. R. Lewis, University of Alabama in Birmingham; Dr. R.B. Paris, Association CEA-Euratom, Fontenay-aux-Roses, France; and Prof. J. Toland, University of Bath. In addition members of the school are active participants in the seminars and colloquia of other third-level institutions in the Dublin area and in conferences overseas. A special significance is placed by the school on the development of close cooperation with industry. This is stimulated by meetings with members of the industrial community and also by the efforts of the institute's liasion office. In this context comment seeking questionnaires have been circulated, aimed at both informing industry of our activities and determining the nature of its present needs. The school is also investigating the possibility of providing a course teaching mathematics through modelling in conjunction with the NIHE Distance Education unit. On a social level, the school is working on the formation of a "Walk-in Numeracy Centre" to cater for the needs of the less numerate in the local community. A preliminary meeting of this new venture on 13 December 1982 was well supported by the community and adult education workers in the area. It was agreed to go ahead with recruiting and training volunteer helpers with a view to operating four two-hour sessions a week.

Staff

Currently there are five full-time staff members (which is expected to steadily increase with student intake over the coming years). Besides individual research interests it has been decided to concentrate combined research in the area of differential equations: theory, analytical and numerical approximations together with applications.

The Staff Members are:

Dr. A.D. Wood (Head of School)	B.Sc. Ph.D. F.I.M.A.	St. Andrews
	(Differential Equations, especially Asymptotic Theory)	
Dr. R. Flood (Lecturer)	B.Sc.	(Queens University, Belfast)
	M.Sc.	(Linacre College, Oxford)
	Ph.D.	(U.C.D.)
	(Interacting Particle Systems, Applications of Stochastic Processes)	

Assistant Lecturers

Dr. M.J. Clancy	B.Sc.	(U.C.G.)
	M.Sc.	(U.C.G.)
	Ph.D.	(University of Notre Dame)
	(Differential Geometry/Applications to Physics)	
Dr. D.W. Reynolds	B.A.	(Oxford)
	M.S.	(Carnegie-Mellon University)
	Ph.D.	(Carnegie-Mellon University)
	(Viscoelasticity, Elasticity, Lifurcation theory, Volterra Integral Equations)	
Dr. G. Moore	B.Sc.	(Bath)
	Ph.D.	(Bath)
	G.I.M.A.	(Bath)
	(Numerical Analysis: Finite Element Approximation. Applied Functional Analysis: Bifurcation Problems)	

Our Lady of Mercy College (Carysfort College)

Our Lady of Mercy College is a College of Education for student teachers for Primary Schools. It was founded by the Sisters of Mercy in 1877 and was transferred to its present location at Carysfort Park, Blackrock, in 1903. Significant changes occurred in 1975 when Carysfort became a recognised College of the National University of Ireland, the B.Ed. degree replaced the old-style N.T. qualification and male students were admitted for the first time. The Mathematics Department was founded in 1975. Of course, Mathematics had been taught as a subject in the N.T. qualification and indeed the late Eamonn De Valera taught Mathematics at Carysfort between 1907 and 1916.

All undergraduates follow a three-year course of studies for the B.Ed. degree, consisting of Education, theoretical and practical, and one elective subject chosen from the range: Irish, English, History, Geography, Mathematics, French and Music. One additional subject from this list is taken in the first year.

The Bachelor of Education degree programme in Mathematics centres around the core areas of Algebra, Analysis and Number Theory. Carysfort also has a well-equipped microcomputer laboratory and Mathematics students take Computer Studies units in their second and final years. They cover topics such as Mathematics-related computing, computer-aided instruction, files and record-keeping. The Mathematics Department also offers courses in Carysfort's Inservice Summer School. The members of the Department are:

Dr. Gerald Ryder	B.S.	(Carnegie-Mellon University)
	M.S.	(Carnegie-Mellon University)
	Ph.D.	(Carnegie-Mellon University)
Dr. John Cosgrave	B.Sc.	(London)
	Ph.D.	(London)
Dr. Michael Barry	B.A., M.A.	(U.C.D.)
	M.S., Ph.D.	(University of Notre Dame)

St. Patrick's College of Education

St. Patrick's College was founded in 1875 for the education of teachers for primary schools and, in 1975, was accepted as a Recognised College of the National University of Ireland. The College offers a three years Honours B.Ed. Degree, a one-year course for graduates leading to a Teacher's Certificate (Primary) and a one-year Diploma in Special Education.

Students following the B.Ed. Course, take Education and one subject chosen from Irish, English, History, Geography, Mathematics, Music, French and Biology (First Year only) for all three years, with an additional subject from the above list in first year.

The B.Ed. programme in Mathematics consists mainly of courses in Analysis, Algebra, Probability and Statistics, covering such topics as real analysis, vector spaces and matrices, groups and rings. Also included is a course in computer science in which a programming language is taught. This course includes application of the computer to problems in number theory and statistics.

The Mathematics Department is involved in an advisory capacity in the mathematics education courses. These form a component of the Education course which all students take. Here the methodology and content of the primary school mathematics curriculum are covered.

The Mathematics Department has a staff of three:

Rev. Brendan Steen (Head of Department)	C.M. M.Sc. H. Dip. Ed.	
Frederick S. Klotz	B.S. Ph.D.	(University of Pittsburgh) (Syracuse University)
Olivia Bree	B.Sc. M.Sc.	(U.C.G.) (U.C.G.)

NUMERICAL METHODS IN DYNAMICAL WEATHER PREDICTION

J.R. Bates

1. Introduction

The scientific problem of forecasting the weather using dynamical methods was first tackled successfully by a group working under the leadership of John von Neumann at the Institute for Advanced Study, Princeton, in the late 1940 s. At that time the first electronic computer, the ENIAC, had just become available. Von Neumann recognised that the new machine was ideally suited to performing the hugh volumes of computations necessary to predict the non-linear development of fluid systems, including the motions of the atmosphere. Using observed initial data derived from balloon ascents over the continental U.S., an integration was performed which succeeded in predicting the main features of the actual evolution of the 500-mb flow for a 24-hour period over the area in question. (Charney, Fjϕrtoft and Von Neumann, 1950). The integration took 24 hours of computer time, however!

An essential ingredient of the success of the integration, due to von Neumann himself, was the development of a computationally stable numerical scheme for representing the differential equations governing the flow. It had been discovered two decades earlier (Courant, Friedrichs and Lewy, 1928) that not all consistent numerical representations of partial differential equations lead to realistic solutions. The method devised by von Neumann was an explicit leapfrog method based on a grid point representation in space and time.

The vast increase in the speed of computers over the past three decades, coupled with progress in devising more efficient numerical schemes for solving the equations of motion, has made it possible to compute the weather fast enough for the forecasts to be used operationally. The computer forecasts have for some time been more accurate than those which can be produced using traditional methods alone. Even with the most powerful computers available today, however, the spatial truncation errors associated with the numerical representation of the governing differential equations are still a factor limiting the accuracy of forecasts. The search for more efficient and accurate schemes therefore remains a central problem in dynamical weather forecasting.

In this article, some of the main numerical techniques used in this field will be briefly outlined.

2. The Simplified Governing Equations

The large-scale motions of the atmosphere are quasi-horizontal and, to a very high degree of approximation, hydrostatically balanced (i.e. the vertical component of the pressure gradient force equals the force of gravity). They are also shallow, in the sense that ~~the~~ vertical scale of the motions is small by comparison with the earth's radius. These facts allow one to adopt simplified versions of the general equations of fluid dynamics for the purpose of weather prediction. A simplified set of equations, which are capable of describing the flow at the atmosphere's middle level (the 500 mb level, approximately) with reasonable accuracy, are the "shallow water" equations,

$$\frac{du}{dt} = -g\frac{\partial h}{\partial x} + fv \quad \ldots. (1)$$

$$\frac{dv}{dt} = -g\frac{\partial h}{\partial y} - fu \quad \ldots. (2)$$

$$\frac{dh}{dt} = h\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] \quad \ldots. (3)$$

where (x,y) are the eastward and northward coordinates, (u,v) are the corresponding velocity components, h is the height of the 500 mb surface, g is the gravitational acceleration, f $(= f_0 + \beta y)$ is the Coriolis parameter representing the effects of the earth's rotation, and $\frac{d}{dt}$ $(= \frac{\partial}{\partial t} + \underset{\sim}{V}_H.\nabla)$ is the

derivative following a fluid particle. In these simplified equations, the effects of friction, thermodynamic forcing and spherical geometry (except for the y-variation of the Coriolis parameter) have all been neglected.

The above equations possess two distinct types of linearized wave solution:

(a) gravity-inertia waves, for which the phase speed in the one-dimensional case (with f regarded as constant) is given by

$$c = \pm\left\{gH + \frac{f^2}{k^2}\right\}^{\frac{1}{2}} \quad \ldots. (4)$$

Here H is the mean height of the surface and k is the wavenumber.

(b) Rossby waves, for which the phase speed in the one-dimensional case (with no perturbation in the height of the surface) is given by

$$c = -\frac{\beta}{k^2} \quad \ldots. (5)$$

The gravity-inertia waves are fast (phase speeds of hundreds of metres per second) but have only very small amplitudes in the atmosphere. The Rossby waves are slow (phase speeds of ten metres per second or less) and are very important in the development of weather systems. In the original governing equations used by the Princeton group, the gravity-inertia waves were filtered out by using a modified version of equations (1)-(3). The filtering procedure introduces some inaccuracies however, and the modern practice is not to use it.

In almost all meteorological applications, an Eulerian approach has been used for solving the equations of motion, i.e. the derivatives ($\frac{d}{dt}$) are expressed as $(\frac{\partial}{\partial t} + \underset{\sim}{V}_H.\nabla)$ and the $\underset{\sim}{V}_H.\nabla$ term is brought to the right hand side of the equations. Only partial derivatives in space and time then occur and one forgets about fluid particles. An alternative is the

Lagrangian approach in which one follows the fluid particles, retaining ($\frac{d}{dt}$) in its original form and doing the numerical calculations accordingly. The Irish Meteorological Service is the first to use a Lagrangian method operationally; we have been using this method for our daily forecasts since May 1982 and find it to be more efficient than the Eulerian approach (see Section 4 below).

3. Main Categories of Numerical Methods used for Solving the Equations in Eulerian Form

The methods used for solving the equations in Eulerian form can be classified under two headings:

(i) The Grid Point Method

Here the partial derivatives in the equations of motion are replaced by finite difference approximations at a discrete set of points regularly distributed in space and time. The difference equations are then solved using algebraic methods.

(ii) Galerkin Methods

The Galerkin procedure represents the dependent variables as a sum of functions that have a prescribed spatial structure. The coefficient associated with each function is then a function of time. This procedure transforms a partial differential equation into a set of ordinary differential equations for the coefficients. These equations are usually solved with finite differences in time. Examples of the Galerkin method are (a) the Spectral Method (using orthogonal functions as basis functions), and (b) the Finite Element Method (using functions that are zero except in a limited region where they are low-order polynomials).

The grid point method has been the most widely used method in meteorology, but spectral methods, using surface spherical harmonics as basis functions, are now being used

for hemispheric or global forecasting models at a number of centres, e.g. the European Centre for Medium Range Weather Forecasts. Some numerical experiments have also been carried out using the finite element method, but so far this has not been found to be competitive in efficiency with the other methods.

Some examples will now be given to illustrate the stability properties of numerical schemes using the grid point method. We consider the equation

$$\frac{\partial \psi}{\partial t} = -\bar{u}\frac{\partial \psi}{\partial x} \qquad \dots. (6)$$

governing the advection of a scalar ψ by a mean flow $\bar{u}$ (here considered a positive constant). Equation (6) contains a subset of the terms of equations (1) - (3). The analytical solution to (6) is

$$\psi = F(x - \bar{u}t)$$

where $F(x)$ is the initial distribution of ψ.

Consider the following simple difference approximation to equation (6):

$$\frac{\left[\psi_j^{n+1} - \psi_j^n\right]}{\Delta t} = -\bar{u}\frac{\left[\psi_{j+1}^n - \psi_{j-1}^n\right]}{2\Delta x} \qquad \dots. (7)$$

where $t = n\Delta t$, $x = j\Delta x$. This is a "forward-in-time, centred-in-space" approximation. We examine the stability of (7) by the von Neumann method, i.e. we assume

$$\psi_j^n = \lambda^n e^{ik(j\Delta x)}\psi_0 \qquad \dots. (8)$$

where λ is the amplification factor. Substituting in (7) then gives

$$\lambda = 1 - i\,\bar{u}\frac{\Delta t}{\Delta x}\,\mathrm{Sin}\,k\Delta x$$

Thus $|\lambda|^2 > 1$ for all values of $(\bar{u}\Delta t/\Delta x)$ for general values of k, i.e. unlike the analytical solution, the numerical solution amplifies with time and the difference approximation (7) is unstable.

Next consider the difference approximation

$$\frac{\left[\psi_j^{n+1} - \psi_j^n\right]}{\Delta t} = -\bar{u}\frac{\left[\psi_j^n - \psi_{j-1}^n\right]}{\Delta x} \quad .... (9)$$

i.e. a "forward-in-time, upstream-in-space" approximation. Again assuming a solution of the form (8), we find that

$$\lambda = 1 - u\frac{\Delta t}{\Delta x}\left[1 - \exp(-ik\Delta x)\right]$$

so that

$$|\lambda|^2 = 1 - 2\bar{u}\frac{\Delta t}{\Delta x}(1 - \bar{u}\frac{\Delta t}{\Delta x})(1 - \operatorname{Cos} k\Delta x)$$

In this case $|\lambda|^2 \leq 1$, (i.e. stability obtains) provided

$$\bar{u}\frac{\Delta t}{\Delta x} \leq 1 \quad .... (10)$$

The difference approximation (9) is thus conditionally stable, the stability criterion being that the distance covered by a particle in the time interval Δt be less than the spatial grid interval.

The difference equations (7) and (9) are both explicit, in the sense that the right-hand sides of both equations involve known quantities, and the unknown ψ_j^{n+1} is obtained by a simple operation at each grid point.

We now consider an implicit numerical representation of (6):

$$\frac{\left[\psi_j^{n+1} - \psi_j^n\right]}{\Delta t} = -\bar{u}\frac{\left[\psi_{j+1}^{n+1} - \psi_{j-1}^{n+1}\right]}{2\Delta x} \quad .... (11)$$

Here the right-hand side involves the unknown quantities at time level (n+1), and the values of ψ_j^{n+1} can only be obtained by a matrix inversion involving the whole grid. Seeking a

solution to (11) of the form (8) we find

$$\lambda = \frac{1}{1 + i(\bar{u}\frac{\Delta t}{\Delta x})\operatorname{Sin} k\Delta x}$$

i.e. $$|\lambda|^2 = \frac{1}{1 + (\bar{u}\frac{\Delta t}{\Delta x})^2 \operatorname{Sin}^2 k\Delta x}$$

so that $|\lambda|^2 \leq 1$ for all values of $(\bar{u}\,\Delta t/\Delta x)$. The difference equation (11) is thus unconditionally stable.

When the remaining terms of (1) - (3) are included, a linearized analysis shows that Eulerian explicit difference representations are in all known cases either unstable or conditionally stable, the stability criterion being that $(c\,\Delta t/\Delta x)$ be less than some number of order unity, where c represents the maximum signal velocity in the fluid. This represents a severe limitation on the allowable time step in the meteorological context because of the fast gravity-inertia wave solutions of the equations. These solutions are essentially only "noise", the meteorologically interesting information being carried by the slow Rossby wave and advective terms in the equations of motion. This problem can in theory be circumvented by adopting an implicit differencing scheme for all terms of the equations of motion, which gives unconditional stability and allows one to choose a time step which is realistic for describing the slow motions.

It turns out, however, that the matrix inversions involved in integrations with implicit schemes are computationally so costly that one is no better off than if one had adopted an explicit scheme with a very small time step.

A successful compromise between these two extremes is to use a semi-implicit approach, where the terms governing the fast motions are treated implicitly while the terms governing the slow motions are treated explicitly (Kwizak and Robert, 1971). This leads to conditional stability, but with a stability criterion which is much more lenient than that for fully explicit methods. At the same time, the matrix inversions are much simplified. The semi-implicit approach is now widely used, with both grid points and spectral models.

An alternative efficient method is to adopt the splitting approach pioneered by Soviet mathematicians (Marchuk, 1974). In this approach the equations (1)-(3) are split into the two sets:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial t} &= -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} \\ \frac{\partial h}{\partial t} &= -u\frac{\partial h}{\partial x} - v\frac{\partial h}{\partial y} \end{aligned}\right\} \quad (12) \qquad \left.\begin{aligned} \frac{\partial u}{\partial t} &= -g\frac{\partial h}{\partial x} + fv \\ \frac{\partial v}{\partial t} &= -g\frac{\partial h}{\partial y} - fu \\ \frac{\partial h}{\partial t} &= -h\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] \end{aligned}\right\} \quad (13)$$

A stability analysis shows that these two sets have independent stability criteria. The set (12) can be stepped forward in time with a long (advective) time step, while the set (13) can be updated successively with a fractional time step. With both sets treated explicitly, a ratio of 3:1 in the respective time steps can be used. This leads to an efficiency comparable to that of the semi-implicit method, with much simpler programming. This method is also widely used by meteorologists.

4. The Semi-Langrangian Method

A fully Lagrangian approach to solving the equations of fluid motion would involve following a fixed set of particles throughout the period of the integration. In atmospheric flow, a set of particles which are initially regularly distributed soon become greatly deformed so it is better to adopt a semi-Lagrangian approach, where a set of particles which arrive at a regular set of grid points are traced backwards over a single time interval to their departure points. The values of the dynamical quantities at the departure points are obtained by interpolation from the surrounding grid points. A new set of particles is then considered at each time step.

A splitting approach can be combined with the Semi-Lagrangian technique by writing (12) in the form

$$\frac{du}{dt} = 0, \quad \frac{dv}{dt} = 0, \quad \frac{dh}{dt} = 0 \qquad \ldots. \ (14)$$

while keeping the remaining terms as in (13). The equations (14) are integrated to give

$$(u,v,h)^{n+1}_{i,j} = (u,v,h)^{n}_{*}$$

where the quantities $(\ )^{n+1}_{i,j}$ are the new values at the grid point (i,j) and the quantities $(\ )^{n}_{*}$ are the old values at the departure point of the particle.

Bates and McDonald (1982) have shown that for linear and quadratic interpolation in the one-dimensional case, and for bilinear and biquadratic interpolation in the two-dimensional case, the above explicit method gives unconditional stability for the advective part of the integration. The use of the semi-Lagrangian method has led to a saving of a third in the computer time required to produce our daily forecasts in the Irish Meteorological Service compared to the Eulerian method previously used, while giving equal accuracy.

5. Conclusions

Only the barest outline has been given above of an extensive field of investigation. Many important questions, such as non-linear computational instability, staggered grids and the maintenance of integral constraints (such as energy and squared vorticity) in numerical integrations have not even been touched on. The governing equations used in practice are much more complicated than the simple shallow water equations which have been discussed here. A comprehensive coverage of numerical methods used in meteorology can be found in

WMO (1976), Chang (1977) and Haltiner and Williams (1980).

Despite the progress that has been made, it appears likely that there is still a long way to go before the ideal numerical method is found which integrates the governing equations and gives clearly maximum accuracy for a given computational cost.

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RECENT DEVELOPMENTS IN LINEAR PROGRAMMING

Fergus J. Gaines

Let A be an mxn matrix, let $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. The basic problem in linear programming is to find, for $x \in \mathbb{R}^n$,

$$\max c^t x, \text{ subject to } Ax \leq b,\ x \geq 0 \qquad (1)$$

For vectors, $x \leq y$ means $x_i \leq y_i$ for all i; $x < y$ means $x_i < y_i$ for all i.)

The standard way of solving this problem is to use the celebrated *simplex method* of G. Dantzig [1]. The idea is to note that the *feasible* solutions of (1), i.e. the $x \in \mathbb{R}^n$ with $Ax \leq b$, $x \geq 0$, form a convex polytope K in $\mathbb{R}^n$. The vertices of K are those feasible x with either x = 0 or such that the positive components of x correspond to linearly independent columns of A. The typical step in the simplex algorithm proceeds from vertex $x^{(k)}$ to a vertex $x^{(k+1)}$ so that $c^t x^{(k+1)} \geq c^t x^{(k)}$. Since max $c^t x$ is attained at a vertex of K, the algorithm eventually gives the answer.

This algorithm is arguably the most widely used algorithm of the present day and it is probably safe to say that most of those who use it do not understand it, whereas most of those capable of understanding it never use it. Its popularity is probably the reason for the widespread, if in many cases inaccurate, coverage in the newspapers given to the discovery in 1979 of a new algorithm for solving (1), the work of a Soviet "unknown" L.G. Khachiyan [2]. (One American newspaper reported bitterly (but incorrectly) that a Soviet mathematician had solved the "travelling salesman problem", despite the fact that the U.S.S.R has no travelling salesmen!)

The immediate reason why Khachiyan's algorithm is important is because it is *in theory* more computationally efficient than the simplex method. One of the noteworthy features of the simplex algorithm (and its variants) is that it is very efficient in all practical cases, i.e. it uses very little machine time. Empirical data show that the number of operations (+,x, etc.) in a typical application is $O(mn^3)$. However, Klee and Minty [3] have produced an example with $m=2n$ where the simplex method requires more than 2^n steps. In contrast, Khachiyan's algorithm is "polynomially bounded" in all cases, but it has serious drawbacks (see below).

But why does the simplex method work so well in practice? In a recent, highly significant paper, [4], Steve Smale has given a very satisfactory explanation. We discuss Smale's result below.

Khachiyan's Algorithm

Since Khachiyan's paper contains no proofs we follow the presentation in [5]. We note that the *linear programme* (LP) (1) can be reduced to the problem of solving a system of linear inequalities. We see this as follows. With LP (1) we can associate the *dual* LP, which is to find, for $y \in \mathbb{R}^m$

$$\min b^t y, \text{ subject to } A^t y \geq c. \quad y \geq 0 \qquad (2)$$

The Duality Theorem says (1) has an optimal solution if and only if (2) has, and in the event, $\max c^t x = \min b^t y$. Thus (1) has a finite optimum if and only if the system of inequalities

$$Ax \leq b, \quad x \geq 0, \quad A^t y \geq c, \quad y \geq 0, \; c^t x \geq b^t y \qquad (3)$$

has a solution. If (x,y) is a solution of (3) then x is an optimal solution of (1). The inequalities (3) can be rewritten

$$Mz \leq d, \qquad z \geq 0$$

where

$$M = \begin{bmatrix} A & 0 \\ 0 & -A^t \\ -c^t & b^t \end{bmatrix} \qquad z = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{and } d = \begin{bmatrix} b \\ -c \\ 0 \end{bmatrix}$$

So (changing notation) we need only solve the problem: find $x \in \mathbb{R}^n$ with $Ax \geq b$, $x \geq 0$.

We describe the algorithm for the problem:

$$\text{find } x \in \mathbb{R}^n \text{ with } Ax < b, \; x \geq 0, \text{ where } A \text{ and } B \text{ have integer entries.} \qquad (4)$$

(The case $Ax \leq b$ can be reduced to this case)

At first sight, the restriction to integer entries does not appear significant as the practical implementation of any algorithm can only involve finite decimals which are equivalent to integers. But this seems to be the root cause of the bad behaviour of the algorithm in practice.

The algorithm determines a sequence $x^{(k)} \in \mathbb{R}^n$ and a sequence of ellipsoids $E^{(k)}$ in $\mathbb{R}^n$ with centre $x^{(k)}$ and

$$\mathrm{vol}(E^{(k+1)}) < \mathrm{vol}(E^{(k)})$$

If L is the length of the binary encoding of (4) the algorithm either gives, for some $k < 4(n+1)^2 L$, an $x^{(k)}$ which is a solution of (4) or, if a solution cannot be found for such k, it shows that no solution exists.

If B_k is a positive definite symmetric matrix then

$$E^{(k)} = \{x \in \mathbb{R}^n : (x-x^{(k)})^t B_k^{-1} (x-x^{(k)}) \leq 1\}$$

is an ellipsoid with centre $x^{(k)}$. The steps in the algorithm are:

1. Set $x^{(0)} = 0$, $B^{(0)} = 2^2 V I$.
2. If $x^{(k)}$ is a solution to (4), terminate. If $k < 4(n+1)^2 L$ go to 3. Otherwise terminate, concluding (4) has no solution.
3. Choose one of the inequalities in (4) not satisfied by $x^{(k)}$, say $\quad a_i{}^t x^{(k)} \geq b_i$ ($a_i{}^t$ is the i th row of A).

Let

$$x^{(k+1)} = x^{(k)} - (1/(n+1))B^{(k)}a_i/(a_i^t B^{(k)} a_i)^{\frac{1}{2}}$$

and

$$B^{(k+1)} = (n^2/n^2-1)[B^{(k)} - (2/n+1)(B^{(k)}a_i)(B^{(k)}a_i)^t/(a_i^t B^{(k)} a_i)]$$

Go to step 2 with k+1 in place of k.

The ellipsoid $E^{(k+1)}$ contains the semi-ellipsoid

$$E^{(k)} \cap \{x \in \mathbb{R}^n : a_i^t(x-x^{(k)}) \leq 0\}$$

Also

$$\mathrm{vol}(E^{(k+1)}) = c(n)\mathrm{vol}(E^{(k)})$$

where

$$c(n)^{2n-2} = \tfrac{1}{2}$$

The ellipsoid algorithm in the worst case is $O(n^3(m+n)L)$ in contrast to the exponential behaviour of the Klee-Minty example. However, the ellipsoid algorithm behaves very badly in practice. As Dantzig points out (cf. [6]) a typical economic planning problem which takes half an hour machine time for the simplex method to solve, would take the ellipsoid algorithm fifty million years! Traub and Wozniakowski [6] give an explanation for the poor performance of Khachiyan's algorithm. They show that for the real number computational model (i.e. $\mathbb{R}$ with exact arithmetic and unit "cost" for each operation) the ellipsoid algorithm in the worst case is not polynomially bounded.

Despite its failure to oust the simplex method, the ellipsoid algorithm appears to have a future in the solution of combinatorial optimization problems other than linear programming. The paper [7] of Grötschel, Lovasz and Schrijver deals with this topic.

Smale's Theorem

Dantzig ([1], p.160) conjectured that for a randomly chosen LP, with fixed number of constraints m, the number of operations in the simplex method grows in proportion to n. Smale [4] not only proved this result but improved on it considerably.

The first problem is to define the average number of steps in the simplex method for a LP. We get a probability measure μ on the unit sphere S^{p-1} in $\mathbb{R}^p$, by normalizing the standard uniform (Lebesgue) measure. The points of S^{p-1} correspond to the rays of $\mathbb{R}^p$. If X is a set of rays in $\mathbb{R}^p$, we define the *spherical measure* of X by $\nu(X) = \mu(X \cap S^{p-1})$. Let A,b,c be as in (1). Then $q = (c,-b) \in \mathbb{R}^N$, where $N=m+n$. Let $\sigma(A,q)$ be the number of steps required to solve (1) by the simplex method. Since $\sigma(A,\lambda q) = \sigma(A,q)$ for $\lambda > 0$, we identify q with a ray in $\mathbb{R}^N$. The average number of steps required to solve (1), with A fixed, is

$$\rho_A = \int_{q \in S^{p-1}} \sigma(A,q)\, d\mu$$

Now identify the space $\mathcal{A}$ of all real mxn matrices with $\mathbb{R}^{mn}$. Since $\sigma(\lambda A,q) = \sigma(A,q)$ for $\lambda > 0$ we identify A with an element of $\mathcal{A}_1$, the set of rays of $\mathcal{A}$. Put a spherical measure ν on $\mathcal{A}_1$. Then the average number of steps required to solve (1) is

$$\rho(m,n) = \int_{A \in \mathcal{A}_1} \rho_A\, d\nu$$

We now have Smale's result.

Theorem Let p be a positive integer. Then depending on p and m, there is a positive constant c_m such that for all n

$$\rho(m,n) \leq c_m n^{1/p}$$

The case p=1 is Dantzig's conjecture.

The proof of the theorem is not easy. Smale considers a version of the simplex method, Lemke's algorithm, applied to the *linear complementarity problem* (LCP): given an NxN real matrix M and $q \in \mathbb{R}^N$, find $w, z \in \mathbb{R}^N_+$, the positive orthant, so that $w^t z = 0$ and $w - Mz = q$. The primal-dual problem (3) is a special case of the LCP. Next he defines a mapping Φ_M on $\mathbb{R}^N$ so that the LCP becomes: find $x \in \mathbb{R}^N$ so that $\Phi_M(x) = q$. If $q_0 = (1, \ldots, 1)^t \in \mathbb{R}^N$, the inverse image of the line segment qq_0, $\Phi_M^{-1}(qq_0)$, is a piecewise linear curve γ in $\mathbb{R}^N$. If γ_0 is the component of γ containing q_0 then Lemke's algor-ithm can be viewed geometrically as "following" γ_0. A pivot of the algorithm corresponds to the intersection of γ_0 with a *facet* (a facet is the intersection of a hyperplane with an orthant Q_S; for

$$S \in \{1, 2, \ldots, N\},\ Q_S = \{x \in \mathbb{R}^N : x_i \geq 0,\ i \in S,\ x_j \leq 0,\ j \notin S\}).$$

There are three main steps in the proof of the theorem. Firstly he derives a formula for ρ_A in terms of the spherical volume of certain cones. Then he derives an estimate for ρ_A. Finally he gets a simplified version of this estimate, when m is fixed and n is large, which gives the result.

The problem of determining the average speed of the simplex method as a function of both m and n still remains. In his Dublin lecture (September 1982) Smale said he felt that his general estimate for ρ_A might be used to solve this prob-lem. However, the basic difficulty to be overcome is that of determining volumes of cones.

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THE INFLUENCE CURVE

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The influence curve of an estimator measures how much an individual observation changes the value of the estimator. Thus, in any estimation problem the role the i th data point say, plays in the analysis, can be made exact. This intuitively appealing idea of Hampels (1974) initiated interest in the influence curve. Now there is a substantial theory on its properties and uses in statistics, of which I give here a preliminary account.

To understand the purpose and nature of the influence curve, we must think of parameters and their corresponding estimators as functionals. Consider a real-valued functional $T(\cdot)$ defined on the space of distribution functions and let the parameter of interest be $\theta = T(F)$ (usually F denotes the 'true' underlying distribution function). To fix ideas look at the following examples.

Example 1: (i) The mean functional is given by

$$\mu(G) = \int x \, dG(x)$$

provided the integral exists.

(ii) The variance functional is given by

$$\sigma^2(G) = \int x^2 \, dG(x) - \left[\int x \, dG(x)\right]^2$$

again provided the integral exists.

In these examples the parameters of interest might be the true mean $\mu(F)$ and the true variance $\sigma^2(F)$. To look at estimators we have to consider $X_1, \ldots, X_n$ a random sample from a population with distribution function $F(\cdot)$. The empirical distribution function of these X's, is $F_n(\cdot)$, where

$$F_n(t) = [X_i\text{'s} \le t]/n \qquad -\infty < t < \infty .$$

In many estimation problems the estimator $\hat{\theta}$ can be put in the same functional form as θ i.e. if $\theta = T(F)$ then $\hat{\theta} = T(F_n)$. Again we can look at familiar examples.

Example 2: (i) The estimator corresponding to the mean is

$$\hat{\mu} = \mu(F_n) = \int x \, dF_n(x) = \sum_{i=1}^{n} X_i/n = \bar{X},$$

i.e. the usual sample mean.

(ii) The estimator corresponding to the variance is

$$\hat{\sigma}^2 = \sigma^2(F_n) = \sum_{i=1}^{n}(X_i-\bar{X})^2/n.$$

We now define the influence curve of a functional $T(\cdot)$ at a point G, $IC(T, G ;\cdot)$, as follows. Let $W = (1-\varepsilon)G + \varepsilon\delta_z$ be a perturbation of G by δ_z, the distribution function for the point mass of one at z, i.e.

$$\delta_z(x) = \begin{cases} 0, & x \le z \\ 1, & x \ge z \end{cases}$$

Then

$$IC(T, G ; z) = \lim_{\varepsilon \to 0} \frac{T(W) - T(G)}{\varepsilon},$$

$$= \frac{d}{d\varepsilon} T(W) \Big|_{\varepsilon=0},$$

provided the limit exists for every $z \in R$. (It is also known as the Gâteau differential of T at δ_z). This derivative measures the effect on the functional T of a small (infinitesimal) change in the weight the distribution function G gives to the point z. Thus, when we consider the estimator $T(F_n)$, its influence curve $IC(T, F_n ; z)$ measures the "influence" on the estimator of an additional observation at the point z. To see this look at the influence curve of the mean.

Example 3: Denote the mean of F by $\mu(F)$. Then we have

$$W = (1-\varepsilon)F + \varepsilon\delta_z ,$$

$$\mu(W) = \int x \, d[(1-\varepsilon)F + \varepsilon\delta_z],$$

$$= (1-\varepsilon)\mu(F) + \varepsilon z ,$$

and

$$\frac{d}{d\varepsilon}\mu(W)\Big|_{\varepsilon=0} = z - \mu(F).$$

So

$$IC(\mu, F ; z) = z - \mu(F).$$

In particular for the sample mean

$$IC(\mu, F ; z) = z - \bar{x} .$$

Thus the effect on the sample mean of an additional observation is directly proportional to the value of the observation as is shown in Fig. 1.

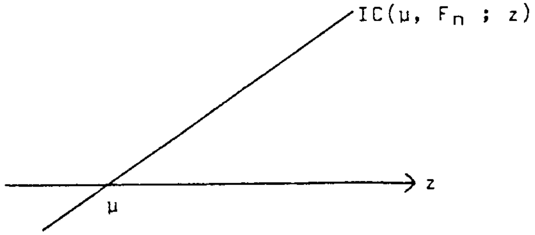

FIGURE 1

The Influence Curve of the Sample Mean.

Here, as is often the case, the influence curve is easy to compute. Note that the distribution functions can be multivariate. In such cases, the point z corresponds to a vector-valued observation. The influence curve has also been defined for vector-valued functionals. For example, the functional given by $T(G) = (\mu(G), \sigma^2(G))^T$ has a vector-valued influence curve defined at F as the pointwise limit:

$$IC(T, F ; z) = \frac{d}{d\varepsilon} T(W)\Big|_{\varepsilon=0} ,$$

$$= (IC(\mu, F ; z), IC(\sigma^2, F; z))^T,$$

provided the limit exists for every z. Now we examine various aspects of the influence curve to gain insight into the nature of an estimator.

Firstly, the shape of the influence curve provides information about the robustness properties of an estimator. In the example above we see the influence curve is unbounded reflecting the fact that the sample mean is sensitive to extremely large or small observations. In contrast to this the influence curve of the median is a step function.

Example 4: The median functional, $m(\cdot)$, is given by

$$m(F) = \tfrac{1}{2}(\sigma^* + \sigma^{**}),$$

where

$$\sigma^* = \sup\{x | F(x) \leq \tfrac{1}{2}\} \text{ and } \sigma^{**} = \inf\{x | F(x) \geq \tfrac{1}{2}\}.$$

Then

$$m(F_n) = \text{median } \{X_1, \ldots, X_n\},$$

is the sample median. The influence curve (see Fig. 2) is given by

$$IC(m, F ; z) = \begin{cases} -\dfrac{1}{f(F^{-1}(\frac{1}{2}))}, & z \quad F^{-1}(\tfrac{1}{2}) \\ 0, & \text{otherwise}, \end{cases}$$

where

$$f(x) = \frac{d}{dx} F(x)$$

This influence curve is bounded and thus is not sensitive to extreme observations and is robust in this sense.

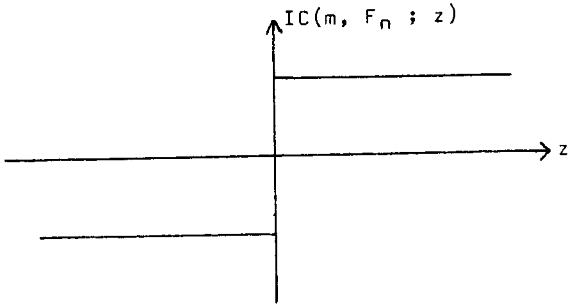

FIGURE 2

The Influence Curve of the Sample Median.

The influence curve is used to derive new estimators with pre-specified robustness properties. The study of various norms connected with the influence curve leads to estimators which are "best" over a large class of distribution functions. This breaks with Fisher's classical theory of estimation which looks for the best estimator with respect to one particular distribution function. The interested reader is referred to Huber (1977).

The influence curve plays an important role in asymptotic theory. The asymptotic variance of an estimator, for example, can be written in terms of the influence function. The usual delta method formula for calculating the asymptotic variance of an estimator $T(F_n)$ is in fact

$$\int IC^2(T, F ; x)\, dF(x)/n.$$

It can be estimated in the usual way by replacing F with F_n i.e. by

$$\sum_{i=1}^{n} IC^2(T, F_n ; X)/n^2.$$

This is because for a wide range of estimators, $T(F_n)$, Von Mises (1947) showed that

$$T(F_n) = T(F) + \sum_{i=1}^{n} IC(T, F ; X_i)/\sqrt{n} + R_n,$$

where

$$P\{\sqrt{n}|R_n| > \varepsilon\} < \varepsilon,$$

for n large. This means

$$\sqrt{n}(T(F_n) - T(F)) = \sum_{i=1}^{n} IC(T, F ; X_i)/\sqrt{n}.$$

So

$$\sqrt{n}(T(F_n) - T(F)) \overset{P}{\to} 0,$$

where $\overset{P}{\to}$ denotes convergence in probability. Then by the Central Limit Theorem

$$\sqrt{n}(T(F_n) - T(F) \overset{d}{\to} N(0, V(F)),$$

where $\overset{d}{\to}$ denotes convergence in distribution and

$$V(F) = \int IC^2(T, F ; x)\, dF(x).$$

Thus, we have

$$\text{Variance } (\sqrt{n}T(F_n)) \to V(F).$$

A simple illustration of this is as follows.

<u>Example 5</u>: For the mean, by Example 3, we have

$$IC(\mu, F ; x) = x - \mu(F).$$

Thus,

$$\int IC^2(\mu, F ; x)\, dF(x) = \int (x-\mu)^2\, dF(x) = \sigma^2(F).$$

So here,

$$V(F) = \sigma^2(F),$$

which is estimated by

$$\sigma^2(F_n) = \int IC^2(\mu, F_n ; x)\, dF_n(x)$$
$$= \sum_{i=1}^{n} (X_i - \overline{X})^2/n.$$

Our formula for the estimated variance of $\overline{X}$ is then

$$\hat{\text{Variance}}\ (\overline{X}) = \sum_{i=1}^{n} (X_i - \overline{X})^2/n^2.$$

A more recent development concerning the influence curve is its use in outlier detection. In many statistical problems it is assumed that the form of the underlying distribution function F is known apart from an unknown parameter θ. The assumed model is then denoted by $F(\cdot\ ;\ \theta)$. A familiar example is the normal distribution with unknown mean θ. Let $\hat{\theta} = \theta(F_n)$ be the estimator and

$$\sum_{i=1}^{n} IC^2(\theta, F_n ; X_i)/n^2$$

its estimated variance. We denote this as

$$\hat{Var}\ (\hat{\theta}) = \sum_{i=1}^{n} IC^2(\theta, F_n ; X_i)/n^2.$$

The statistic

$$D_i = IC^2(\theta, F_n ; X_i)/\sum_{i=1}^{n} IC^2(\theta, F_n ; X_i)$$

can be interpreted as a measure of the 'goodness of fit' of the i th data point to the model $F(\cdot\ ;\ \theta)$. It can be shown that for n large $D \overset{d}{\sim} F(1, n-1)$, where $F(1, n-1)$ is the F-distribution with 1 and n-1 degrees of freedom. The symbol "$\overset{d}{\sim}$" denotes "is asymptotically distributed as" when $n \to \infty$. Thus D_i provides a measure of fit of the i th data point in terms of descriptive levels of significance. For p vector-valued influence curves, IC^2 in D_i is replaced by $IC^T IC$ and then $D_i \overset{d}{\sim} F(p, n-p)$. Another interesting interpretation of D_i is as follows. Let $\hat{\theta}_{-i}$ be $\hat{\theta}$ with the i th observation omitted. Now

$$(n-1)(\hat{\theta}-\hat{\theta}_{-i}) = \left.\frac{\theta(W) - \theta(F)}{\varepsilon}\right|_{\varepsilon = \frac{1}{n-1}}, \qquad (1)$$

and this together with the definition of the influence curve implies

$$D_i \overset{d}{\sim} \frac{(\hat{\theta} - \hat{\theta}_{-i})}{Var\ \hat{\theta}}.$$

{Equation (1) also provides the connecting link between the influence curve and the jackknife; c.f. Miller (1974).} This can be used in the following way. Let $F(1, n-1, 1-\alpha)$ denote the $(1-\alpha)$ th probability point of the $F(1, n-1)$ distribution. Then for example, if $D = F(1, n-1, 5)$, removal of the i th data point moves the estimator to the edge of the 50% confidence region for θ based on $\hat{\theta}$. Measures of large residuals from regression models surveyed by Atkinson (1982) can be shown to be all versions of the statistic D_i above.

The following example serves as a demonstration of the use of D_i. No attempt at a complete analysis is made.

<u>Example 6</u>: Miller (1982) presented simultaneous pairs of measurements of serum kanamycin levels in blood samples drawn from twenty babies. One of the measurements was obtained by a heelstick method (X), the other using an umbilical catheter (Y). The heelstick method had been customarily used but due to the necessity of frequently drawing samples, this left neonates with badly bruised heels. The aim of the experiment was to see if the two methods measured the same levels except for error variability. If true, this would eliminate the unnecessary trauma to the newborn of repeated venapunctures. Since both measurements

are subject to error, an error in variables rather than regression analysis is used (c.f. Kendall and Stuart). It was assumed the true F was the bivariate normal and that the points followed a line with unknown slope β and intercept α. The parameter of interest then is $\underset{\sim}{\theta} = (\alpha, \beta)^T$ and the influence curve is bivariate.

The twenty pairs of heelstick and catheter values are presented in Table 1.

Baby	Heelstick	Catheter
1	23.0	25.2
2	33.2	26.0
3	16.6	16.3
4	26.3	27.2
5	20.0	23.2
6	20.0	18.1
7	20.6	22.2
8	18.9	17.2
9	17.8	18.6
10	20.0	16.4
11	26.4	24.8
12	21.8	26.8
13	14.9	15.4
14	17.4	14.9
15	20.0	18.1
16	13.2	16.3
17	28.4	31.3
18	25.9	31.2
19	18.9	18.0
20	13.8	15.6

TABLE 1

Serum kanamycin levels in blood samples drawn simultaneously from an umbilical catheter and a heel venapuncture in twenty babies

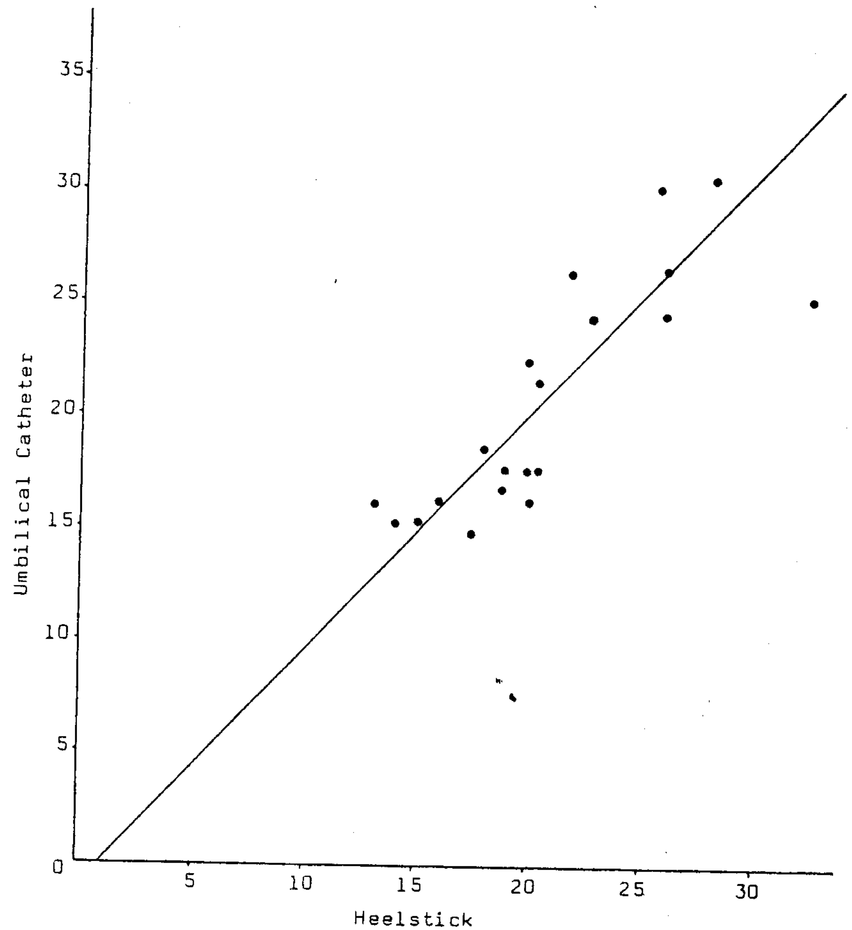

FIGURE 3

Plot of the Twenty Pairs of Serum Kanamycin levels from Table 1

TABLE 2: Estimates of the influence curve and deleted estimates, for each data point

Baby	Intercept: $IC(\alpha, F_n; Z)$ Sample Influence	Intercept: $\hat{\alpha}_{-i}$ Deleted Estimate	Slope: $IC(\beta, F_n; Z)$ Sample Influence	Slope: $\hat{\beta}_{-i}$ Deleted Estimate	D_i Normalised Influence
1	-31.68	-.96	.26	1.06	.012
2	61.66	-5.26	-3.36	1.29	.768
3	-1.64	-1.07	.06	1.07	.001
4	-1.06	-1.10	.08	1.07	.000
5	1.07	-1.21	.09	1.07	.051
6	-6.34	-.83	.20	1.06	.046
7	.77	-1.20	.03	1.07	.012
8	-7.35	-.77	.26	1.06	.041
9	3.36	-1.34	-.12	1.08	.010
10	-14.71	-.39	.52	1.04	.175
11	8.02	-1.61	-.49	1.10	.025
12	-10.85	-.59	.74	1.03	.088
13	4.21	-1.40	-0.17	1.08	.009
14	-14.93	-.34	.59	1.04	.117
15	-6.34	-.83	.20	1.06	.046
16	23.83	-2.57	-.98	1.13	.290
17	-16.20	-.14	.88	1.01	.045
18	-30.47	.58	1.68	.97	.145
19	-3.75	-.96	.13	1.06	.013
20	14.40	-2.01	-.59	1.04	.103

A graphical display of these twenty pairs is reproduced in Fig. 3. The estimates of intercept and slope from the analysis are

$$\hat{\theta} = -1.16, \quad \hat{\beta} = 1.07$$

The line with $\hat{\alpha}$ and $\hat{\beta}$ is drawn in Fig. 3. Table 2 presents the influence curve of the slope and intercept at each data point. The estimates of α and β obtained by deleting each data point in turn are also tabled as well as the values of D_i. We see Babies 2 and 16 have the largest values of the influence curve and have a negative influence on the slope estimate. If we look again at Fig. 3 we realise how difficult it is to detect and agree on what an 'outlier' is, without some objective measure. From Table 3, we have $D_2 = .688 = F(2, 18; .45)$, so removal of Baby 2 moves the estimate of $\underset{\sim}{\theta}$ to approximately the edge of a 55% confidence region around $\underset{\sim}{\hat{\theta}}$. Removal of Baby 16 moves the estimate of $\underset{\sim}{\theta}$ to the edge of a 40% confidence region around $\underset{\sim}{\hat{\theta}}$.

The influence curve is easy to explain and interpret in consultancy work and we could make the argument that it become an integral part of all data analysis. For this reason, I have emphasised mathematical rigor less than intuitive meaning in this article. There are still many open mathematical details, like regularity conditions, to be addressed. Many other influence curves of widely used estimators, need to be derived, studied and interpreted.

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FEIGENBAUM'S NUMBER

J. Kennedy

In 1975 Mitchell J. Feigenbaum [2] of the Los Alamos National Laboratory, whose work concerns the transition from periodic to aperiodic bahaviour, discovered a new universal constant which has since been called Feigenbaum's number. He had been using a programmable calculator to examine the iteration of one-parameter families of maps of a finite interval into itself. One map he looked at was $x \to f_B(x) = Bx(1-x)$; another was $x \to B \sin \pi x$, both on the interval $[0,1]$. Feigenbaum observed some common features of the parameter dependence of these maps which he suspects would not have been noticed had the calculations been carried out on a large computer rather than a small calculator. The theory of these maps has been extended by Pierre Collet of Paris, Jean-Pierre Eckmann of Geneva and H. Koch of Harvard. The topic is reviewed in the book by Collet and Eckmann [3] on which this note is based.

For the most part, Collet and Eckmann consider mappings $x \to f(x)$ which are C^1-unimodal. A mapping f of the interval $[-1,1]$ into itself is C^1-unimodal if f is continuous; $f(0)=1$; f is strictly decreasing on $(0,1]$ and strictly increasing on $[-1,0)$; and f is once continuously differentiable with $f'(x) \neq 0$ when $x \neq 0$.

Denoting by f^0 the identity, $f^1=f$, $f^2=f \circ f$, $f^n = f \circ f^{n-1}$ the sets of iterates of points $x \in [-1,1]$

$$O_f(x) = \{x, f(x), f^2(x), f^3(x) \ldots\}$$

are called the orbits of f. A point $x \in [-1,1]$ is called a periodic point for f if $O_f(x)$ is a finite set. The cardinality of this set is called the period of x and $O_f(x)$ is called the periodic orbit of x. It is then also the periodic orbit of

$f^n(x)$ for all $n \geq 0$.

It is well known that if $\bar{x}$ is a fixed point of $x \to g(x)$ it will be a stable fixed point provided $|Dg(\bar{x})| < 1$ where Dg means $\frac{dg}{dx}$. Then iterations of the map which start in the neighbourhood of $\bar{x}$ will eventually converge to $\bar{x}$. If P is a periodic orbit of period p for f the orbit is called a stable periodic orbit when, for $x \in P$, $|Df^p(x)| < 1$. By the chain rule, $Df^p(x)$ takes the same value for all $x \in P$, and all the points of P are then fixed points of f^p. Thus in the case of a stable periodic orbit, many starting points give rise to similar behaviour as the number of iterations becomes large. A periodic orbit P is called superstable if $0 \in P$. From the definition of a C^1-unimodal function this means that a periodic orbit is superstable iff $Df^p(x) = 0$ for $x \in P$. Singer [4] has used the concept of negative Schwartzian derivative defined by

$$Sf(x) \equiv \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2$$

to discuss the existence of stable periodic orbits. The requirement that $Sf(x) < 0$ for all $x \in [-1,1]$ is a more precise statement of the shape of functions considered than C^1-unimodality. Ignoring some technical details, Singer concludes that a function f which is C^1-unimodal and for which $Sf(x) < 0$ for all $x \in [-1,1]$ has at most one stable periodic orbit plus possibly a stable fixed point in the interval $[-1, f(1))$. If 0 is not attracted to a stable periodic orbit, then f has no stable periodic orbit in $[f(1), 1]$. Further, there exist functions satisfying Singer's conditions which do not have a stable periodic orbit and maps of such functions have ergodic properties.

Collet and Eckmann use numerical results for the mapping

$$x \to f_B(x) = 1 - Bx^2,$$

where B is a parameter in the range $[0,2]$ to illustrate these ideas. For a value of $B \leq B_1 = .75$ iteration of the mapping

converges to a single stable fixed point. For $.75 = B < B \leq B_2 = 1.25$ initial points from the interval $[-1,1]$ are eventually attracted to a stable periodic orbit of period 2. B_1 is called a bifurcation value of the parameter B. At $B_2 = 1.25$ both of the branches bifurcate again and a stable periodic orbit of period $2^2 = 4$ appears. Further bifurcations from period 2^{n-1} to 2^n take place at values B_n which converge to $B_\infty = 1.401$ .... . For most values of B in the range $(B_\infty, 2]$ orbits which have very large periods or which are aperiodic appear. For some values of B in this range not only do orbits with period 2^k appear but also orbits of period 6, 5 and 3. Much is known about this type of behaviour but it will not be discussed further here except to say that what happens as the parameter approaches and then passes B_∞ is typical of the transition from regular to chaotic or turbulent behaviour of many systems [3].

The first few values of B_n can be found by using a programmable calculator although the convergence to stable periodic orbits is slow when the parameter is near a bifurcation point. For higher values of B_n multiple-precision arithmetic is required. Collet and Eckmann give the following table.

n	B_n	$\frac{B_n - B_{n-1}}{B_{n+1} - B_n}$
1	.75	
2	1.25	4.233738275
3	1.3680989394	4.551506949
4	1.3940461566	4.645807493
5	1.3996312389	4.663938185
6	1.4008287424	4.668103672
7	1.4010852713	4.668966942
8	1.401140214699	4.669147462
9	1.401151982029	4.669190003
10	1.401154502237	4.669196223
11	1.401155041989	

Not only do the B_n converge to a limit B_∞ but the ratios

$$\frac{B_n - B_{n-1}}{B_{n+1} - B_n}$$

also seem to converge to a limit δ.

It has been established ([2], [5]) that, for sufficiently smooth families of maps, the number δ does not in general depend on the family and families similar to the one discussed here produce asymptotically

$$|B_n - B| \sim \text{const. } \delta^{-n},$$

where

$$\delta = 4.6692016 \ldots$$

is Feigenbaum's number. More precisely, the universality of δ is somewhat relative in that its value does depend on the function space in which the mapping functions are assumed to lie.

Feigenbaum discovered the universality of δ experimentally and then proposed an explanation suggested by the renormalization group approach to critical phenomena. Collet, Eckmann and Lanford [5] have proved rigorously, at least in a certain limiting regime, the existence of the scheme outlined by Feigenbaum.

There is another type of scaling which can again be illustrated by the mapping.

$$x \to 1 - Bx^2.$$

For large n it is found that as each bifurcation point is passed the pattern of the orbits exhibits a certain regularity. This may be exemplified by considering the parameter range $(B_n, B_{n+1}]$ where stable orbits of period 2^n occur. It is found that there is always one value of B in this range which gives rise to a superstable orbit containing the point $x = 0$. It is further found that on such a superstable orbit the distance from the point at zero to its nearest neighbour is, for large n, λ^n where λ is another universal constant

$$\lambda = -0.3995 \ldots$$

Indeed if α is defined by

$$\delta^{\alpha} = |\lambda|$$

and a plot is made of the points of the stable orbit for different values of B using a vertical scale $x(B-B_\infty)^{\alpha}$ and a horizontal scale $\log|B-B_\infty|$ then, for large n, a periodic diagram appears. Once again this behaviour is independent of the one parameter family in question.

Finally the value, B_∞, of the parameter at which transition to ergodic behaviour occurs is, in general, different for different maps of the class considered, as are the functions f_B. However the function

$$f(x) = \lim_{n\to\infty} f_{B_\infty}^{2^n}(\lambda^n x)$$

is a universal function up to a change of scale. If scaled to $f(0) = 1$ it has the expansion

$$f(x) = 1 - 1.52763x^2 + 0.104815x^4 - 0.267057 \ldots x^6 + \ldots.$$

By construction f satisfies

$$f \circ f(\lambda x) = \lambda f(x)$$

which evaluated at $x=0$ gives

$$\lambda = f(1)$$

Collet and Eckmann [3], [5] also show that f may be used to characterize δ as the largest eigenvalue of the linear operator (on the function space in question)

$$h(.) \to \frac{1}{\lambda}h(f(\lambda.)) + \frac{1}{\lambda}f'(f(\lambda.))h(\lambda.).$$

Recently Rollins and Hunt [6] have modelled a simple, nonlinear, electronic system which exhibits the period-doubling route to chaos with universal scaling.

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PATHS IN A GRAPH

Johannes Siemons

In a connected graph any two vertices can be joined by a sequence of edges. This is the definition of connectedness for graphs. However, how do you find a path joining a given pair of vertices, and how do you decide effectively if a graph is connected? These are the questions I shall discuss in this note. The graphs we consider are finite, undirected and have no loops or multiple edges. A path is a sequence $\{v',v_1\} = e_1$, $\{v_1,v_2\} = e_2$, ..., $\{v_{r-1},v''\} = e_r$ of edges without repetition (of edges: vertices *may* occur repeatedly). The vertices v' and v" are the end vertices of the path.

A popular version of this problem is to find the exit in a maze. We have to distinguish two cases. In the first instance, imagine that we are actually inside a maze without knowing its overall design. Here the only solution seems to be trial and error. A successful route to the exit is very unlikely to be a path according to our definition. In fact, the probability to reach the exit on a path is less than 2^{-c}, where c is the number of intermediate junctions on a path to the exit (provided that there is only one such path in the maze). In other words, it is almost impossible to avoid walking into a cul-de-sac! However, most commonly, maze puzzles are done with paper and pencil, and the design of the maze is right in front of your eyes. In this situation, can you avoid a cul-de-sac? The answer is yes, there is a *construction* for a path to the exit!

From a set P of edges let V(P) be the set of end vertices of edges in P. For a vertex v in the graph, let $d_P(v)$ be the number of edges in P that end at v. A cycle is a path that ends in its initial vertex. Our construction is based upon the following simple observation:

Lemma Let v' and v'' be two vertices in a graph and let P be a set of edges such that $d_P(v')$ and $d_P(v'')$ are odd, while $d_P(v)$ is zero or even for all remaining vertices. Then $P = P(v',v'') \cup C_1 \cup \ldots \cup C_r$ where $P(v',v'')$ is (suitably arranged) a path from v' to v'' while each C_i is a cycle that has no vertex in common with $P(v',v'')$.

Proof Let $G_0, G_1 \ldots$ be the connected components of the subgraph with vertices $V(P)$ and edges P. As $d_P(v')$ is at least 1, v' is a vertex in one of the G_i, say in G_0. But then also v'' belongs to G_0, for otherwise the total degree sum in G_0 would be odd, which is impossible: In any graph the total degree sum is even. Therefore G_0 is a path from v' to v'' and the remaining components are cycles.

How can we effectively determine such a set of edges? And, secondly, how can we ensure that P does not contain cycles? (From a practical point of view, the second problem is less relevant, for if we start our path in v' we will reach v'' without entering any of the cycles C_i) We shall say that a set P as in the lemma is *short* if none of its subsets is a cycle. Thus a short path from v' to v'' is a path where none of the intermediate vertices is repeated.

We order the vertices of G in some way $v_1, \ldots, v_n$ and also order its edges $e_1, \ldots, e_m$. The graph now can be represented by its incidence matrix I. This is the matrix whose rows are indexed by vertices and whose columns are indexed by edges, such that $(I)_{v,e}$ is 1 if e ends at v and $(I)_{v,e} = 0$ otherwise. A set S of vertices is represented by a 0-1-vector $\underline{S}$ of length n where $(\underline{S})_i = 1$ iff v_i belongs to S. In the same way, an edge set P is represented by a 0-1-vector $\underline{P}$ of length m. The incidence matrix associates a vertex vector to any edge vector: $I \cdot \underline{P}^t$ is a vector of length n and its i th component is easily

seen to be $d_P(v_i)$. Now we realise that a set P has the property of the lemma exactly if $\underline{P}$ satisfies a linear congruence modulo 2.

Path Construction: A set P of edges consists of a path $P(v',v'')$ and a number of cycles disjoint from $P(v',v'')$ if and only if $I \cdot \underline{P}^t \equiv \underline{S}$ modulo 2 where $S = \{v',v''\}$.

Thus a path from v' to v'' can be constructed by solving this linear congruence, for instance by Gauss elimination. This is particularly simple in characteristic 2 where we only need to add rows and possibly permute rows and columns of I. Note also that cycles and unions of cycles correspond to 0-1-vectors in the kernel of I modulo 2. In order that the graph is connected, this congruence has to be solvable for any choice of S. This will be the case if and only if the rank of I is at least $n-1$ in characteristic 2. However, as each column of I adds up to 2, the rank will be $n-1$ exactly. Therefore, we obtain a criterion for connectedness in a graph.

The number of connected components in a graph is the number of vertices minus the rank of I in characteristic 2.

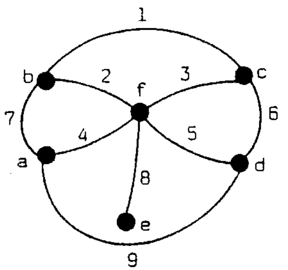
Short paths: Now we shall see that $I \cdot \underline{P}^t \equiv \underline{S}$ can be solved in such a way that a solution automatically will be short, that is, P does not contain a cycle. Using Gauss elimination, the congruence can be transformed into

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & * & * & * & \cdots & * \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & * & * & * & \cdots & * \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & * & * & * & \cdots & * \\ \vdots & & & & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \cdots 0 & 1 & * & * & * \cdots & * \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n-1} \\ p_n \\ \vdots \\ p_m \end{pmatrix} \equiv \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \text{ mod } 2$$

We now choose $p_n = p_{n+1} = \ldots = p_m = 0$ and hence have $p_i = s_i$ for $i = 1, \ldots, n-1$. If P is determined in this way, none of its subsets can satisfy the homogeneous congruence and therefore P does not involve any cycle. Thus P is a short path from v' to v". Of course, the above tableau can usually be achieved in a number of distinct ways. This corresponds to the fact that a short path is unique only if the graph contains no cycle.

Maximal and Minimal Short Paths: In the above tableau, the entries s_i are calculated from $\underline{S} = \{v', v''\}$ during the subsequent row operations. The number of $s_i \neq 0$ is, as we have seen above, the length $\ell(P)$ of the short path from v' to v". Therefore, the minimum value obtainable for $\ell(P)$ in any tableau is the distance from v' to v". As a short path passes through any vertex at most once, $\ell(P) + 1$ is the number of vertices en route, v' and v" included. Thus $\ell(P)$ is at most n-1, but this may or may not be obtainable in a tableau. For, if $\ell(P) = n-1$, then P passes through all the vertices of the graph. Such a path is called *hamiltonian*. No satisfactory criteria for the existence of such paths exist for graphs in general. In a particular case, however, we notice that hamiltonian paths correspond to tableaux of the above form in which $s_1 = s_2 = \dots = s_{n-1} = 1$.

As an example consider the graph in Fig. 1. It has 6 vertices a,b, ..., f, 9 edges 1,2, ..., 9 and its incidence matrix is the 6x9 matrix I given below. We form the 6x15 matrix (I,Id) where Id is the 6x6 identity matrix.



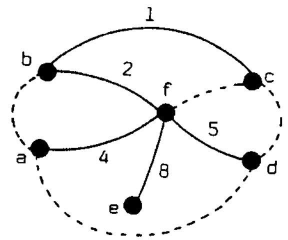
$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array}
\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\left[\begin{array}{ccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0
\end{array}\right] \end{array} = I$$

FIGURE 1

On this matrix subsequently Gauss elimination is performed (allowing permutations of the columns of I) and we obtain

$$(I^+, S) = \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 4 & 5 & 8 & 3 & 6 & 7 & 9 \end{array} \\
\left[\begin{array}{ccccccccc|cccccc}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right] \end{array}$$

In this process at least 1 column permutation has to take place, e.g. edge 8 has to be included among the first 5 edges. In the event these are the edges 1,2,4,5,8. This means that they are the only edges effectively used in the construction. As no cycle can be formed from them, they automatically build a *spanning tree*.



$P(a,e) = \{4,8\}$

$P(a,c) = \{4,2,1\}$

$P(a,d) = \{4,5\}$

FIGURE 2

The matrix S in a certain way is a generalised inverse of I. For a given pair x,y of vertices the path P = P(x,y) with edges among 1,2,4,5,8 is unique and can immediately be read off from $I^+ \cdot \underline{P} = S \cdot \{x,y\}$.

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IS NEVANLINNA THEORY DEAD? - AN ESSAY

Richard M. Timoney

First of all, what is (or was) Nevanlinna theory? It is a far-reaching elaboration of the Picard theorem mentioned in all first courses on complex analysis: A non rational meromorphic function defined on the complex plane $\mathbb{C}$ takes (nearly) every value in the extended plane $\hat{\mathbb{C}}$ infinitely often, with at most two exceptional values.

The question of obtaining further information about the solutions of the equation $f(z) = a$, was studied by various people after Picard obtained his result (1880). If $f(z)$ is rational and non-constant, then there are a finite number (≥ 1) of solutions of $f(z) = a$, for all $a \in \hat{\mathbb{C}}$. (To be exact we must include $z = \infty$ in the domain of f to make this statement). Furthermore, if we count solutions of $f(z) = a$, according to their multiplicity, then the number of solutions is independent of a. Picard tells us that, for non-rational $f(z)$, if we avoid exceptional values a, then the number of solutions of $f(z) = a$, is countably infinite and thus independent of a.

However, something more exact is true about the "number" of solutions. Consider the following examples,

(i) The solutions of $e^{iz} = a$ are $z = \alpha + 2n\pi$, $n \in \mathbb{Z}$, where α is one solution (if $a \neq 0, \infty$).

(ii) $e^{iz^2} = a$ has solutions, $z = \pm\sqrt{\beta^2 + 2n\pi}$, $n \in \mathbb{Z}$, where β is one solution $(a \neq 0, \infty)$.

Intuitively, there seem to be "more" solutions in the second example, in the sense that the solutions are packed more densely.

A concept which expresses this is the counting function

$n_f(r,a) = n(r,a) =$ number of solutions of $f(z) = a$, in $|z| < r$. In the first example $n(r,1)$ is roughly $2r+1$, while in the second $n(r,1)$ is about $4r^2+1$. In both cases $n(r,1)$ and $n(r,a)$ have the same behaviour for large r (as long as $a \neq 0, \infty$). Nevanlinna, as we shall see, found a way to express the independence of $n_f(r,a)$ from a and the relationship of $n_f(r,a)$ to the size of f (for general f).

Using the counting function, Hadamard found a relation between the size of f and the size of $n(r,a)$ in the case of the entire function $f(z)$ (without poles). The <u>order</u> ρ of an entire function corresponds to the exponent of z in our examples (i) and (ii). It is defined to be

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}$$

where $M(r,f) = \sup\{|f(z)| : |z| \leq r\}$. The order $\rho(a)$ of $n(r,a)$ is

$$\rho(a) = \limsup_{r\to\infty} \frac{\log n(r,a)}{\log r}$$

Using infinite products, Hadamard (1893) showed that $\rho(a) \leq \rho$, for all $a \in \mathbb{C}$. Borel (1897) proved that $\rho(a) = \rho$, for (nearly) all $a \in \mathbb{C}$, with at most one exceptional $a \in \mathbb{C}$. (Note that $a = \infty$ is automatically exceptional in this context). An exceptional a could exist only for ρ a positive integer or $\rho = \infty$.

Borel's result was, of course, a considerable strengthening of Picard's theorem for the case of entire functions. Notice that it includes our two examples. Rolf Nevanlinna's celebrated contribution (1925) was to find a way to cope with the case of arbitrary meromorphic functions $f(z)$. He replaced the counting function $n(r,a)$ by a logarithmic integral

$$N(r,a) = \int_0^r n(t,a)\,\frac{dt}{t}$$

(changes are needed if $f(0) = a$). The difficulty was to find a replacement for the maximum modulus $M(r,f)$ which would measure the size of a meromorphic $f(z)$.

Nevanlinna's $T(r,f)$ - called the <u>characteristic function</u> of f - is more a generalization to the meromorphic case of $\log M(r,f)$ than of $M(r,f)$. Nevanlinna's idea was based on the following formula due to Jensen.

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta + N(r,\infty) - N(r,0)$$

Writing $\log^+ x$ to mean $\max(\log x, 0)$ and $\log^- x = \log^+(\frac{1}{x})$ (if $x > 0$), Nevanlinna rearranged the above equation

$$\frac{1}{2\pi}\int_0^{2\pi} \log^-|f(re^{i\theta})|\,d\theta + N(r,0) + \log|f(0)|$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \log^-|f(re^{i\theta})|\,d\theta + N(r,\infty).$$

He defined

$$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta + N(r,\infty)$$
$$= m(r,\infty,f) + N(r,\infty)$$

(again modifications needed if $f(0) = \infty$).

Now Jensen's formula says that

$$T(r,f) - T(r,\tfrac{1}{f}) = \log|f(0)|$$

(if $f(0) \neq 0,\infty$) and it is a simple matter to modify the argument to show that

$$T(r,f) - T(r, 1/(f-a))$$

is a bounded function of $r > 0$ (for each a). This is a statement along the lines that $f(z) = a$, and $f(z) = \infty$, have the same number of solutions, except that it is cluttered up with $m(r,\infty,f)$ and $m(r,a,f) = m(r,\infty,1/(f-a))$.

The term $m(r,\infty,f)$ can be explained as measuring the average growth of $\log|f|$ on the set where $|f| \geq 1$. Its role in $T(r,f)$ is subservient to that of $N(r,\infty)$, unless ∞ is an exceptional value - that is, unless $N(r,\infty)$ is not as large as $N(r,a)$ usually is. This was shown in a precise form by Nevanlinna.

Before elaborating on this, we note a basic fact about the characteristic function. The function $f(z)$ is rational if and only if

$$\liminf_{r\to\infty} \frac{T(r,f)}{\log r} < \infty .$$

This may be viewed as a generalization of Liouville's theorem (f entire, $M(r,f) \leq cr^n + c$, implies f a polynomial) because, if $f(z)$ is entire,

$$T(r,f) \leq \log^+ M(r,f) \leq 3\,T(2r,f)$$

(This inequality can be shown using the fact that $\log|f|$ is subharmonic).

Nevanlinna called

$$\delta_f(a) = \delta(a) = 1 - \limsup_{r\to\infty} \frac{N(r,a)}{T(r,f)}$$

the <u>deficiency</u> of the value $a \in \hat{\mathbb{C}}$. It is easy to see that, if f is not rational, then $\delta(a) < 1$ implies $f(b) = a$ has infinitely many solutions. Also $0 \leq \delta(a) \leq 1$ is always true. Nevanlinna showed that (if f is not constant) $\delta(a) \neq 0$ for at most countably many $a \in \hat{\mathbb{C}}$ and

$$\Sigma_a\, \delta(a) \leq 2$$

This is a quantitative version of Picard's theorem.

Nevanlinna's characteristic function $T(r,f)$ became the magic tool for studying the distribution of $f^{-1}(a) \subseteq \mathbb{C}$, for f(z) meromorphic. All sorts of results were obtained under various restrictions on the function - mainly restrictions on the order

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

Relations between f and f' were also investigated as were simultaneous solutions of $f(z) = a$ and $g(z) = a$. Ahlfors developed a geometrical approach to the characteristic function. The beauty of the field was that there were interesting results to be proved which were simple to state, but required ingenuity to derive. Nevanlinna's theory can justifiably be described as one of the greatest of mathematical theories. It is a marvellous simplification of the difficult problem of studying solutions of $f(z) = a$, which nevertheless has great depth.

Now, however, this great industry started by Nevanlinna seems to be suffering from the worldwide economic recession. One might argue that David Drasin hammered the last nail in the coffin when he settled one of the most fundamental outstanding questions. He showed (1977) that Nevanlinna's defect inequality $\Sigma\delta(a) \leq 2$ told the full story when f is unrestricted. Given a sequence $(a_n)_n$ of distinct elements of $\mathbb{C}$ and positive numbers d_n satisfying $\Sigma d_n \leq 2$, it is possible to find f(z) meromorphic with $\delta_f(a_n) = d_n$ and $\delta(a) = 0$ for a not one of the a_n's.

But can such a wonderful theory die? One might ask whether Euclidean geometry died centuries ago. After all, it is hard to find a major unsolved problem in Euclidean geometry. Of course the story changes considerably if we look at differential geometry, Riemannian geometry, Kähler manifolds, etc., which are the subjects one might imagine Euclid considering if he were around in this century.

So it seems to me to be unreasonable to point to the scarcity of really central open problems in value distribution

theory of functions of one variable and conclude that the field is dead. Rather, we should ask where we can use Nevanlinna's inspiration today. The answer is not yet completely clear, and it could be that future mathematicians may indeed say that Nevanlinna's theory lived only from 1925 to 1977.

There have been promising developments in the field of several complex variables. Considerable progress has been made in establishing defect relations for holomorphic maps $f: M \to N$ between certain non-compact complex manifolds M and certain compact manifolds N. The requirement on M is that it has a parabolic exhaustion function (example: $M = \mathbb{C}^m$) and N is a projective algebraic variety (example: $N = P_n\mathbb{C}$ = n-dimensional complex projective space). Deficiencies are defined with respect to subvarieties of N. It seems clear that the most general result has yet to be obtained, but here is a sample of what has been done. If $H_1, H_2, \ldots, H_q$ are hyperplanes in $P_n\mathbb{C}$ in "general position" and $f:\mathbb{C}^m \to P_n\mathbb{C}$ is a "transcendental mapping", then

$$\sum_{j=1}^{q} \delta(H_j) \leq n + 1.$$

The deficiency $\delta(H) = \delta_f(H)$ has, as before, the properties $0 \leq \delta(H) \leq 1$ and $f(\mathbb{C}^m) \cap H = \emptyset$ implies $\delta(H) = 1$.

The results obtained to date in several variables, though not completely satisfactory, have already yielded new nontrivial results about complex manifolds. So, Nevanlinna theory is not dead, but its state of health is uncertain.

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THE MEAN VALUE THEOREM FOR VECTOR VALUED FUNCTIONS: A SIMPLE PROOF

William S. Hall and Martin L. Newell

It is well known that the mean value theorem in one dimension extends readily to real-valued functions of several variables, but fails for the vector-valued case. For example, let $f(t) = (\cos t, \sin t)$ and suppose there is a point ξ in $(0,2\pi)$ such that $f'(\xi) = 0$. Then $-\sin\xi = \cos\xi = 0$, an impossible situation. A useful and correct generalization is the inequality

$$|f(y) - f(x)| < \sup_{0<t<0} ||f'(x+t(y-x))|| \, |y-x|$$

where $f:D\subset\mathbb{R}_n \to \mathbb{R}_m$ is a differentiable vector-valued function on a convex open set D, f' is the matrix $\partial f_i/\partial x_j$, $i = 1,2 ..m$, $j = 1,2 ..n$, $|\ |$ is the appropriate norm (in $\mathbb{R}_n$, or in $\mathbb{R}_m$), $||\ ||$ is the usual norm in the set of linear maps from $\mathbb{R}_n$ to $\mathbb{R}_m$, and x,y are arbitrary points in the domain D.

Many undergraduate calculus and analysis texts prove the mean value theorem in the real case but omit the result above. Those that do present this more general form usually give either a "sloppy" proof, using components, or a "slick" proof with the Hahn-Banach Theorem. Here we present a direct approach, requiring only the chain rule and the mean value theorem in $\mathbb{R}$. It is worth noting that f' at each point is a linear map (given by the Jacobian matrix) and that the usual norm for a linear map (matrix) is given by

$$\sup_{|x|=1} |Ax|.$$

However, other norms such as $(\Sigma\alpha_{ij}^2)^{\frac{1}{2}}$ where $A = (\alpha_{ij})$ are frequently used in advanced calculus courses. All we really use is that $|Ax| \leqslant ||A|| \, |x|$.

The result is certainly true if $f(y) = f(x)$. If not, form the function $\phi(t)$ by

$$\phi(t) = \langle f(y)-f(x), f(x+t(y-x))\rangle / |f(y)-f(x)|$$

where $\langle , \rangle$ denotes the inner product in $\mathbb{R}_m$. Then

$$\phi(1) = \langle f(y)-f(x), f(y)\rangle / |f(y)-f(x)|,$$

$$\phi(0) = \langle f(y)-f(x), f(x)\rangle / |f(y)-f(x)|,$$

$$\phi'(t) = \langle f(y)-f(x), f'(x+t(y-x))(y-x)\rangle / |f(y)-f(x)|.$$

In the last line we used the chain rule twice, once because f is differentiable, and once because the inner product is also differentiable. Of course, ϕ itself is well-defined because D is convex.

By the usual mean value theorem,

$$\phi(1)-\phi(0) = \phi'(t) = \langle f(y)-f(x), f(y)-f(x)\rangle / |f(y)-f(x)| = |f(y)-f(x)|,$$

and by the Schwarz inequality,

$$\phi'(t) < |f(y)-f(x)|\,|f'(x+t(y-x))(y-x)| / |f(y)-f(x)|$$

$$< \|f'(x+t(y-x))\|\,|y-x| \leqslant \sup_{0<t<1} \|f'(x+t(y-x))\|\,|y-x|.$$

This proves the theorem.

It is clear that if $\mathbb{R}_n$ is replaced by any Banach space and $\mathbb{R}_m$ is replaced by any real Hilbert space, then the method of the proof remains valid.

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A SIMPLE PROOF OF TAYLOR'S THEOREM

Raymond A. Ryan

This proof uses only the Fundamental Theorem of Calculus in the form:

$$\frac{d}{dx}\int_x^b g(t)\,dt = -g(x)$$

Taylor's Theorem: If f is n+1 times continuously differentiable in an open interval containing the points a and b, then

$$f(b) = f(a) + f'(a)(b-a) + \ldots + \frac{1}{n!} f^{(n)}(a)(b-a)^n + \frac{1}{n!}\int_a^b f^{(n+1)}(t)\,(b-t)^n\,dt$$

Proof:

Let $F(x) = f(x) + f'(x)(b-x) + \ldots + \frac{1}{n!} f^{(n)}(x)(b-x)^n + \frac{1}{n!}\int_x^b f^{(n+1)}(t)(b-t)^n dt$

Then F is differentiable, and

$$F'(x) = f'(x) - f'(x) + f''(x)(b-x) - \ldots - \frac{1}{(n-1)!} f^{(n)}(x)(b-x)^{n-1} + \frac{1}{n!} f^{(n+1)}(x)(b-x)^n - \frac{1}{n!} f^{(n+1)}(x)(b-x)^n$$

$$= 0$$

Hence $F(x)$ is constant. Furthermore, $F(b) = f(b)$, and so

$$f(b) = F(b) = F(a) = f(a) + \ldots + \frac{1}{n!} f^{(n)}(a)(b-a)^n + \frac{1}{n!}\int_a^b f^{(n+1)}(t)(b-t)^n\,dt$$

Q.E.D.

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LINEAR INDEPENDENCE FOR CORK

R.E. Harte

In Summer 1978, Question 4 of the Second Arts Matrix Theory paper at U.C.C. begins as follows:

> "Explain what it means for a finite subset of a real vector space to be linearly independent. If $\{x_1, x_2, x_3\}$ is linearly independent and t is real, show that
>
> A $\{x_1 - tx_2, x_2 - tx_3, x_3 - tx_1\}$ linearly independent $\Longleftrightarrow t \neq 1$".

The embarrassing fact is that the solution is wrong, but in a rather subtle way. In fact provided the three vectors x_1, x_2 and x_3 are distinct from one another, the statement (A) is correct, as can be and was verified by Second Year Arts students. Suppose however

B $\qquad x_1 \neq x_2 \neq x_3$

still assuming that the set $\{x_1, x_2, x_3\} = \{x_1, x_2\}$ is linearly independent, then:

C $\{x_1 - tx_2, x_2 - tx_3, x_3 - tx_1\}$ linearly independent $\Longleftrightarrow t = 0$.

Finally if $x_1 = x_2 = x_3 \neq 0$ then the statement (A) is again valid.

MORAL: linear independence should be defined for sequences, not sets.

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COMPUTER SCIENCE AND THE MATHEMATICS CURRICULUM

Anthony Karel Seda

> "As soon as an Analytical Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will then arise - By what course of calculation can these results be arrived at by the machine in the shortest time?"
>
> Charles Babbage, 1864.

§1. Introduction

In any discussion of computer science and its relationship with mathematics, from an educational viewpoint, certain obvious questions come to the fore:

(1) What is the role of mathematics in computer science education?

(2) What is the role of computer science in mathematics education?

(3) What is, or has been, the response of mathematicians to computer science in relation to the mathematics curriculum?

There are two viewpoints, at least, from which these questions can be contemplated. One is that of the computer scientist engaged in teaching/research in a third level institution peering over the ramparts at the mathematicians. The other, which is ours, is that of the mathematician similarly engaged in teaching/research and similarly peering at the computer scientists. Having thus declared my vantage point, and for reasons of space, I wish to concentrate here on Question 1, and only to touch on Questions 2 and 3. Specifically, I wish to bring to the attention of readers of the *Newsletter* the discussion contained in the articles [4] and [5] of Professor Anthony Ralston. Ralston's conclusion is,

> "It is time to consider (i.e., try)
> an alternative to the standard
> undergraduate mathematics curriculum
> which would give discrete analysis an
> equivalent role to that now played
> by calculus in the first two years
> of the undergraduate curriculum".

In §3 I have listed the topics which Ralston proposes in order to achieve his aim. Actually, [4] is a detailed version (83 pages) of [5], and [5] will suffice to support the main thread of the argument here.

In the quotation above, Babbage is of course talking about algorithms, and algorithms in the words of Knuth [3],are "... really the central core of the subject (computer science), the common denominator which underlies and unifies the different branches". Indeed, Knuth has, just prior to writing this, chosen to describe computer science as "the study of algorithms". Now, as confirmed by Knuth, the study of algorithms is very mathematical and it is worth stating this fact in order to dispose of the short, negative reply to Question 1 which just might be proposed from the other vantage point! Further confirmation of this fact, i.e. of the mathematical nature of computer science, can be gained by consulting the list of topics in Section 68 of the 1980 Subject Classification of Mathematical Reviews, or by actually reading some recent reviews in this section; see also [1].

2. Some History and Some Educational Philosophy

Whilst our main discussion centres on Question 1, it will not be out of place to devote a few words to Questions 2 and 3.

One might wonder why it is today that there is a division

between computer scientists and mathematicians, and that there is not more sympathy shown by each for the other's subject. After all, computer science grew out of mathematics and in its early days, some twenty five-thirty years ago, it was necessarily closely bound to mathematics. However, today, digital computers vastly predominate over analogue computers and digital computers are essentially discrete. What, though, is being taught in most mathematics departments? I suspect that it is largely either continuous mathematics, such as analysis, or relatively abstract mathematics, to the great exclusion of discrete mathematics. Certainly this is true in U.C.C., but may be less so in non-university departments. Indeed, Ralston [4] argues that in American universities the present-day structure of the mathematics curriculum (mainly calculus/linear algebra - at least in the first two years) has come about for reasons more to do with history and inertia (human) than with a judicious choice of topics to meet the educational requirements of those students other than majors in physical science and engineering.

As far as Question 3 is concerned, there are at least three discernible responses:

(a) Ignore the problem - maybe it will go away.

(b) Continue teaching traditional material but illuminate it with examples/projects worked on the computer.

(c) Meet the problem head-on and design/update courses to more nearly meet the needs of those students studying computer science.

Response (a) needs no comment; (b) is outside the scope and limits of this note but surely has a lot of merit, see [2] and its references for some experiments, and also elsewhere in this *Newsletter*; (c) is the main topic of this discussion, see §3.

Before leaving this section, there is another aspect worth noting. Mathematics courses are widely held to be educational, irrespective of their content, for purposes of training the mind. Can the same be said of computer science? This touches on Question 2, because the solution Ralston has in mind for (c) is best framed in terms of a mathematical sciences degree programme and, naturally, the educational value of such a programme, over and above its content, has to be considered. To quote G.E. Forsythe, see [3], "The most valuable acquisitions in a scientific or technical education are the general-purpose mental tools which remain serviceable for a lifetime. I rate natural language and mathematics as the most important of these tools, and computer science as a third". Some of Knuth's own views on this can also be found in [3].

3. Ralston's Proposals for the Mathematics Curriculum

I want, now, to list the topics which Ralston believes could form a suitable basis for the discrete component in a better balanced curriculum for mathematics students, computer science students and others. The headings below are taken from [4] and [5] and the topics from [4].

i) Algorithms and their Analysis. Topics: the notion of an algorithm; notation for expressing algorithms; basic analysis of algorithms.

ii) Introductory Mathematical Logic. Topics: the notion of mathematical proof; the propositional calculus; Boolean algebra; the notation of the predicate calculus; introduction to the verification of algorithms.

iii) Limits and Summation. Topics: the notion of infinite processes; ideas of convergence and limits; limits of discrete functions; summation.

iv) Mathematical Induction. Topics: the principles of induction; examples of induction proofs.

v) The Discrete Number System. Topics: real numbers and finite number systems; definition and laws of the discrete number system; number bases other than 10.

vi) Basic Combinatorial Analysis. Topics: the binomial theorem and Stirling numbers; permutations and combinations; simple combinatorial algorithms.

vii) Difference Equations and Generating Functions. Topics: recurrence relations; linear difference equations and their solution; generating functions.

viii) Discrete Probability. Topics: basic laws; discrete probability distributions; random number generation; queueing theory; probability and algorithm analysis.

ix) Graphs and Trees. Topics: basic definitions and theorems of graph theory; path and colouring problems; tree enumeration and binary trees.

x) Basic Recursion and Automata Theory. Topics: basic definitions; recursive algorithms; recursive functions; regular sets and expressions; finite state machines; languages and grammars; Turing machines.

In connection with this list, the following points should be noted:

(A) These topics are only suggestions. Moreover, it is assumed by Ralston that they will be presented in some combination with abstract algebra, linear algebra, analysis etc., for in [4] it is observed that "... there are numerous areas of computer science where calculus plays an important role..." Moreover, a better balanced curriculum is being argued for, but not a complete reversal in favour of discrete mathematics.

(B) These topics are, with the possible exception of some in viii), mathematics subjects and as such are best taught by mathematicians.

(C) Due to the differences between the educational systems here and in America, certain additions and subtractions might need to be made were these proposals to be adapted to fit into our context (Probably extra more advanced material such as more computability theory or computational complexity could be added for, say, honours students).

(D) These proposals are at least worthy of consideration, for Professor Ralston has wide experience in both computer science and mathematics and backs up his suggestions with an exhaustive study.

More questions are asked here than are answered. For example, consideration needs to be given to the feasibility of such topics for various types of student, ranging from students of management through to honours mathematics students. But space permits no more comment, and for answers to such questions the reader must either consult [4] and [5] or, if Ralston [5] page 484 is correct, undertake experiment for himself or herself.

Educational problems are not usually very well defined; they are likely to be controversial and to raise temperatures. Indeed it may be that Ralston~~'s criticism~~ does not apply here and that all is well. If not, and this article creates some discussion or starts people thinking about the problems raised here, then it will have achieved its purpose. We hardly need reminding in 1983 that computer science is a major undergraduate subject. But what has perhaps not been widely recognised yet is the fact that the next generation of students will be taught computer science in secondary schools by those currently studying it at third-level. Future incoming students may therefore elect to study computer science "because it is familiar" just as many do now, I suspect, in the case of mathematics.

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USING MICROCOMPUTERS IN UNIVERSITY MATHEMATICS TEACHING

A.W. Wickstead

Until the last few years, most attempts to involve computers in the teaching process involved sitting a student at the terminal of a large computer and attempting to perform the whole teaching process by presenting information which was then tested. Apart from being very limited in what could be taught in this way, such projects have tended to be very expensive. They involve relatively large computers, many terminals to them and a lot of (expensive) programming effort. All of this expenditure has to be incurred before the technique can be tried, so its use has, of course, been very limited.

The advent of cheap microcomputers in recent years has enabled the less well financed University Mathematics department to acquire a computer and, experiment with its use in teaching. With such limited resources (and no programming assistance) it would be foolish to hope to emulate the large projects that have been attempted in the past. There are however uses that can be made of a single microcomputer in front of a class, especially if reasonably good quality graphics is available. All our uses of microcomputers in teaching Pure Mathematics at Q.U.B., since we started three years ago, have been of this nature, although other departments make microcomputers available to students for numerical work in statistics, numerical analysis etc. Our approach was easy to introduce as it fitted easily into our existing, fairly traditional, teaching methods. This meant that the technique could be tried by one individual without needing any departmental policy decision.

Typically I have used a microcomputer to introduce a topic before teaching it in the normal way. The idea has been to augment the normal exhibiting of a few examples to illustrate a definition, such as that of uniform convergence. The computer can draw graphs more accurately and quicker than I can, so that many more examples can be provided to drive home the idea of uniform convergence before trying to work with it. In order that students have a record of the session they must either be given time to copy what they see or printed copies of the graphical output of the computer. The temptation to proceed too fast is even worse here than with pre-prepared transparencies! One of the most useful features of using a microcomputer for illustrative purposes, rather than many drawings prepared in advance, is that students can suggest examples that they would like to see and then have them drawn immediately. I have found students far more willing to make suggestions in such a context than they are in any normal teaching session and that the increased interaction seems to persist beyond the actual session (at least for a few days).

Topics that we have approached in this way include uniform convergence, Riemann integration, continuity, differentiation, Turing machines, approximations of the Binomial distribution and a naive approach to Fourier series. We hope

to add to this list in the next few months. The bias towards Analysis in this list of topics reflects the teaching commitments of myself and others interested in using the computer rather than any inherent limitation in the technique. We have had most success with students who might have difficulty even understanding the definition of a concept (more than might be supposed). These can progress further and more quickly than might otherwise be the case. The good students gain little (this reflects the results of a survey done with schoolchildren recently) but it does mean that we are helping those who need it most. Opportunities for using microcomputers seem to lessen as the level of courses increases, although there are exceptions.

All our teaching has been on an ITT2020 (virtually identical to an APPLE II except for improved graphics). This works at a reasonable speed and can provide tolerably good black and white graphics (but not such good colour). We have found it desirable to improve its speed by adding an Arithmetic Processor Card, and (at the time of writing) have just taken delivery of improved graphics for it. We have not found much use for colour, but some programs certainly benefit from it. For example, one of our programs compares the graph of a continuous function with that of a polynomial approximation to it. Once the approximation is reasonably good, colour is essential to distinguish the two graphs. It is not realistic to load a sizeable program from cassette tape in front of a class, so disk storage (or something similar) is vital. We have also found it useful to have a printer available. This has greatly aided program development and is also useful for providing a printed record for the class of some of the computer's output. In some cases, such as simulation of the action of a Turing machine, it would not be possible for students to make a copy as they watched.

From our experience, not only in teaching at University level using a microcomputer but also from training teachers to use them, we have isolated several critical features of programs that are to be useful. First and foremost, the program should teach rather than do. There are many programs that multiply two matrices together, but very few that try to teach how it is done. A good program is as flexible as possible. This allows the maximum interaction with a class and also allows more people to fit it into their own teaching style. Often it is little more work to write a program that will handle many examples rather than just one, although that is not always the case. If many people are to use a program it should be as easy to use as possible. No-one is going to read a twenty-page manual before using a program for half an hour, yet that is what some programmers expect! Choices that need to be made should be clearly stated and a relatively small number presented at once. Far more could be done to guide people through a program than is being done. For instance, there is little point in offering the choice of saving a graph onto a disk file to a person who has not yet drawn a graph! By such restrictions it is possible to lead a person gently into a program. The final feature of a program that we have found desirable is frequent opportunities to save its output onto disk files. If this has been done in advance of a class they can be printed for distribution to a class or many of them can be displayed to a class in far less time than it took to produce them. Before we obtained our Arithmetic Processor Card we frequently found it necessar to do this in order to present the number of examples that we wanted in the time available.

There is, unfortunately, no source of programs for teaching mathematics at the University level. About eighteen months ago I contacted about twenty Universities, Polytechnics and Teacher Training Colleges in the U.K., that I knew to have an interest in the field, with a view to setting up a program exchange. The response was negligible. What has been done elsewhere tends to be on larger machines or on exotic hybrid systems. Anyone setting up such a system should be reconciled to "going it alone" on program development as any specific combination will be rare. Even though little is avail-

able now, it is worth bearing the possibility of exchanging software in mind when selecting a system and when programming for it. For instance, all our programs can be modified in a few minutes to run on a standard APPLE II, yet we can also use our extra features if they are available. If Universities in the Republic of Ireland were to standardise on a microcomputer and exchange software for it, they would soon find themselves in a far more favourable position in this respect than the U.K. Universities.

Our programs require a 48 K APPLE II with a language card and at least one disk drive. Anyone seeking copies is welcome to write to me for further details. The programs are all written in APPLE PASCAL 1.1 and I have described some of the programming difficulties in "Interactive Mathematical Programming", in COMPUTER AGE 14 (January 1981) 15-17.

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"DEVELOPING MATHEMATICAL THINKING"

By *Ann Floyd*

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This book comprises a selection of articles, some of which have been published elsewhere, and some for the first time in this publication. The major objective is to make explicit strategies for developing mathematical thinking in primary and post-primary schools. It attempts to achieve this in the first place by examining the nature of mathematical thinking itself which is seen largely as the type of thinking that is needed to solve unfamiliar mathematical problems. While recognition is given to the importance of computational skills as mathematical tools, the essential emphasis is on the acquisition of generalised methods of attacking mathematical problems not previously met.

This concern with problem solving is not a present-day phenomenon; educators over the past hundred years have expressed a similar concern and official statements over this period in the United Kingdom display a nagging unease about a perceived imbalance between pupils' ability to deal with computation on the one hand and problem-solving on the other. Recent investigations by the Drumcondra Research Centre and the Department of Education Inspectorate suggest that the problem is equally grave in this country.

The durability of this concern is well substantiated in the first section of this book which sets the problem in its historical setting quoting from H.M. Inspectorate's reports dating back to 1875, the Hadow Report (1931), the Plowden Report (1966), Biggs (1967), Primary Education in England (1978) et al. Readers will no doubt marvel at the self-assurance of the H.M.I. in 1876 who considered that mathematics "is a subject which seems to be beyond the comprehension

of the rural mind". Insights of this nature are not the exclusive prerogative of the Victorian mind. Readers will not be less impressed by the confident assertion of a contemporary author that "the reasons for teaching the standard written algorithms are out of date and that it is time we took note of this". However this book is "designed to evoke the critical understanding of students" and to this extent at least it should be successful. The first section on the book provides the student with a carefully selected compendium of extracts from the major reports on mathematics teaching and learning in the United Kingdom over the past fifty years. Most teachers and student teachers will find these articles absorbing reading, and some will be tempted to delve back into the original reports and a few may be tempted to engage in similar research in this country.

The second section introduces the reader to two major reports by H.M. Inspectors of Schools - Primary Education in England and Aspects of Secondary Education in England. These reports describe contemporary mathematics programmes and classroom procedures in English schools and provide interesting comments ranging from the efficacy of group teaching in the primary school to the dilemma of postprimary mathematics teachers attempting to meet the contradictory demands of a wide range of mathematics in newer schemes as against demands from local communities for concentration on a narrow range of traditional arithmetical skills. The dilemma will not be unknown to Irish teachers of mathematics. The final survey in this section (Mathematical Development, Foxman et al.) provides a national picture of pupils' mathematical capabilities.

Section three is concerned with the nature of mathematical thinking and strategies for developing it. Skemp's article on concept formation is of enduring interest. He holds that primary concepts are not acquired through analysing definitions but rather as a result of abstracting a general truth or law from a variety of experiences and observations. The school curriculum must therefore provide adequate opportunity for active exploration so that the pupil will construct the mathematical rule before applying it. Such an approach is very much in the Piagetian tradition and is indeed the foundation stone on which the Irish Primary School Curriculum 1971 is built. This is not to say that Piaget is given unquestioning fealty in this reader. An article by E.R. Hughes describes a study of the order of acquisition of the concepts of weight, area and volume which casts serious doubts on some of the findings of Piaget.

Section four describes a wide range of activities designed to develop mathematical thinking. The merits of informal mathematical procedures are extolled and on the whole this emphasis is to be welcomed. But the reader may feel that the evidence suggests that we may build on the informal in order to construct the more economic formal algorithms rather than discard them altogether. The broad thrust of this section would appear to be a general denigration of the value of the traditional algorithms. Such an approach can be justified in the interest of evoking the "critical understanding of students". It is to be hoped, however, that the students will value the baby rather more than the bathwater.

Fielker's article on primary school geometry will be of considerable interest to primary and postprimary teachers wishing to think in a unified structural way about geometry rather than presenting pupils with an assortment of enjoyable but unrelated experiences of shapes.

The final section is concerned with the views of practising teachers who have tried to develop mathematical thinking in their classrooms using "do, talk and record" approaches. Practical considerations such as grouping within classes, pupil discussions, the creation of classroom atmosphere, the development of a consistent school policy etc. are discussed in the light of teachers' experience at primary and postprimary level.

This volume must surely be regarded as essential reading for students in education departments in Universities and Colleges of Education. It immerses the student in the major contemporary problems facing teachers of mathematics at primary and junior cycle postprimary levels. It will also be invaluable for many mathematics teachers' study groups interested in identifying, investigating and researching problems in mathematics teaching and contributing towards the development of better mathematics curricula. Finally it should help to minimize the feeling of isolation experienced by many mathematics teachers striving to enthuse their students with an awareness of the order, unity and beauty of mathematics.

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PROBLEMS

First, the solutions to the December problems.

1.
$$I = \int_0^{\frac{1}{2}\pi} \frac{dt}{1+\tan^{\sqrt{2}} t} = \frac{\pi}{4}.$$

The trick with this is to substitute $x = \frac{1}{2}\pi - t$. Then

$$I = \int_0^{\frac{1}{2}\pi} \frac{dx}{1+\cot^{\sqrt{2}} t} = \int_0^{\frac{1}{2}\pi} \frac{\tan^{\sqrt{2}} x}{\tan^{\sqrt{2}} x+1}\, dx ,$$

so that

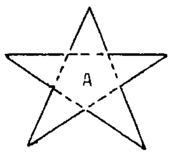
$$2I = \int_0^{\pi} \frac{1+\tan^{\sqrt{2}} t}{1+\tan^{\sqrt{2}} t}\, dt = \frac{\pi}{2}.$$

2. For a set A in $\mathbb{R}^n$, aSb means that the line segment joining a to b lies in A. Does the condition

(*) "If $a,b,c \in A$ then at least one of the following holds
aSb, bSc, cSa ".

imply that A is the union of two convex sets?

The answer is <u>no</u> as the following set shows.



To see that A cannot be written as the union of two convex sets, consider where the five outer vertices would end up.

Now, some more problems.

1. Let Γ be a C^1 closed curve in $\mathbb{R}^3$. Must there exist a pair of points in Γ at which the tangent vectors to Γ are parallel but of opposite sense?

2. An electrician is faced with a bundle of n (>2) unmarked identical wires running from the ground floor to the top floor of a block of flats.

With the aid of a bell and a battery, and making only one visit to the top floor, how should he proceed to match up and label the top and bottom ends of the wires?

3. Prove, or disprove, that

$$\inf\{|n \sin n| : n=1,2,3,\dots\} = 0$$

(submitted by *T. Laffey*)

4. The following problem arises in Control Theory (Simultaneous Stabilization):

Given rational functions d_1, d_2, n_1 and n_2 which have no poles in the closed unit disk $\bar{D}$ and whose zero sets are pairwise disjoint in $\bar{D}$, when is it possible to find a rational function m without poles in $\bar{D}$ such that neither of the functions d_1+mn_1, d_2+mn_2 has any zeros in $\bar{D}$?

If each of the functions d_1, d_2, n_1 and n_2 is self-conjugate (a function f is self-conjugate if $f(\bar{z}) = \overline{f(z)}$), when is it possible to find a self conjugate function m as above?

This problem is equivalent to the following (at least when the zeros of d_1, d_2, n_1 and n_2 are simple): When can one find a rational function r without poles or zeros in $\bar{D}$ such that

$$r(z) = \frac{n_1(z)}{n_2(z)}$$

if and only if $(n_1d_2 - n_2d_1)(z) = 0$.

Remark: If we only need n_1+md_1 to be non-vanishing, there is no restriction on solving the problem without the self conjugation condition: with this condition, the problem can be solved if and only if the values of d_1 at all <u>real</u> zeros of n_1 in $\overline{D}$ have the same sign.

(submitted by *J.J. Murray*)

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WORD CONSERVATION

To counter the recent British changeover from the English billion, which is worth one million million, to the American billion, of one thousand million, we advocate the introduction of the Irish thousand, worth one hundred hundred, as a substitute for the English and American thousand, which is ten hundred. This would give an Irish billion of ten to the power of sixteen, which the Americans would have to call ten quadrillion. By the time we in Ireland have used up one new word therefore the Americans will have used up three, and be well on the way to their fourth.

AUTONOMOUS NOTATION

$e^{\text{xponential}}$; $\log_a\text{rithm}$; $\int e^x = f(u_n)$; $\text{CAT}_{\text{A}_{\text{STROPHE}}}$

ALGEBRA CONFERENCE

MARY IMMACULATE COLLEGE OF EDUCATION, LIMERICK

12-13 NOVEMBER, 1982

The principal invited speaker was Professor John Thompson (Cambridge) who is well known to algebraists for his many fundamental contributions to the theory of finite groups and in particular to the classification of finite simple groups. His first lecture took the form of an anecdotal tour through the background to the Odd Order Paper highlighting the decisive influence of Frobenius, Burnside, Brauer and Suzuki. Without the individual achievements of these great mathematicians the project could not have been undertaken.

In his second, more technical, lecture Professor Thompson sought to demonstrate the fundamental and intimate connexions that exist between the theory of modular functions and finite group theory. The precise nature of the relationship is still not clearly understood and new facts are constantly emerging. In particular Professor Thompson announced his recently proved result that the group known as Monster fits remarkably well into this scheme as a Galois group over the rationals. He discussed the possibility that there may lie the foundation of an overall theory which would put the twenty six sporadic simple groups into a unified context, thereby satisfying those who, like himself, dislike regarding these exceptions as mere 'bumps in the universe'.

There were three other invited lectures given, appropriately, by speakers from Cork, Dublin and Galway.

Des McHale (University College Cork) described the lives of Boole and Hamilton seeking to explain why there was so little contact between them, despite the fact that they lived in Ireland at the same time for a period of fourteen years. In a detailed analysis drawn from his own work on Boole and a recently published biography of Hamilton by Hankins, he showed that this was due essentially to a conflict of personalities and to differences in social class.

Tom Laffey (University College Dublin) gave a survey of results on similarity and congruence of matrices. In this most informative talk Professor Laffey referred particularly to the similarity (well known) and congruence (recently proved by Gow among others) of a matrix with its transpose. He mentioned various refinements of these and concluded with a discussion of orthogonal similarity.

The third invited lecturer was Martin Newell (University College Galway) who has been investigating metabelian groups of exponent p^n in general and of exponent 8 in particular. He showed that for $m \geq 4$ the free group of exponent 8 on m generators has class at most 3m + 1. This combined with a result of Liebeck gives the class as exactly 3m + 1. In doing so he gave us some insight into the trials and tribulations of the commutator calculus!

Eight shorter talks were given at the conference on subjects variously algebraic. The speakers were: M.J. Curran (Otago, N.Z. and Oxford), D.A. Towers (Lancaster), A.R. Prince (Edinburgh), B. Goldsmith (Dublin Institute of Technology), J. Hannah (U.C.G.), F. Holland (U.C.C.), R. Gow (U.C.D.) and D.W. Lewis (U.C.D.)

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SEMINAR ON MATHEMATICS EDUCATION

This seminar, which was held in St. Patrick's College, Drumcondra, on November 13th, 1982, was organised by the Irish Mathematics Teachers Association and was aimed at teachers of mathematics at all levels. It focused on important aspects of mathematics teaching including pupil characteristics, motivation, talented pupils, and quality of textbooks. Approximately 150 teachers from primary, post-primary and third-level schools and colleges attended the seminar.

For the first lecture, Dr. Vincent Greaney and Miss Mary Hegarty of the Educational Research Centre, St. Patrick's College, presented their findings of a research study of factors relating to achievement in mathematics in a sample of fifth grade pupils of above average ability. Their analysis suggested that pupil variables, such as verbal ability and reading attainment, together with background variables, such as pressure for achievement and mother's educational level, account for (predict) more than two-thirds of the variation in mathematics attainment at this grade-level. It also suggested that home background variables are more related to success in computation and problem-solving, whereas personal variables are more related to success with mathematical concepts. The findings highlight the apparent importance of non-school factors in determining the performance of fifth grade pupils in mathematics.

The second session consisted of a talk by Mr. Peter McGrattan of St. Mary's College of Education, Belfast, on techniques for motivating students to learn mathematics. He demonstrated to the audience how he makes abstract topics in mathematics more meaningful and interesting to his students by guiding them to comprehend the significance of the topics in such everyday recreational activities as golf, snooker, darts, etc.

Mr. McGrattan was follwed by Mr. Francis Douglas of the U.C.C. Education Department who talked about teaching mathematics creatively to ordinary and mathematically talented students. He emphasised the importance of encouraging divergent thinking and generalisation on the part of students in their approach to problem-solving and illustrated his points with examples from the teaching of conic sections and the volume of regular shapes. He also outlined a viewpoint on the role of the teacher in catering for the mathematically talented students.

For the final session Miss Catherine Mulryan of the Education Department in St. Patrick's College, Dublin, presented the findings of an analysis of four mathematics textbook series currently in use in Irish primary schools. The study examined the textbooks under the following headings: content coverage and continuity, presentation and consolidation, readability and technical characteristics such as use of colour, type size etc. The results indicated substantial differences among the four series of textbooks on all of these groups of characteristics.

The large attendance and the number of positive comments in the follow-up questionnaire suggested that there is much interest among mathematics teachers in the area of mathematics education.

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CONFERENCE ANNOUNCEMENTS

Finite Element Programming with Special Emphasis on Semiconductor Device and Process Modelling, Galway, Ireland, June 13-14, 1983

In association with the NASECODE III Conference, an International Short Course on Finite Element Programming with special emphasis on Semiconductor Device and Process Modelling will be held in the Great Southern Hotel, Galway, Ireland on June 13th and 14th, 1983. The conference itself will take place on the three subsequent days.

The 16 invited lecturers are:

T. Arnborg	Royal Institute of Technology, Stockholm
K. Board	University College, Swansea
A.R. Boothroyd	Carleton University, Ottawa
W.T. Cochran	Bell Laboratories, Allentown
W.T. Coffey	Trinity College, Dublin
R.W. Dutton	Stanford University, Stanford
W. Fichtner	Bell Laboratories, Murray Hill
A.F. Kravchenko	Institute of Semiconductor Physics, Novosibirsk
P. Mole	General Electric, Wembley
E. Palm	Catholic Unversity of Louvain, Louvain la Neuve
D.N. Pattanayak	Rockwell International Corporation, Anaheim
D.J. Rose	Bell Laboratories, Murray Hill
M. Rudan	Institute of Electronics, Bologna
A.V. Rzanov	Academy of Science, Novosibirsk
C.W. Trowbridge	Rutherford Laboratories, Didcot
N.N. Yanenko	Institute for Theoretical and Applied Mechanics, Novosibirsk

Each lecturer will present a tutorial or review lecture on a topic in which he has a special interest or knowledge. The course will be suitable therefore both to those people wishing to enter this area for the first time, and to those who feel the need for a review of recent results in areas other than their own.

An exhibition of relevant books, journals and mathematical software will be held in conjunction with the short course.

For a complete programme of the Short Course and the Conference please contact:

NASECODE III Secretariat,
c/o Boole Press Ltd.,
P.O. Box 5,
51 Sandycove Road,
Dun Laoghaire.
Co. Dublin,

Telephone 808025.

Summer School on Combinational Optimization

This Summer School will be held at the National Institute for Higher Education, Dublin, from July 4th to July 15th,1983.

Each of the invited speakers will deliver an introductory lecture on his chosen area and a lecture devoted to recent research in this area. A limited number of contributed papers will complement the invited papers.

The following list of topics to be covered is liable to some modification: Vehicle Routeing, Distribution Problems, Polyhedral Combinatorics, Probabilistic Analysis of Deterministic and Probabilistic Algorithms, Networks and (Poly)Matroids, Parallel Algorithms for Combinatorial Problems, Integer Programming (Decomposition Methods), Integer Programming (L.P. Based Methods), Polytopes and Complexity, Scheduling and Complexity, Blocking and Antiblocking.

The invited speakers include: N. Christofides (London), M.L. Fisher (Philadelphia), M. Grotschel (Bonn), R.M. Karp (Berkeley), E.L. Lawler (Berkeley), J.K. Lenstra (Amsterdam), G.L. Nemhauser (Ithaca), M.W. Padberg (New York), C.H. Papadimitriou (Athens), A.H.G. Rinnooy Kan (Rotterdam), L.E. Trotter (Ithaca).

For further information contact Secretariat, Summer School on Combinatorial Optimization, N.I.H.E., Glasnevin, Dublin 9.

Galway Group Theory Conference

The next Group Theory conference at University College, Galway, will be held on May 13th, 14th, 1983. The list of speakers includes:

Peter Cameron (Oxford)
Gerard Enright (Limerick)
John Hannah (Galway)
Colin Walter (Dublin)
Richard Watson (Maynooth)

For information contact Dr. T. Hurley, Mathematics Department, University College, Galway.