IS NEVANLINNA THEORY DEAD? - AN ESSAY

Richard M. Timoney

First of all, what is (or was) Nevanlinna theory? It is a far-reaching elaboration of the Picard theorem mentioned in all first courses on complex analysis: A non rational meromorphic function defined on the complex plane $\mathbb{C}$ takes (nearly) every value in the extended plane $\hat{\mathbb{C}}$ infinitely often, with at most two exceptional values.

The question of obtaining further information about the solutions of the equation $f(z) = a$, was studied by various people after Picard obtained his result (1880). If $f(z)$ is rational and non-constant, then there are a finite number (≥ 1) of solutions of $f(z) = a$, for all $a \in \hat{\mathbb{C}}$. (To be exact we must include $z = \infty$ in the domain of f to make this statement). Furthermore, if we count solutions of $f(z) = a$, according to their multiplicity, then the number of solutions is independent of a. Picard tells us that, for non-rational $f(z)$, if we avoid exceptional values a, then the number of solutions of $f(z) = a$, is countably infinite and thus independent of a.

However, something more exact is true about the "number" of solutions. Consider the following examples,

(i) The solutions of $e^{iz} = a$ are $z = \alpha + 2n\pi$, $n \in \mathbb{Z}$, where α is one solution (if $a \neq 0, \infty$).

(ii) $e^{iz^2} = a$ has solutions, $z = \pm\sqrt{\beta^2 + 2n\pi}$, $n \in \mathbb{Z}$, where β is one solution ($a \neq 0, \infty$).

Intuitively, there seem to be "more" solutions in the second example, in the sense that the solutions are packed more densely.

A concept which expresses this is the counting function

$n_f(r,a) = n(r,a)$ = number of solutions of $f(z) = a$, in $|z| < r$. In the first example $n(r,1)$ is roughly $2r+1$, while in the second $n(r,1)$ is about $4r^2+1$. In both cases $n(r,1)$ and $n(r,a)$ have the same behaviour for large r (as long as $a \neq 0, \infty$). Nevanlinna, as we shall see, found a way to express the independence of $n_f(r,a)$ from a and the relationship of $n_f(r,a)$ to the size of f (for general f).

Using the counting function, Hadamard found a relation between the size of f and the size of $n(r,a)$ in the case of the entire function $f(z)$ (without poles). The <u>order</u> ρ of an entire function corresponds to the exponent of z in our examples (i) and (ii). It is defined to be

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}$$

where $M(r,f) = \sup\{|f(z)| : |z| \leq r\}$. The order $\rho(a)$ of $n(r,a)$ is

$$\rho(a) = \limsup_{r\to\infty} \frac{\log n(r,a)}{\log r}$$

Using infinite products, Hadamard (1893) showed that $\rho(a) \leq \rho$, for all $a \in \mathbb{C}$. Borel (1897) proved that $\rho(a) = \rho$, for (nearly) all $a \in \mathbb{C}$, with at most one exceptional $a \in \mathbb{C}$. (Note that $a = \infty$ is automatically exceptional in this context). An exceptional a could exist only for ρ a positive integer or $\rho = \infty$.

Borel's result was, of course, a considerable strengthening of Picard's theorem for the case of entire functions. Notice that it includes our two examples. Rolf Nevanlinna's celebrated contribution (1925) was to find a way to cope with the case of arbitrary meromorphic functions $f(z)$. He replaced the counting function $n(r,a)$ by a logarithmic integral

$$N(r,a) = \int_0^r n(t,a)\,\frac{dt}{t}$$

(changes are needed if $f(0) = a$). The difficulty was to find a replacement for the maximum modulus $M(r,f)$ which would measure the size of a meromorphic $f(z)$.

Nevanlinna's $T(r,f)$ - called the <u>characteristic function</u> of f - is more a generalization to the meromorphic case of $\log M(r,f)$ than of $M(r,f)$. Nevanlinna's idea was based on the following formula due to Jensen.

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta + N(r,\infty) - N(r,0)$$

Writing $\log^+ x$ to mean $\max(\log x, 0)$ and $\log^- x = \log^+(\frac{1}{x})$ (if $x > 0$), Nevanlinna rearranged the above equation

$$\frac{1}{2\pi}\int_0^{2\pi} \log^-|f(re^{i\theta})|\,d\theta + N(r,0) + \log|f(0)|$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log^-|f(re^{i\theta})|\,d\theta + N(r,\infty).$$

He defined

$$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta + N(r,\infty)$$

$$= m(r,\infty,f) + N(r,\infty)$$

(again modifications needed if $f(0) = \infty$).

Now Jensen's formula says that

$$T(r,f) - T(r,\tfrac{1}{f}) = \log|f(0)|$$

(if $f(0) \neq 0,\infty$) and it is a simple matter to modify the argument to show that

$$T(r,f) - T(r, 1/(f-a))$$

is a bounded function of $r > 0$ (for each a). This is a statement along the lines that $f(z) = a$, and $f(z) = \infty$, have the same number of solutions, except that it is cluttered up with $m(r,\infty,f)$ and $m(r,a,f) = m(r,\infty,1/(f-a))$.

The term $m(r,\infty,f)$ can be explained as measuring the average growth of $\log|f|$ on the set where $|f| \geq 1$. Its role in $T(r,f)$ is subservient to that of $N(r,\infty)$, unless ∞ is an exceptional value - that is, unless $N(r,\infty)$ is not as large as $N(r,a)$ usually is. This was shown in a precise form by Nevanlinna.

Before elaborating on this, we note a basic fact about the characteristic function. The function $f(z)$ is rational if and only if

$$\liminf_{r\to\infty} \frac{T(r,f)}{\log r} < \infty .$$

This may be viewed as a generalization of Liouville's theorem (f entire, $M(r,f) \leq cr^n + c$, implies f a polynomial) because, if $f(z)$ is entire,

$$T(r,f) \leq \log^+ M(r,f) \leq 3\,T(2r,f)$$

(This inequality can be shown using the fact that $\log|f|$ is subharmonic).

Nevanlinna called

$$\delta_f(a) = \delta(a) = 1 - \limsup_{r\to\infty} \frac{N(r,a)}{T(r,f)}$$

the <u>deficiency</u> of the value $a \in \hat{\mathbb{C}}$. It is easy to see that, if f is not rational, then $\delta(a) < 1$ implies $f(b) = a$ has infinitely many solutions. Also $0 \leq \delta(a) \leq 1$ is always true. Nevanlinna showed that (if f is not constant) $\delta(a) \neq 0$ for at most countably many $a \in \hat{\mathbb{C}}$ and

$$\Sigma_a\, \delta(a) \leq 2$$

This is a quantitative version of Picard's theorem.

Nevanlinna's characteristic function $T(r,f)$ became the magic tool for studying the distribution of $f^{-1}(a) \subseteq \mathbb{C}$, for f(z) meromorphic. All sorts of results were obtained under various restrictions on the function - mainly restrictions on the order

$$\rho(f) = \lim_{r\to\infty} \sup \frac{\log T(r.f)}{\log r}$$

Relations between f and f' were also investigated as were simultaneous solutions of $f(z) = a$ and $g(z) = a$. Ahlfors developed a geometrical approach to the characteristic function. The beauty of the field was that there were interesting results to be proved which were simple to state, but required ingenuity to derive. Nevanlinna's theory can justifiably be described as one of the greatest of mathematical theories. It is a marvellous simplification of the difficult problem of studying solutions of $f(z) = a$, which nevertheless has great depth.

Now, however, this great industry started by Nevanlinna seems to be suffering from the worldwide economic recession. One might argue that David Drasin hammered the last nail in the coffin when he settled one of the most fundamental outstanding questions. He showed (1977) that Nevanlinna's defect inequality $\Sigma\delta(a) \leq 2$ told the full story when f is unrestricted. Given a sequence $(a_n)_n$ of distinct elements of $\mathbb{C}$ and positive numbers d_n satisfying $\Sigma d_n \leq 2$, it is possible to find f(z) meromorphic with $\delta_f(a_n) = d_n$ and $\delta(a) = 0$ for a not one of the a_n's.

But can such a wonderful theory die? One might ask whether Euclidean geometry died centuries ago. After all, it is hard to find a major unsolved problem in Euclidean geometry. Of course the story changes considerably if we look at differential geometry, Riemannian geometry, Kähler manifolds, etc., which are the subjects one might imagine Euclid considering if he were around in this century.

So it seems to me to be unreasonable to point to the scarcity of really central open problems in value distribution

theory of functions of one variable and conclude that the field is dead. Rather, we should ask where we can use Nevanlinna's inspiration today. The answer is not yet completely clear, and it could be that future mathematicians may indeed say that Nevanlinna's theory lived only from 1925 to 1977.

There have been promising developments in the field of several complex variables. Considerable progress has been made in establishing defect relations for holomorphic maps $f:M \to N$ between certain non-compact complex manifolds M and certain compact manifolds N. The requirement on M is that it has a parabolic exhaustion function (example: $M = \mathbb{C}^m$) and N is a projective algebraic variety (example: $N = P_n\mathbb{C}$ = n-dimensional complex projective space). Deficiencies are defined with respect to subvarieties of N. It seems clear that the most general result has yet to be obtained, but here is a sample of what has been done. If $H_1, H_2, \ldots, H_q$ are hyperplanes in $P_n\mathbb{C}$ in "general position" and $f:\mathbb{C}^m \to P_n\mathbb{C}$ is a "transcendental mapping", then

$$\sum_{j=1}^{q} \delta(H_j) \leq n + 1.$$

The deficiency $\delta(H) = \delta_f(H)$ has, as before, the properties $0 \leq \delta(H) \leq 1$ and $f(\mathbb{C}^m) \cap H = \emptyset$ implies $\delta(H) = 1$.

The results obtained to date in several variables, though not completely satisfactory, have already yielded new non-trivial results about complex manifolds. So, Nevanlinna theory is not dead, but its state of health is uncertain.

Reading List

[1]. Carlson, J. and Griffiths, P., A Defect Relation for Equidimensional Holomorphic Mappings Between Algebraic Varieties. Ann. Math. (2) 95 (1972), 567-584.

[2]. Griffiths, P. and King, J., Nevanlinna Theory and Holomorphic Mappings Between Algebraic Varieties, Acta Math. 130 (1973), 143-220.

[3]. Hayman, W.K., Meromorphic Functions, Oxford (1962).

[4]. Hayman, W.K., Rolf Nevanlinna (obituary), Bull. London Math. Soc. 14 (1982), 419-436.

[5]. Nevanlinna, R., Analytic Functions, Springer-Verlag (1970).

[6]. Stoll, W., Introduction to Value Distribution Theory of Meromorphic Maps, in Complex Analysis (Proceedings, Trieste 1980), Springer Lecture Notes in Mathematics 950, 210-359.

School of Mathematics,
Trinity College,
Dublin 2.

THE MEAN VALUE THEOREM FOR VECTOR VALUED FUNCTIONS: A SIMPLE PROOF

William S. Hall and Martin L. Newell

It is well known that the mean value theorem in one dimension extends readily to real-valued functions of several variables, but fails for the vector-valued case. For example, let $f(t) = (\cos t, \sin t)$ and suppose there is a point ξ in $(0,2\pi)$ such that $f'(\xi) = 0$. Then $-\sin\xi = \cos\xi = 0$, an impossible situation. A useful and correct generalization is the inequality

$$|f(y) - f(x)| < \sup_{0<t<0} ||f'(x+t(y-x))|| \, |y-x|$$

where $f: D \subset \mathbb{R}_n \to \mathbb{R}_m$ is a differentiable vector-valued function on a convex open set D, f' is the matrix $\partial f_i/\partial x_j$, $i = 1,2 \,..m$, $j = 1,2 \,..n$, $|\ |$ is the appropriate norm (in $\mathbb{R}_n$, or in $\mathbb{R}_m$), $||\ ||$ is the usual norm in the set of linear maps from $\mathbb{R}_n$ to $\mathbb{R}_m$, and x,y are arbitrary points in the domain D.

Many undergraduate calculus and analysis texts prove the mean value theorem in the real case but omit the result above. Those that do present this more general form usually give either a "sloppy" proof, using components, or a "slick" proof with the Hahn-Banach Theorem. Here we present a direct approach, requiring only the chain rule and the mean value theorem in $\mathbb{R}$. It is worth noting that f' at each point is a linear map (given by the Jacobian matrix) and that the usual norm for a linear map (matrix) is given by

$$\sup_{|x|=1} |Ax|.$$

However, other norms such as $(\Sigma\alpha_{ij}^2)^{\frac{1}{2}}$ where $A = (\alpha_{ij})$ are frequently used in advanced calculus courses. All we really use is that $|Ax| \leqslant ||A||\,|x|$.