PROBLEMS

First, the solutions to the December problems.

1. \[
\int \frac{\frac{1}{2} \, dt}{\sqrt{1 + \tan^2 t}} = \frac{1}{2} \tan^{-1} \tan^2 t.
\]

The trick with this is to substitute \( x = \frac{1}{2} \tan t - t \). Then

\[
\int \frac{x \, dx}{\sqrt{1 + \cot^2 t}} = \frac{1}{2} \tan^{-1} \tan^2 t, \quad \tan^{-1} \tan^2 t = \frac{1}{2} \tan^{-1} x + t
\]

so that

\[
2I = \int \frac{1 + \tan^2 t \, dt}{1 + \tan^2 t} = \frac{1}{2} \tan^{-1} \tan^2 t.
\]

2. For a set \( A \) in \( \mathbb{R}^n \), \( a \subseteq b \) means that the line segment joining \( a \) to \( b \) lies in \( A \). Does the condition

\((x)\) "If \( a, b, c \subseteq A \) then at least one of the following holds \( a \subseteq b, b \subseteq c, c \subseteq a \)."

imply that \( A \) is the union of two convex sets?

The answer is no as the following set shows.

\[\begin{align*}
A &= \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1\} \\
&\quad \cup \{(x, y) \in \mathbb{R}^2 : \sqrt{(x-1)^2 + y^2} < 1\}
\end{align*}\]

To see that \( A \) cannot be written as the union of two convex sets, consider where the five outer vertices would end up.

Now, some more problems.

1. Let \( \Gamma \) be a \( C^1 \) closed curve in \( \mathbb{R}^3 \). Must there exist a pair of points in \( \Gamma \) at which the tangent vectors to \( \Gamma \) are parallel but of opposite sense?

2. An electrician is faced with a bundle of \( n \) (>2) unmarked identical wires running from the ground floor to the top floor of a block of flats.

With the aid of a bell and a battery, and making only one visit to the top floor, how should he proceed to match up and label the top and bottom ends of the wires?

3. Prove, or disprove, that

\[
\inf \{ |n \sin n| : n = 1, 2, 3, \ldots \} = 0
\]

(Submitted by T. Laffey)

4. The following problem arises in Control Theory (Simultaneous Stabilization):

Given rational functions \( d_1, d_2, n_1 \) and \( n_2 \) which have no poles in the closed unit disk \( \mathbb{D} \) and whose zero sets are pairwise disjoint in \( \mathbb{D} \), when is it possible to find a rational function \( m \) without poles in \( \mathbb{D} \) such that neither of the functions \( d_1 + mn_1, d_2 + mn_2 \) has any zeros in \( \mathbb{D} \)?

If each of the functions \( d_1, d_2, n_1 \) and \( n_2 \) is self-conjugate (a function \( f \) is self-conjugate if \( f(z) = \overline{f(z)} \)), when is it possible to find a self conjugate function \( m \) as above?

This problem is equivalent to the following (at least when the zeros of \( d_1, d_2, n_1 \) and \( n_2 \) are simple): When can one find a rational function \( r \) without poles or zeros in \( \mathbb{D} \) such that

\[
r(z) = \frac{n_1(z)}{n_2(z)}
\]

if and only if \( (n_1 d_2 - n_2 d_1)(z) = 0 \).
Remark: If we only need \( n_1 + m_1 \) to be non-vanishing, there is no restriction on solving the problem without the self-conjugation condition; with this condition, the problem can be solved if and only if the values of \( d_1 \) at all real zeros of \( n_1 \) in \( D \) have the same sign.

(submitted by J.J. Murray)

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WORD CONSERVATION

To counter the recent British changeover from the English billion, which is worth one million million, to the American billion, of one thousand million, we advocate the introduction of the Irish thousand, worth one hundred hundred, as a substitute for the English and American thousand, which is ten hundred. This would give an Irish billion of ten to the power of sixteen, which the Americans would have to call ten quadrillion. By the time we in Ireland have used up one new word therefore the Americans will have used up three, and be well on the way to their fourth.

AUTONOMOUS NOTATION

\[ e^{\text{exponential}} : \log \text{arith} : \int e^x = f(v_0) : \text{CAT}_{\text{STROPHE}} \]