IRISH MATHEMATICAL SOCIETY

NEWSLETTER

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No. 5 SEPTEMBER 1982

CONTENTS

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Editorial

As the recently appointed editor of the Irish Mathematical Society Newsletter, I would like to take this opportunity to comment briefly on plans for the publication.

The Newsletter should serve the membership of the I.M.S. as a forum for articles on all aspects of Mathematics, pure and applied, as also for articles on research and teaching. The membership of the Society spans the broad range of Mathematicians from researchers to teachers in all areas of Mathematical Science. Accordingly, the Newsletter should reflect this and in doing so help the Society establish a cohesion and identity of its own.

The present plans are to produce the Newsletter on a regular basis, three times during the 1982/83 Academic Year, and quarterly after that. The main sections will be articles, mathematical education, book reviews, problems, conference reports and news items.

Articles should be of wide interest, written in an expository manner and should be accessible to a majority of the membership. The section on education should contain articles on varying aspects of teaching with a special emphasis on the third level sector. Book reviews will cover undergraduate textbooks and research level texts. The conference reports will be primarily devoted to conferences held at Irish venues. Finally the news section will include affairs of the Society and personal items.

The success of the Newsletter will depend entirely on the contributions. I hope that anyone who wishes to make a contribution in any section will get in contact with me, or better still, write and submit.

Finally, I feel honoured to be given the opportunity of serving the Society as Editor of its Newsletter and hope that I will help in getting it established as a reputable publication.

Irish Mathematical Society

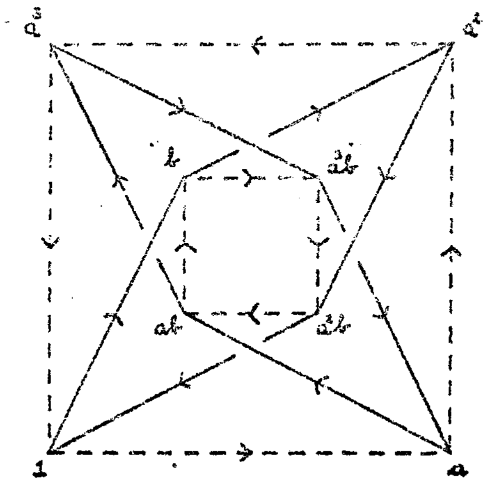
Logo

DES MacHALE (U.C.C.)

I approached the challenge of designing a logo for the Irish Mathematical Society in the following manner. Mathematics is created by people and the personalities of mathematicians are a unique source of impetus to the subject. Hamilton, who was proud to be an Irishman, was the greatest mathematician this country has produced and Quaternions were his greatest contribution to Mathematics. The Quaternion group generated by the elements i, j and k, gives eight elements in all and can conveniently be represented by a di-graph which is called a Cayley diagram. This fact also commemorates the Cayley-Hamilton Theorem (or is it the Hamilton-Cayley Theorem?) Another aspect of the logo I finally chose is that of Hamiltonian Circuits which are an important branch of Graph Theory and are currently an active area of research.

The logo is thus an abstraction of the Cayley di-graph of the quaternion group given by

$< a,b \mid a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^3 >$ and it looks as follows when fully directed and labelled:-



NEWS AND ANNOUNCEMENTS

International Mathematical Union

At the General Assembly of the I.M.U. held in Warsaw on 8th and 9th August, 1982, the following awards were announced.

FIELDS MEDALS

- A. CONNES (I.H.E.S., Paris)
- W. THURSTON (PRINCETON UNIVERSITY)
- S.T. YAU (I.A.S., PRINCETON)

NEVANLINNA PRIZE

- R. TARJAN (Stanford)

The Nevanlinna Prize is a new prize in Information Science to be awarded every four years on the occasion of International Congresses of Mathematicians. The University of Helsinki is funding the prize.

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It was also announced that the 1986 ICM will be held at Berkeley, California.

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PERSONAL ITEMS

New Appointments.

Dr. R. Gimson of the Computer Science Dept., U.C.D. has been appointed to a Research Position at Oxford University.

Dr. J. Hannah, a Post Doctoral Fellow at the Mathematics Department, U.C.D. has been appointed to the staff of the Mathematics Department, U.C.G. Dr. Hannah works in Ring Theory.

Dr. J. Morris (Ph.D., T.C.D.) has been appointed to the staff of the Computer Science Department, U.C.D. Dr. Morris' speciality is Program Construction.

Dr. D. Ó'Mathuna of the U.S. Department of Transport has been appointed a Research Associate at T.C.D. Dr. Ó'Mathuna's field of interest is Mechanics.

Dr. D. Reynolds of the Mathematical Physics Department, U.C.D. has been appointed to the staff of N.I.H.E., Dublin.

Dr. P. Rippon of the Mathematics Department, U.C.C. has been appointed to a position in the Open University. Dr. Rippon works in Complex Analysis.

Promotions.

Dr. J. Kelly of the Computer Science Department, U.C.D. has been promoted to College Lecturer.

Dr. T. Tchrakian of the Mathematical Physics Department, Maynooth College, has been promoted to Senior Lecturer.

Dr. J.B. Twomey of the Mathematics Department, U.C.C. has been promoted to Associate Professor.

Study/Sabbatical Leave

Dr. J. Adam of the New University of Ulster will be on study leave from January to September 1983 at the University of Rochester, New York.

Dr. H. Seifert of the Mathematics Department, U.C.D. will be on Sabbatical leave at the University of Paris for the Academic Year 1982-1983.

Dr. D.A. Walsh of the Mathematics Department, Maynooth College will be on Sabbatical leave from January to September 1983 at Edinburgh University with brief visits in England and France.

Visitors.

Dr. P.B. Chapman of the Mathematics Department, University of Western Australia will visit T.C.D. from May to August 1983. His field of study is Numerical Analysis.

Irish National Mathematics Contest

The Fourth Irish National Mathematics Contest was held on March 9, 1982 and attracted about 1,400 entries from 75 schools. The total represents a decrease of about 300 entrants from 1981. This decrease is possibly accounted for by the increase in fee and the difficulty of the 1980 and 1981 Contests which may have deterred some schools from participating.

The results to hand indicate that the 1982 Contest was more difficult than the 1981 edition. The number of those who scored 80 or more was 26 - and includes 5 girls - as against 45 in 1981; the average mark for the top scorers was 84.9 versus 85.4 in 1981. Nevertheless, the highest score achieved so far by an Irish student was recorded this year by our winner:

David A. Donnelly, St. Michael's College, Omeath, Co. Louth,

who scored 115. This is a highly commendable achievement.

The top three scorers will be presented with suitable prizes later on in the year.

A fuller analysis of the results will appear in the Newsletter published by the Irish Mathematics Teachers Association, co-sponsors with the Irish Mathematical Society of the 1982 Contest.

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Newsflash

Rumour has it that a postgraduate student in Oxford has proved the generalized Poincare conjecture in dimension 4. (The n-dimensional Poincare conjecture is that any closed n-manifold with the homotopy of an n-sphere is homeomorphic to an n-sphere.) If the rumour is true, this means that the only outstanding case is the original one of Poincare when n = 3. (The cases n ≥ 5 were done by Smale in 1956.)

CONFERENCE ANNOUNCEMENTS

LIMERICK MATHEMATICS 1982.

A two-day Algebra Conference will be held at Mary Immaculate College, Limerick on November 12th and 13th, 1982. The main speakers are Prof. J.G. Thompson (Cambridge), Prof. T.J. Laffey (U.C.D.), Prof. M.L. Newell (U.C.G.) and Dr. D. MacHale (U.C.C.). Persons seeking further details or willing to give short talks of up to 20 minutes duration are invited to contact the organiser Dr. G.M. Enright, Mathematics Department, Mary Immaculate College, South Circular Road, Limerick. Phone (061)44588.

NASECODE III.

The Third International Conference on the Numerical Analysis of Semiconductor Devices and Integrated Circuits will be held in Galway from 15th to 17th June 1983, under the auspices of the Numerical Analysis Group and cosponsored by the Electron Devices Society of the IEEE, the Institute for Numerical Computation and Analysis and the Irish Mathematical Society.

Contributed papers are solicited from engineers, physicists and mathematicians on any topic relevant to the numerical analysis, modelling and optimisation of electronic, opto-electronic and quantum electronic semiconductor devices and integrated circuits. The deadline for the receipt of abstracts and preliminary versions of 20-minute contributed papers is 18th February, 1983. All correspondence should be addressed to: NASECODE Conference, 39 Trinity College, Dublin 2. Phone (01) 772941 ext. 1869/1949.

I.C.M.E. - V

The Fifth International Congress on Mathematics Education will be held in Adelaide, South Australia, in August 1984. The chairman of the International Program Committee is Dr. M.F. Newman, Department of Mathematics, Australian National University, Canberra.

INFINITE EXPONENTIALS

P.J. Rippon

A question of the following type appeared during 1981 in a Regional Math. Contest for high school students in the U.S.A. and caused some difficulties for the referees present.

"Find all the real numbers a such that

$$a^{a^{a^{\cdot^{\cdot^{\cdot}}}}} = 8." \qquad (1)$$

The expected answer, presumably, was that

$$a^{a^{a^{\cdot^{\cdot^{\cdot}}}}} = a^{\left(a^{a^{\cdot^{\cdot^{\cdot}}}}\right)} = a^8 = 8\,,$$

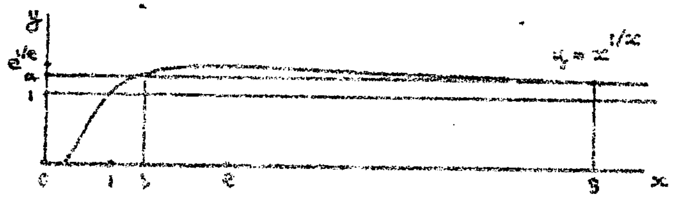
and so $a = 8^{1/8}$. Here (1) has been taken to mean that the sequence $a, a^a, a^{(a^a)}, \ldots$, which we shall denote throughout by

$$a_1 = a\,,\quad a_{n+1} = a^{a_n}\,, \qquad (n = 1, 2, \ldots) \qquad (2)$$

converges to 8. Now it is not immediately obvious that the sequence a_n will be convergent when $a = 8^{1/8} \approx 1.3$ and this difficulty occurred to the referees at the Math. Contest. In fact it turns out that the sequence is convergent. Unfortunately however it does not converge to 8 and the question was very nearly scrubbed!

In this article I shall attempt to explain the somewhat surprising facts outlined above and survey a number of the known results about the convergence of such 'infinite exponentials'. Both the real and complex cases will be discussed and some new results given. I am greatly in debt to the survey by Knoebel [4] which appeared coincidentally at about the same time as the Math. Contest and which contains a huge bibliography on this topic. I am also grateful to Leon Greenberg, whose ingenious approach to the convergence of a_n (for real a) first kindled my interest in this problem, and to many colleagues and students at U.C.C. for advice and encouragement, particularly with the computer.

To see why $a = 8^{1/8}$ is not a solution of (1) we consider the graph of $y = x^{1/x}$, $x > 0$.



If b is the unique number less than e satisfying

$$b^{1/b} = a,$$

then $1 < a < b$. Since $a^b = b$ we deduce by induction that the increasing sequence a_n is bounded above by b and so converges to some number c, $1 < c \leq b$. Because $a^c = c$ we must have $c = b$. A little work with a calculator shows that this limit b is approximately 1.46.

It turns out that the behaviour of the sequence a_n for $a > 0$ was studied as long ago as 1777 by Euler [3]. He, it seems, was already aware of the fact (since rediscovered many times) that the sequence is convergent if and only if

$$(1/e)^e \leq a \leq e^{1/e}.$$

The proof of convergence when $1 \leq a \leq e^{1/e}$ uses the argument given above for $8^{1/8}$. On the other hand if a_n converges to b then $a = b^{1/b}$ and so convergence is impossible if $a > e^{1/e}$.

The case $0 < a < 1$ is less straightforward since the sequence a_n is no longer monotonic. In fact

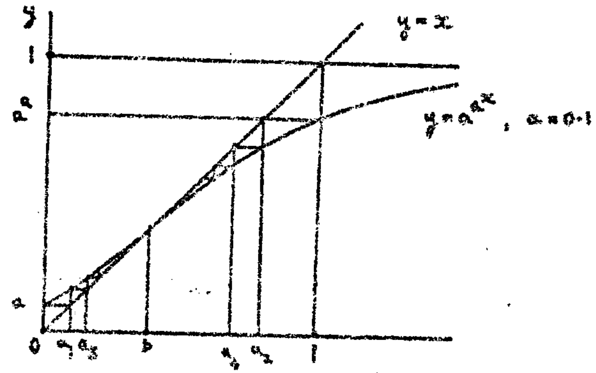
$$0 < a_1 < a_3 < a_5 < \ldots < a_6 < a_4 < a_2 < 1,$$

that is, a_{n+1} lies strictly between a_{n-1} and a_n for each $n = 2, 3, \ldots$. The proof is by induction, starting from $0 < a < 1$, using repeated exponentiation. It is natural then to study the graph of $y = a^{a^x}$, $x > 0$. As before we let b denote the (unique) solution of $a^x = x$, which will also be a solution of $a^{a^x} = x$. In this case $a < b < 1$.

By elementary calculus,

$$\frac{d}{dx}\left(a^{a^x}\right) \leq (-\log a)/e, \qquad (a > 0)$$

with equality only at the single point of inflection of $y = a^{a^x}$, which occurs when $a^x = (-\log a)^{-1}$. If $(1/e)^e \leq a < 1$, therefore, the graph of $y = a^{a^x}$ crosses the graph of $y = x$ exactly once, at $x = b$.

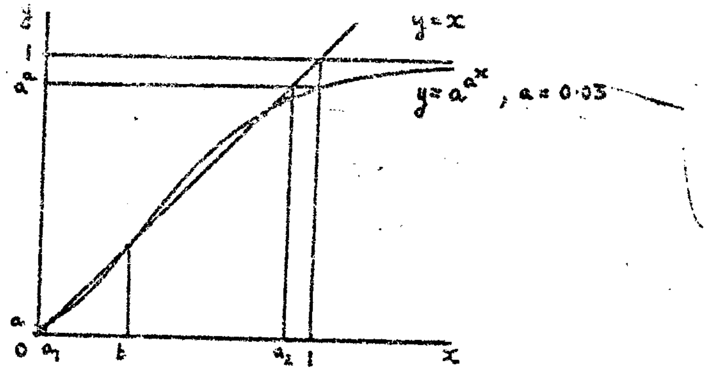


The convergence of the sequence a_n to b is then immediate from the graph.

On the other hand we have

$$\frac{d}{dx}\left(a^{a^x}\right)\bigg|_{x=b} = (\log b)^2,$$

which is strictly greater than 1 if $0 < b < 1/e$, that is if $0 < a < (1/e)^e$. This shows that the graph of $y = a^{a^x}$ crosses the graph of $y = x$ exactly three times if $0 < a < (1/e)^e$.



In this case it is clear from the graph that the sequence a_n does not converge.

The problem becomes much harder if we allow a to assume complex values. In this case we put $a_1 = a$ and

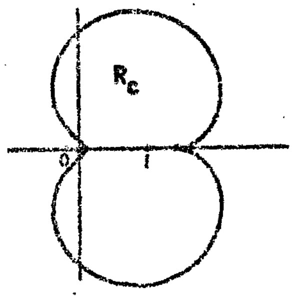
$$a_{n+1} = \exp(a_n \log a) = a^{a_n}, \quad (n = 1, 2, \ldots) \qquad (3)$$

where the principal value of log a is taken. This excludes numbers on the negative real axis from consideration, though we note that $a = -1$ is an exceptional special case. Quite a lot is known about the convergence problem for complex a but there are still a number of interesting open questions.

The main problem concerns the set

$$R_C = \{ e^{\xi e^{-\xi}} : |\xi| \le 1 \},$$

which is illustrated below.



This set meets the real axis precisely in the interval $R_R = [(1/e)^e, e^{1/e}]$ on which th[e] sequence a_n is convergent. Computing evidence suggests that a_n may in fact be convergent throughout R_C and we shall survey a number of partial results towards this.

The set R_C was first identified by Carlsson in his thesis [2] of 1907, where he showed that if $a_n \to w$ as $n \to \infty$ and if $a_n \neq w$, $n = 1,2, \ldots$, then $a \in R_C$. In fact, by (3), we have $w = \exp[w \log a]$ and on putting $\xi = w \log a$ we obtain $w = e^{\xi}$ and $a = e^{\xi e^{-\xi}}$. It only remains to show that $|\xi| \le 1$. To do this we let

$$b_n = a_n w^{-1} - 1, \qquad (n = 1,2, \ldots) \tag{4}$$

so that, by (3),

$$(1+b_{n+1})w = \exp[(1+b_n)w \log a]$$

and hence

$$b_{n+1} = \exp[\xi b_n] - 1. \tag{5}$$

Since $b_n \neq 0$, $n = 1,2, \ldots$, and $b_n \to 0$ as $n \to \infty$ we have

$$\frac{b_{n+1}}{b_n} = \xi + o(1), \quad n \to \infty. \tag{6}$$

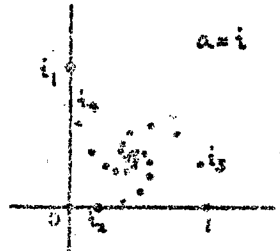
Thus

$$|\xi| = \lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| \le 1,$$

as required. We also observe from (4) and (6) that if $\arg \xi \neq 0$ then

$$\arg\left(\frac{a_{n+1} - w}{a_n - w}\right) = \arg\left(\frac{b_{n+1}}{b_n}\right) \to \arg \xi, \quad n \to \infty.$$

This shows that the sequence a_n converges to w in a spiral-like manner. As an example we illustrate the convergence of the sequence $i, i^i, i^{(i^i)}, \ldots$, which will be proved later.



We shall deduce positive results from the following

Lemma. Let Ω be any domain bounded by a simple closed curve Γ and let $f: \bar{\Omega} \to \bar{\Omega}$ be a continuous function, analytic in Ω, which does not map Ω conformally onto itself. If f has no fixed point on Γ then f has a unique fixed point z_0 in Ω and for any z_1 in Ω the sequence

$$z_{n+1} = f(z_n), \qquad (n = 1,2, \ldots) \tag{7}$$

converges to z_0.

I am grateful to Peter Walker for pointing out how this lemma can be used to prove the convergence of $i, i^i, i^{(i^i)}, \ldots$. It also serves to unify and simplify many of the approaches which have been made to the general problem. For other results of this type we refer to Burckel [1].

To prove the lemma first note that f must have at least one fixed point z_0 in Ω by Brouwer's theorem (in any particular application the existence of an interior fixed point can often be demonstrated more directly). We assume, as we may by the Riemann mapping theorem, that Ω is the unit disc and that $z_0 = 0$. By

Schwarz's lemma

$$|f(z)| < |z|, \qquad (0 < |z| < 1) \tag{8}$$

the inequality being strict because f is not a rotation of Ω. Thus $|z_n|$ is decreasing and so

$$r = \lim_{n\to\infty} |z_n|$$

exists. Some subsequence of z_n is convergent, to z say, and by the continuity of f we have

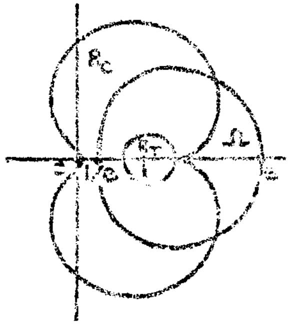
$$|f(z)| = r = |z|.$$

According to (8), therefore, we must have $z = 0$ and so $z_n \to 0$, as required.

As a first application we show that

$$R_T = \left\{ a : |\log a| \le e^{-1} \right\}$$

is a set of convergence for a_n. This is due to Thron [7].



Letting

$$\Omega = \left\{ z : |\log z| < 1 \right\},$$

we find that

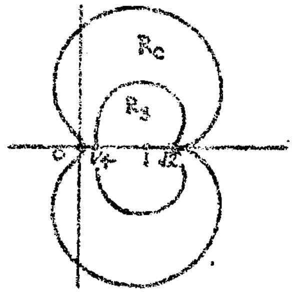
$$|\log(a^z)| = |z \log a| \le 1, \qquad (z \in \bar{\Omega},\ a \in R_T)$$

since $|z| \le e$ when $z \in \bar{\Omega}$. The inequalities here are strict, except when $z = e$,

and so a^z maps Ω properly into itself. The only possible fixed point of a^z on $\partial\Omega$ is $z = e$ and this can only occur when $|\log a| = e^{-1}$. This would mean that $a = e^{1/e}$ and we already know that a_n is convergent for this value of a. For any other a in R_T the lemma gives the desired convergence of $1, a, a^a, \ldots$.

Another major contribution towards the solution of this problem was made by Shell in his thesis of 1959, part of which was published in [6]. We shall prove here one result of his, that a_n is convergent when a lies in

$$R_S = \left\{ e^{\xi e^{-\xi}} : |\xi| \le \log 2 \right\}.$$



For any $a = e^{\xi e^{-\xi}}$, $|\xi| \le \log 2$, consider the disc

$$\Omega_\xi = \left\{ z : |z - e^\xi| < |e^\xi| \right\}.$$

If $z \in \bar{\Omega}_\xi$ then $|ze^{-\xi} - 1| \le 1$ and so (†)

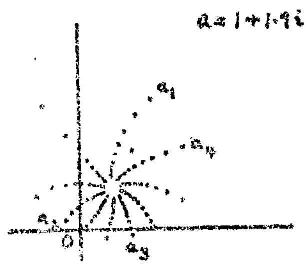
$$\begin{aligned} |a^z - e^\xi| &= |\exp(z \log a) - e^\xi| \\ &= |e^\xi|\ |\exp[\xi(ze^{-\xi} - 1)] - 1| \\ &\le |e^\xi|(\exp[|\xi|\,|ze^{-\xi} - 1|] - 1) \\ &\le |e^\xi|. \end{aligned} \tag{9}$$

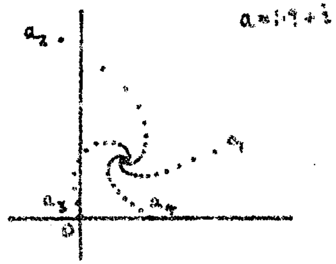
It follows that a^z maps $\bar{\Omega}_\xi$ into itself and evidently has $z = e^\xi$ as a fixed point. This mapping is not a rotation (for instance, its derivative at e^ξ is equal to ξ) and it is easy to check that $z_1 = 1$ lies in Ω_ξ except when $\xi = -\log 2$ (corresponding to $a = \frac{1}{4}$, where the convergence of a_n is already known). According to the proof of the lemma the sequence $1, a, a^a, \ldots$ converges to e^ξ, as required.

(†) The inequality $|e^z - 1| \le e^{|z|} - 1$, which we use here, follows immediately from the Taylor series for e^z.

A more careful study of (9) reveals [6] that if $s = |\xi| < 1$ then there is a number $\rho(s) > 0$ (which depends continuously on s and tends to zero as s tends to 1) such that a^z maps the disc Δ_ξ, with centre e^ξ and radius $\rho(s)$, into itself. Thus if $a_n \in \Delta_\xi$ for any n we must have $a_n \to e^\xi$ as $n \to \infty$. By continuity we deduce that the set of interior points of R_C for which the sequence a_n is convergent forms an open set.

By now the reader will appreciate that the approach being used is to find a 'domain of invariance' for the mapping a^z which is large enough to include the poi[nt] a itself. To give some idea of the types of domains of invariance which might be en[-] countered we illustrate the sequence a_n in two particular cases where its convergen[ce] has not been established.





An equivalent way to look at this problem is to make the substitution used earlier in (4), namely,

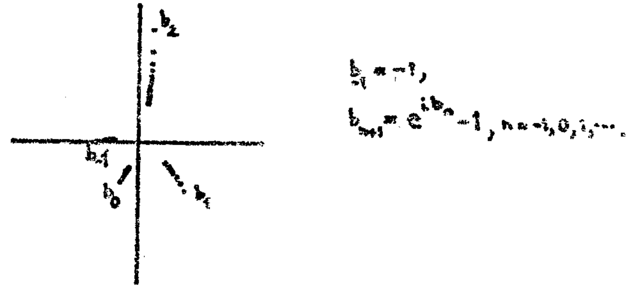
$$b_n = a_n e^{-\xi} - 1, \qquad (n = 1,2, \ldots)$$

where $a = e^{\xi e^{-\xi}}$. We then have $b_1 = ae^{-\xi} - 1 = \exp[\xi(e^{-\xi} - 1)] - 1$, and

$$b_{n+1} = \exp[\xi b_n] - 1, \qquad (n = 1,2, \ldots)$$

and it is natural to also put $b_0 = e^{-\xi} - 1$ and even $b_{-1} = -1$. The problem then is to show that $b_n \to 0$ whenever $|\xi| \leq 1$. Shell's argument showed that this was true when $|\xi| \leq \log 2$ by observing that the unit disc is invariant under the mapping $e^{\xi z} - 1$, for such ξ.

To provide some food for thought we illustrate the spiral-like behaviour of the sequence b_n when $\xi = i$ (which corresponds to $a \approx 1.99 + 1.19i$).



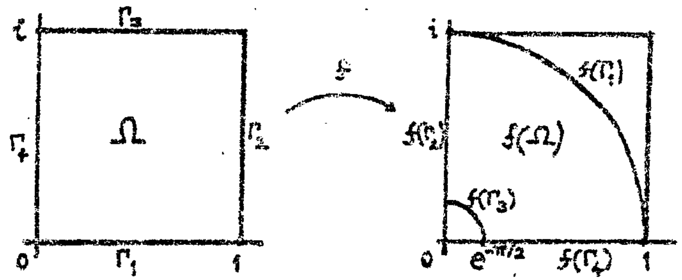
It seems clear that this sequence converges to 0, albeit slowly. For instance, calculations on a computer indicate that when $n = 200{,}000$ the modulus of b_n is about 0.1. In this case however there would be no hope of finding a suitable domain of invariance Ω for $f(z) = e^{iz} - 1$ since $|f'(0)| = 1$, which would force f to map Ω conformally onto itself.

Neither of the positive results proved earlier cover the case $a = i$, corr[-] esponding to $\xi \approx -0.6+0.7i$, which is naturally of particular interest. Apparently a proof of the convergence when $a = i$ was given in Shell's thesis, but this does not appear in his subsequent paper. However the following stunningly simple proof was published by Macintyre [5] in 1966. Let

$$\Omega = \{x+iy : 0 < x,y < 1\}.$$

Then $f(z) = i^z$ maps Ω properly into itself, since

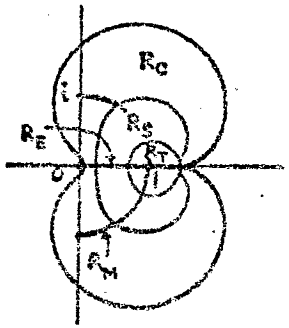
$$i^z = e^{iz\pi/2} = e^{-y\pi/2}e^{ix\pi/2}.$$



By inspection f has no fixed points on $\partial\Omega$ and so we can apply the lemma with $z_1 = i_3 = \exp[i\frac{\pi}{2}e^{-\pi/2}]$, which lies in Ω. The same argument shows that the sequence a_n converges whenever a is of the form $e^{i\theta}$, $0 \le \theta \le \pi/2$. By conjugacy, therefore, we have convergence throughout

$$R_M = \{e^{i\theta}:\ |\theta| \le \pi/2\}.$$

To get an impression of the results so far we illustrate the sets R_C, R_E, R_T, R_S and R_M together and remind the reader that the set of convergence is open in the interior of R_C.



It is interesting that there is no relationship of containment amongst R_E, R_T, R_S and R_M.

We finish with an intriguing fact that does not seem to have been mentioned in the literature. The sequence a_n is in fact convergent for many numbers lying outside R_C. An obvious example is $a = -1$ but there are also, for instance, unbounded sequences of numbers a in the first and fourth quadrants each having the property that $a^a = a$. For such numbers we clearly have $a_1 = a_2 = a_3 = \ldots$.

To demonstrate their existence we show that, for each positive integer k, there is a number z lying in

$$S = \{x+iy : x > 0,\ 0 < y < \pi/2\},$$

such that

$$z(e^z - 1) = 2\pi k i. \tag{10}$$

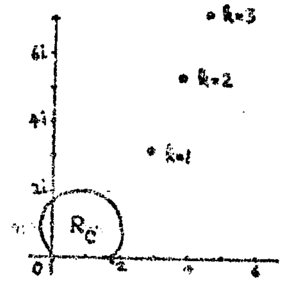
In that case $a = e^z$ lies in the first quadrant and

$$a^a = \exp[a \log a] = \exp[ze^z] = \exp[z + 2\pi k i] = a.$$

Moreover the solutions z of (10) will be unbounded with k. One way to show that these solutions exist is to rewrite (10) as

$$z = \log\left(1 + \frac{2\pi k i}{z}\right).$$

The function on the right-hand side of this equation maps the first quadrant into S and has no fixed points on the boundary. Apart from proving that (10) has a unique solution z in S for each k, the lemma allows us to compute approximate values for these solutions and for the corresponding numbers a. The first few of these are illustrated below and it is clear that they all lie outside R_C, a fact which can also be proved analytically.



Further work along these lines (details of which will appear elsewhere) leads to the existence of other numbers such that $a_2 = a_3$, $a_3 = a_4$, ... We give some examples, which the reader is invited to check-preferably with the help of Fortran IV! Due allowance should be made for round-off errors.

$a_1 = a_2 = \ldots$	$a_2 = a_3 = \ldots$	$a_3 = a_4 = \ldots$	$a_4 = a_5 = \ldots$
2.8629 + 3.2233i	2.4293 + 0.55465i	1.9813 + 0.16031i	1.78285 + 0.082166i
3.7273 + 5.3180i	2.6921 + 0.58785i	2.0599 + 0.16702i	1.81660 + 0.071851i
4.4332 + 7.1938i	2.8513 + 0.60067i	2.1024 + 0.15843i	1.83403 + 0.066124i

Finally I should be grateful if someone would disprove (or, better still, prove) the following (rather wild) conjecture which has been bothering me for some time now. Could it be that the set of numbers a such that the sequence a_n converges is actually dense in the complex plane?

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Majorization and Schur Functions

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The concepts of majorization and Schur functions lay the basis for a rich and elegant theory in which many classical and applicable inequalities may be viewed. In this expository paper the basic definitions and properties of majorization and Schur functions are presented, together with a variety of applications emphasizing in particular some in reliability theory. For a thorough and recent account of majorization and Schur functions, the interested reader should consult the excellent Inequalities: Theory of Majorization and its Applications by Marshall and Olkin (1979).

1. Majorization

Given a vector $\underline{x} = (x_1,\ldots,x_n) \in R^n$, let $x_{[1]} \geq \ldots \geq x_{[n]}$ denote a decreasing rearrangement of $x_1,\ldots,x_n$.

Definition 1.1 If $\underline{x},\underline{y} \in R^n$, then $\underline{x} \prec \underline{y}$ if

$$\sum_1^j x_{[i]} \leq \sum_1^j y_{[i]} \qquad \text{for } j = 1,\ldots,n-1$$

and

$$\sum_1^n x_{[i]} = \sum_1^n y_{[i]}.$$

If $\underline{x} \prec \underline{y}$, we say that $\underline{x}$ is majorized by $\underline{y}$. Note that if $\underline{x} \prec \underline{y}$, then the components of $\underline{y}$ are more "spread out" than those of $\underline{x}$. For example $(\frac{1}{n},\ldots,\frac{1}{n},\frac{1}{n}) \prec (\frac{1}{n-1},\ldots,\frac{1}{n-1},0) \prec (1,0,\ldots,0)$, and $(\bar{x},\bar{x},\ldots,\bar{x}) \prec (x_1,\ldots,x_n)$ where $\bar{x} = \Sigma x_i/n$.

One of the origins of majorization can be found in the work of Schur (1923) on Hadamard's determinant inequality (which states that for any $n \times n$ positive semi-definite Hermitian matrix $M = (m_{ij})$, $\det M \leq \prod_i^n m_{ii}$). Preliminary to proving this result, Schur showed that the diagonal elements $m_{11}, \ldots, m_{nn}$ of a positive semi-definite Hermitian matrix M are majorized by the characteristic roots $(\lambda_1, \ldots, \lambda_n)$. Horn (1954) later showed that this relationship actually characterizes those vectors $\underline{m} = (m_{11}, \ldots, m_{nn})$ and $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ that can arise together as respectively the diagonal and characteristic root vectors of the same Hermitian matrix.

Many basic inequalities reduce to an inequality of the form $f(\bar{y}, \ldots, \bar{y}) \leq f(y_1, \ldots, y_n)$ for some appropriate f. This suggests perhaps considering comparisons of the type $f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)$ where $\underline{x} \prec \underline{y}$. Hardy, Littlewood and Polya (1923) asked the following question: what conditions on $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{y} = (y_1, \ldots, y_n)$ ensure that

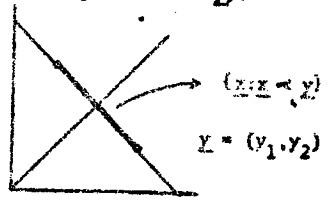
$$\Sigma g(x_i) \leq \Sigma g(y_i)$$

for all convex functions $g: R \to R$? They proved in fact that * is true for all convex g if and only if $\underline{x} \prec \underline{y}$.

Majorization is inherent in the work of economists studying income inequality in the early part of this century. Lorenz (1905) did so in introducing what is now known as a "Lorenz curve". Let $\underline{x} = (x_1, \ldots, x_n)$ be the wealth vector for a population of size n, i.e. x_i is the wealth of individual i. We let $S_0 = 0$, and $S_k = \sum_{i=n-k+1}^{n} x_{[i]}$ be the total wealth of the k poorest individuals. If we plot the points $(k/n, S_k/S_n)$ for $k = 0, \ldots, n$ and connect them in a linear fashion we obtain the Lorenz curve for the wealth vector $\underline{x}$. The Lorenz curve is always convex and is a straight line if and only if the wealth is uniformly distributed. Suppose now $\underline{y}$ is another wealth vector from a population of size n. One would say that the wealth distribution of $\underline{x}$ is more equal than that of $\underline{y}$ if the Lorenz curve

of $\underline{x}$ lies above that of $\underline{y}$. When the total wealth of the two populations is the same this is equivalent to saying that $\underline{x} \prec \underline{y}$.

Hardy, Littlewood and Polya (1929) showed that $\underline{x} \prec \underline{y}$ if and only if there is a doubly stochastic matrix P (a matrix with nonnegative elements whose rows and columns both sum to one) such that $\underline{x} = \underline{y}P$. "Hitting" a vector $\underline{y}$ with a doubly stochastic matrix P has the effect of averaging or smoothing out its components. Birkhoff (1946) proved that the set of doubly stochastic matrices is the convex hull of the permutation matrices (and moreover that the permutation matrices are the extreme points of this set). Birkhoff's result together with the above characterization of Hardy, Littlewood and Polya enable one to show that for a given $\underline{y}$, $\{\underline{x}: \underline{x} \prec \underline{y}\}$ is the convex hull of the orbit of $\underline{y}$ under permutations (the set of points obtained by permuting the components of $\underline{y}$).



2. Schur Functions

Majorization represents a partial ordering on R^n. A Schur function is a real valued function which is monotone with respect to this ordering.

Definition 2.1 A function f satisfying the property that $f(\underline{x}) \leq (\geq) f(\underline{y})$ whenever $\underline{x} \prec \underline{y}$ is Schur convex (concave). Functions which are either Schur convex or Schur concave are called Schur functions.

Note that a Schur function is necessarily symmetric or permutation invariant, that is $f(\underline{x}) = f(\underline{x}^{\Pi})$ where $\underline{x}^{\Pi}$ is an arbitrary rearrangement of the coordinates of $\underline{x}$.

The terminology "Schur convex (concave)" is rather misleading. Although a symmetric convex function on R^n is Schur convex, a Schur convex function may be far from convex in any usual sense of the word. "Schur increasing" would perhaps be more appropriate than "Schur convex", although (unfortunately) for historical reasons the later term is now conventional.

A useful characterization of Schur functions is given by the following result of Ostrowski (1952), sometimes referred to as the Schur Ostrowski condition.

Theorem 2.2 Let $A \subset R^n$ be convex and permutation invariant with non empty interior. If $f: A \to R$ is continuously differentiable on the interior of A and continuous and symmetric on A, then

f is Schur convex (concave)

<=>

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i}(\underline{x}) - \frac{\partial f}{\partial x_j}(\underline{x})\right) \geq (\leq)\ 0 \quad \text{for all } i \neq j.$$

Example 2.3 Let $S_k(\underline{x})$ be the k^{th} elementary symmetric function of $\underline{x}$, for $k = 0,1,\ldots,n$. That is $S_0(\underline{x}) \equiv 1$, $S_1(\underline{x}) = \Sigma x_i$, $S_2(\underline{x}) = \sum_{i<j} x_i x_j$, $S_3(\underline{x}) = \sum_{i<j<k} x_i x_j x_k, \ldots,$ $S_n(\underline{x}) = x_1 \cdots x_n$. Verifying the Schur Ostrowski condition, one can see that $S_k(\underline{x})$ is Schur concave on $(0,\infty)^n$ for $k = 0,1,\ldots,n$.

Example 2.4 If g is a convex (concave) function of one real variable, then $f(\underline{x}) = \Sigma g(x_i)$ is Schur convex (concave). This enables one to construct many Schur functions.

a) $f(\underline{x}) = \Sigma \frac{1}{x_i}$ is Schur convex on $(0,\infty)^n$. One may use this result to prove an inequality due to Schweitzer (1914):

Let $0 < m \leq a_i \leq M$ for $i = 1,\ldots,n$. Then

$$\left(\frac{1}{n}\Sigma a_i\right)\left(\frac{1}{n}\Sigma \frac{1}{a_i}\right) \leq \frac{(M+m)^2}{4mM}$$

b) $H(\underline{p}) = -\Sigma p_i \log p_i$ is Schur concave on $(0,1)^n$. $H(\underline{p})$ is called the entropy of $\underline{p}$ when $\Sigma p_i = 1$. Hence the entropy of $\underline{p}$ increases as the p_i's become "more equal".

c) $s(\underline{x}) = \left(\frac{1}{n}\Sigma(x_i - \bar{x})^2\right)^{1/2}$ is Schur convex on R^n. $s(\underline{x})$ is the sample "standard deviation" for the sample vector $\underline{x}$.

One may show that a convex symmetric real valued function f is Schur convex. If f is convex, there are various methods of symmetrizing f while preserving its convexity (and hence generating a Schur convex function). Techniques of this sort enable one to prove a famous inequality due to Muirhead (1903) and Hardy, Littlewood and Polya (1934).

Theorem 2.5 Let $\underline{y} = (y_1,\ldots,y_n)$ where $y_i > 0$ for $i = 1,\ldots,n$. If $\underline{a} \prec \underline{b}$, then

$$\sum_{\Pi} y_{\Pi(1)}^{a_1} \cdots y_{\Pi(n)}^{a_n} \leq \sum_{\Pi} y_{\Pi(1)}^{b_1} \cdots y_{\Pi(n)}^{b_n}.$$

Note in particular that taking $\underline{a} = \left(\frac{1}{n},\ldots,\frac{1}{n}\right)$ and $\underline{b} = (1,0,\ldots,0)$, one obtains the arithmetic-geometric mean inequality:

$$(y_1 \cdots y_n)^{1/n} \leq \Sigma y_i / n.$$

3. Applications in Reliability Theory

Definition 3.1 A system with n independent components which functions if and only if at least k of the components function is a k out of n system.

A parallel system is a 1 out of n system, an $n-1$ out of n system is called a fail-safe system, and an n out of n system is a series system.

If $\underline{p} = (p_1,\ldots,p_n)$ is the vector of components reliabilities (that is p_i = probability that component i functions), then

$$h_k(\underline{p}) = \sum_{\epsilon_1+\cdots+\epsilon_n \geq k} p_1^{\epsilon_1} \cdots p_n^{\epsilon_n} (1-p_1)^{1-\epsilon_1} \cdots (1-p_n)^{1-\epsilon_n}$$

(where ϵ_i is either 1 or 0)

is the probability that k or more of the components function. $h_k(\underline{p})$ is called the reliability function for a k out of n system. Note also that $h_k(\underline{p})$ may be interpreted as the probability of k or more successes in n independent Bernoulli trials with respective success probabilities $p_1,\ldots,p_n$.

Using the Schur Ostrowski characterization of Schur functions and a monotonicity result, one obtains the following Theorem (Boland-Proschan, 1982):

Theorem 3.2 The reliability function $h_k(\underline{p})$ of a k out of n system is Schur convex in $\left[\frac{k-1}{n-1}, 1\right]^n$ and Schur concave in $\left[0, \frac{k-1}{n-1}\right]^n$.

If k=1, that is we are considering a parallel system, the above result says that $h_k(\underline{p})$ is Schur convex on $[0,1]^n$. This means that subject to the constraint that Σp_i is constant, the more spread out the component reliabilities are the more reliable the system is. When considering a series system (k=n), the opposite is true - subject to the constraint that Σp_i is constant, the more equal the component reliabilities are the more reliable the system is.

Example 3.3 Let us consider a 3 out of 4 system. If $\underline{p} = (p_1,p_2,p_3,p_4)$ is the vector of component reliabilities, then

a) (1.0,.9,.8,.7) yields higher reliability than (.95,.95,.75,.75) which in turn is superior to (.85,.85,.85,.85).

b) (.6,.5,.3,.2) is inferior to (.6,.4,.4,.2) which in turn is inferior to (.4,.4,.4,.4).

If $h(\underline{p})$ is the reliability function of a system, one can measure the importance of a component in contributing to system reliability by the rate at which system reliability improves as the reliability of the component improves. More specifically one can define the reliability importance $I_h(j)$ of component j as $I_h(j) = \frac{\partial h}{\partial p_j}(\underline{p})$. (See Barlow-Proschan, 1975).

Now let us consider again a k out of n system. Without loss of generality let us assume that component reliabilities $\underline{p} = (p_1,\ldots,p_n)$ are such that $p_1 \leq p_2 \leq \cdots \leq p_n$. Using the Schur Ostrowski condition and Theorem 3.2, it follows that

a) whenever $\underline{p} \in \left[\frac{k-1}{n-1}, 1\right]^n$ the most reliable component (component n) is the most important to the system, and

b) whenever $\underline{p} \in \left[0, \frac{k-1}{n-1}\right]^n$ the least reliable component is the most important to the system.

Note that this says that for example for parallel systems, the component with highest reliability is the most important to the system. This is intuitively clear as the system functions if only one component functions. On the other hand for a series system, the weakest component is the most important to the system. This reflects the well known adage that a chain is as strong as its weakest link.

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NEWS FOR COMPLEX VARIABLES TEACHERS

Aongus Ó Cairbre and Anthony G. O'Farrell

It is good pedagogical practice to mix solid applications (inside or outside mathematics) with the development of general theory [3]. This is routinely done in general courses, but it is just as important in honours courses, because otherwise the students may get the wrong idea of what mathematics is and how it is done. Mathematics is best done with a specific problem in mind.

People who agree with this point of view will be interested to learn that two major applications of complex function theory have recently been simplified to the point where both can now be presented to average honours undergraduates. Hitherto they were, realistically speaking, first year postgraduate material. The results are the big Picard theorem and the prime number theorem.

The original proof of Picard's theorem, using the elliptic modular function and monodromy, remains firmly at postgraduate level. Of course it was undergraduate material long ago, when it was acceptable to be vague about topological problems. Until last year, the proof normally used was basically that in Landau's "Neuere Ergebnisse", via the theorems of Bloch and Schottky. The new proof is a simplification of this latter proof. It gets the result in one page after Schottky's theorem. The entire proof, assuming the maximum principle, Rouché's theorem, and a knowledge of the logarithm and complex powers, may be presented in two lectures. The new idea is due to Bridges, Calder, Julian, Mines, and Richman, and is explained in [2]. Curiously enough, they found this simpler proof, not because they were trying to, but because they sought a constructive proof, i.e. one not using the apparatus of normal families.

The new proof of the prime number theorem is due to Newman [6]. Until it appeared, the simplest proof was that in Heins' "Topics". The latter proof involved the Riemann-Lebesgue lemma and many technical convergence details. Newman actually offered two proofs. He started by giving an ingenious proof of a Tauberian theorem of Ingham. He observed that Landau's equivalent form of the prime number theorem follows at once. He went on to give the details of another proof, based on the fact that the existence of the limit

$$= \lim_{n\to\infty} \sum_{\substack{p<n \\ p \text{ prime}}} \frac{\log p}{p} - \log n$$

implies the prime number theorem. Korevaar [4] has produced a variation on Newman's method. Korevaar's version is shorter than either of Newman's proofs. It can be presented in three or four lectures, including the basic facts about the Riemann zeta function. Newman's first proof, with all the details included, takes about five lectures. Some of the details are sufficiently straightforward to be left to students. We are inclined to favour Newman's first proof, even though it takes more time, because the proof of the sufficiency of Landau's equivalent form depends upon two gems of number theory, namely the Möbius inversion formula and Dirichlet's estimate

$$d(1) + d(2) + \ldots + d(n) = n \log n + (2\gamma - 1)n + O(\sqrt{n})$$

where $d(n)$ is the number of divisors of n, and γ is Euler's constant. The proof of Landau's equivalent form is in [5]. A more recent reference is [1], which contains a complete account, with all the details, of Newman's first proof.

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Group Theory and Other Abstract Tripe

J.W. Bruce

After completing the marking of the summer examinations, a disturbing experience for most of us, ones thoughts inevitably become rather philosophical. Why am I here? What is it all for? Especially - what are we teaching our students? Not a lot, usually, but what I want to discuss here is the course design rather than our success or otherwise in teaching these courses (although the two are clearly closely linked.) I will concentrate on the honours courses for the same reasons that we usually over emphasise this aspect of our teaching: we have a free hand here in the course content, we can avoid getting involved in anything unsavoury like questions of applications to other subjects and we all went through honours courses ourselves. (I should explain that the "we" used here refers to the mathematical community at large, and since this article is rather critical of this community's policies the "we" probably doesn't include the reader and definitely does not include the author.)

The first hint honours students have that something may be amiss usually comes with the traditional ε, δ analysis course. Hint here is perhaps understating the case, the experience seems more akin to being hit over the head with an intellectual sledge hammer. This is part of the well known disorientation process first year students undergo, where any connections between this new (learning?) experience and school mathematics are minimized. Moreover great pains are taken to prove results which may appear rather trivial to the uninitiated (as well as Newton and Gauss probably) but which use a definition which takes a year to understand. As a colleague[+] put it, "having jumped in the lift and expecting to be shot up to the top floor the student is immediately taken down to the basement." Thankfully this rather austere start is complemented by some light relief in the shape of a course on abstract (what else?) algebra, usually an introduction to group theory. This is also a new experience, so new in fact that it appears hardly to be mathematics at all, more a sort of contrived parlour game (which we call TRIPE). The game involves defining as many new words as possible and relating them with equally many three (but better if you can manage two) line theorems. (Lines here refers to the number of lines of proof; extra marks are awarded if they are exceeded by the number of lines of statement). You know the sort of thing, injective, kernel, subgroup, homomorphism (Patented TRIPE could do more damage to the image of mathematics than the Rubic cube.)

[+]Why should he remain anonymous - it was Brian Twomey.

The thinking behind the game, its educational value is that it impresses upon the student the importance of structure, a key word. (In retrospect it seems rather amazing that Gauss managed without this experience. Yes, I know there are not many Gauss's(?) about, but if you admit he was good, and structure is important, why didn't he spend more time playing TRIPE?)

The second year is usually the worst in the standard three year course. With the alibi of giving final year options the relevant springboard (and usually with no reference to what is actually taught there) we squeeze in large amounts of good solid (= dreadful?) mathematics. The sort of stuff absolutely essential for the safe release of any self respecting honours student (always assuming there are any such students left by year two). The most important ingredient appears to be the "all you ever wanted to know about linear algebra (and a great deal more besides)" course, and the "advanced (now that you don't understand elementary) calculus" course (on an arbitrary Banach space if you are lucky.) From these huge machines, erected with a great deal of trouble there emerges after a ponderous crank of the handle all sorts of wonderful things! Why, did you know one can classify quadrics up to Euclidean motions? Impressed - well just wait until Lagrange multipliers change your life! Probably the saving grace, universally, is the standard second year course on complex analysis. For my fellow students this was the first thing that looked remotely like school mathematics since they started their university career, and contour integrals were devoured with the same relish with which a starving man takes his first decent meal.

And then we have the ultimate intellectual experience, the final year honours courses, which usually come in a collection of rather standard packages. Group Theory, by now fairly popular with the players of TRIPE, supplies even longer definitions and connecting theorems. To prove that these antics aren't the private property of the algebraists a course on Topology is often given (If you thought connected meant in one piece, wait until you have seen our definition!). Adn then there is Functional Analysis. I remember attending a course which started with measure spaces, integration and went on to L^2 spaces, separable Hilbert spaces, the Riesz representation theorem, and discussed unitary, normal, self adjoint (and other) operators on these Hilbert spaces. Feeling rather like Oliver when he asked for more, I remember asking, at the end of the course, "why?" (or words to that effect.) Our lecturer immediately started talking authoratively about Sturm-Louiville theory and differential equations, and I remember feeling very impressed. The only example of a measure he had introduced was the counting measure on a finite set: solving D.E.'s using that would certainly be a good trick. I <u>could</u> go on (what about the courses on algebraic number theory which don't appear to have anything to do with numbers?) and so could you.

Well what are we teaching our students? In a word, or rather two, formal trips. How can one justify teaching a final year course simply on group theory. No mention of Galois theory or geometric transformations, just dreary theorem after dreary theorem. (If the Jordan Holder theorem does something for you, my condolences.) Yet look at the standard undergraduate textbooks on group theory. Similarly should we really let people who want to teach courses on Hilbert spaces and their operators without mentioning differential equations loose on the streets? What is the point of teaching algebraic number theory when we so often duck out of any course on elementary number theory? Why do we do it? One reason is undoubtedly that this type of course <u>is</u> neat and easily available in nice tidy packages. Another may be in that we are the weirdos who actually liked the stuff; or at least did well at it. Just in case you think the author is an exception, may I shamefacedly admit that I lapped this abstract rubbish up as an undergraduate (but couldn't do a contour integral to save my life.) In common with many others my conversion took place during the first year I spent doing research. Having soaked up courses of the above type I had the vague idea that by writing out a suitable list of axioms all sorts of wonderful theorems would pop out and make me famous; mathematics had become a formal game. It took me a year to grow out of this illusion, to realise that a complete understanding of two good examples is worth (and will probably result in) ten good theorems. That the simple version of the theorem is the important one and that any fool could generalize (and I often did).

The effect of our teaching programme is frequently disastrous, for exactly these reasons. The students have no sense of history of the subject, nor its origins. They see no relationship between mathematics and the world in which we live. They are continually confronted with definitions and theorems completely cut off from their historical and quite valid origins. (Abstraction, structure have their place, but at the right level, and this is usually <u>post</u>graduate.) But worst of all we kill any enthusiasm our students have for the subject, which we present as a logical and pedestrian development of results from an apparently arbitrary base made up of some axioms. (We know on the other hand from our research experience, that the subject is anything <u>but</u> logical, that anyone sticking with axioms also sticks with three line theorems, that these axioms have their origins in some very important concrete theorems and examples.) Of course we have our occasional successes as well, but the students involved are often so good it is unclear whether they have succeeded because of or despite our efforts. Where we fail, and fail quite dismally, is with

the hard working students of average ability. These people, who constitute the majority of our honours classes frequently leave the universities disillusioned and perplexed, having gained little or nothing from their three year stay. (An extra reason for concern on our part is that they frequently go on to teach in secondary schools; but our bad influence there is another story.)

In overstating my case, I hope I have trodden on as many corns as possible, and look forward to the ensuing criticism. What I would like to see is an approach to the syllabus taught more in line with the historical development of the subject (ε and δ emerged from problems concerned with Fourier series not calculus). The further one is from physics, number theory and geometry, the three main sources of good mathematics, the more careful one must be about the material taught (perhaps even the research one does?). Abstraction for its own sake can quickly degenerate into irrelevant and trivial nonsense. Probably the worst crime of all to commit is not to give many really good examples in courses. Examples first and unifying concepts (much) later. If we must set out on a new theory we should always be able to justify the journey on the grounds that the material developed solves some interesting problem lying outside the course itself. The need for good exercises is of course also well established. Mathematics is not a spectator sport; but too often now examinations are passed on bookwork, and exercises set during the year for the abstract courses involve definition juggling only, so that the problem solving aspect of our students education suffers.

Mathematics is fascinating, vibrant alive - you all believe this. Yet we (and remember what we means!) seem incapable of persuading our students that this is the case. Despite Russell's assertion to the contrary mathematics is <u>not</u> the subject in which we never know what we are talking about, nor whether what we are saying is true. We just teach it as if it was.

THE COCKROFT REPORT: A review

A Committee of inquiry was established in 1978 under the chairmanship of Dr. W.H. Cockroft, vice chancellor of the New University of Ulster, with the following terms of reference:

> "To consider the teaching of mathematics in primary and secondary schools in England and Wales with particular regard to the mathematics required in further and higher education, employment and adult life generally, and to make recommendations."

The Report "Mathematics Counts", published in January 1982, comprises three main sections: first the mathematical needs of adult life, employment and further education are examined, then a detailed analysis is carried out of the current state of mathematical education in primary and secondary schools, and finally facilities for the training of mathematics teachers both initial and in-service are discussed.

A large number of recommendations is included in the body of the Report and a great deal of statistical information is appended. Accordingly, I can do no more here than present a small sample of this wealth of information and analysis in the hope of stimulating the reader to further investigation. Clearly, also, I cannot hope to make precise the relevance of the Report to the Irish context: such a task would be of an order of magnitude altogether different from this review. However, I believe that its relevance should not be underestimated. Perhaps it could provide a timely stimulus for an investigation into the teaching and learning of mathematics in Irish schools along similar lines to that undertaken by the Committee: perhaps taking less time, perhaps not so broad in its brief, but nevertheless with sufficient influence to recommend changes which may well be long overdue. Is such an investigation necessary?

There is a clear recognition in the Report of the spread of ability in mathematics found among schoolchildren. For example at the primary level

> "It therefore seems that there is a 'seven year difference' in achieving an understanding of place value which is sufficient to write down the number which is 1 more than 6399. By this we mean that, whereas an 'average' child can perform this task at age 11 but not at age 10, there are some 14 year olds who cannot do it and some 7 year olds who can. Similar comparisons can be made in respect of other topics" (342)

[the number in brackets refers to the paragraph in the Report from which the quote is taken]

The principal implications of this lie in the organisation of classroom eaching to cope with the range of ability found within each class. The Report s quite firm in recommending that efforts should be made to adapt the curriculum nd pace of learning *as far as possible* to suit the individual child. How far these fforts can go will obviously depend on the school concerned and on the enthusiasm nd ability of the teacher. However certain helpful aids are discussed such as the se of workcards and textbooks, the deployment of teaching staff in a variety of lternative modes (for example the use of team teaching) and the provision of a mathematics area" within the school. There is also a detailed discussion of the elative merits of the organisation of classes on the one hand into "sets" and on he other into "mixed ability" groups, with the former being regarded as the more ppropriate at the secondary level. Even within the "sets" the range of ability s quite wide enough for any ordinary teacher to cope with!

The Report emphasises repeatedly the need to foster positive attitudes towards athematics from the earliest days at school. The extremes of attitude which can and occur are described thus:

> "By the end of the primary years a child's attitude to mathematics is often becoming fixed and will determine the way in which he will approach mathematics at the secondary stage. He may thoroughly enjoy his work in mathematics, or he may be counting the days until he can stop attending mathematics lessons. He may have learned that mathematics provides a means of understanding, explaining and controlling his environment, or he may have failed to realise that it has any relevance outside the classroom. He may have learned the importance of exploration and perseverance when tackling a problem and experienced the pleasure which comes from finding its solution, or he may regard mathematics as a series of arbitrary routines to be carried out at the teacher's behest, with no opportunity for initiative or independent thought. He may be well on the way to mastering some of the mathematician's skills, or he may already see mathematics as an area of work which he cannot understand and in which he always experiences failure." (346)

It should therefore be the aim of any teaching programme to instil confidence the pupil in his ability to use what mathematics he knows (and however little it y be) and to apply it in his everyday life. In the case of children whose attainent is low, advances should be made very gradually with frequent opportunity for reetition and reinforcement. Use should be made of "extensive and varied practical and oral work related to everyday situations such as measurement, shopping and the use of money" (335)

In particular the Report provides a "foundation list of topics" or core which should be the basis for work in the secondary years (but which should of course not be regarded as limiting the scope of courses provided for children of higher attainment). The broad headings are: number, money, percentages, use of calculator, time, measurement, graphical and pictorial represenation, spatial concepts, ratio and proportion and statistical ideas. Again the need to make mathematics *at all levels* of attainment relevant to everyday life is stressed. This is brought home very clearly by an extract from one of the submissions received by the Committee.

> "Mathematics lessons in secondary schools are very often not about anything. You collect like terms, or learn the laws of indices, with no perception of why anyone needs to do such things. There is excessive preoccupation with a sequence of skills and quite inadequate opportunity to see the skills emerging from the solution of problems. As a consequence of this approach, school mathematics contains very little incidental information. A French lesson might well contain incidental information about France - so on across the curriculum; but in mathematics the incidental information which one might expect (current exchange and interest rates; general knowledge on climate, communications and geography; the rules and scoring systems of games; social statistics) is rarely there, because most teachers in no way see this as part of their responsibility when teaching mathematics." (462)

The Committee's abhorrence of the practice of teaching mathematical routines divorced from the everyday lives of the pupils is evident in the prominence it gives to this extract.

In building on the foundation list of topics the aim should be to provide a "differentiated curriculum" whose content is matched to the level of attainment and rate of learning of the pupils with the aim of improving their mastery of the mathematics they study and inspiring them to greater confidence in their approach to mathematics. This should be achieved by providing two "reference levels" for examination purposes (rather like the Higher and Lower level syllabuses for Intermediate and Leaving Certificate courses). The Report questions the validity of including many of the topics at present appearing on examination syllabuses. For example

> "We cite ... multibase arithmetic. Although this is a topic which offers opportunity for interesting and often challenging work at a variety of levels in the hands of a skilled teacher, and which can therefore appropriately find a place in some classrooms, we do not believe that a question of the kind

> 'evaluate 27 x 3 in base 8' is suitable as an examination question at any level."

One aspect of mathematics which in the opinion of the Committee has been largely ignored in recent years is the use of mental arithmetic. Those who can use mathematics effectively in their daily lives usually employ mental calculations with ease and these are very often based on quite different techniques from the ones they learned at school. Oral and mental work should therefore form a significant component of any mathematics programme. However, although a facility with addition and multiplication tables up to the 10 times table is desirable, the Committee totally reject the call for a "back to basics" approach to teaching mathematics. The movement towards such an approach has

> "encouraged some primary teachers and some teachers of low-attaining pupils in secondary schools to restrict their teaching largely to the attainment of computational skills ...However, we hope that the argument ... makes it clear that the ability to carry out a particular numerical operation and the ability to know when to make use of it are not the same; both are needed ... An excessive concentration on the purely mechanical skills of arithmetic for their own sake will not assist the development of understanding in those other areas. It follows that the results of a 'back to basics' approach ... are most unlikely to be those which its proponents wish to see, and we can in no way support or recommend an approach of this kind." (278)

A detailed discussion of the Committee's comprehensive analysis of the examination system is inappropriate here as it deals of course with the current position in England and Wales. However their description of the spirit of an examination is perhaps worthwhile recording.

> "We believe that there are two fundamental principles which should govern any examination in mathematics. The first is that the examination papers and other methods of assessment which are used should be such that they enable the candidates to demonstrate what they do know rather than what they do not know. The second is that examinations should not undermine the confidence of those who attempt them." (521)

One of the "other methods of assessment" adverted to is indicated:

> "... we believe that provision should be made for an element of teacher assessment to be included in the examination of pupils at all levels of attainment." (535)

However even this combined with formal examinations may be inappropriate for lower-attaining pupils and the Report recommends that there be further research into the assessment of such pupils.

The subject of teacher training is similarly unsuitable for detailed discuss here. However it may again be worthwhile to pick out one paragraph: this one deali with the aims of courses provided by the universities for intending teachers of mat matics. According to the Committee these should be

"to develop knowledge and mastery of mathematics substantially beyond the level at which they will be teaching and also, where appropriate, provide opportunity to pursue some topic in depth;
to develop enjoyment of mathematics and confidence in its application;
to provide an historical perspective of mathematics;
to provide an appreciation of the relationship between mathematics and other fields of study;
to develop the ability to communicate mathematical ideas both orally and in writing." (643)

There is also a great deal of room for improvement in the number of girls taking mathematics at degree level or as the principal subject in a teacher traini course. Indeed, the Report includes a substantial appendix on "Differences in mat matical performance between girls and boys", which may point the way to the resolu of the problems of disparity of numbers and acute shortage of well-qualified teach of mathematics.

The Committee urge a significant improvement in the in-service support provided for teachers

" ... any improvement in the standards of mathematics in schools must come largely as a result of the efforts of those teachers who are already in pos [and therefore] all those who teach mathematics need continuing support throughout their careers in order to be able to develop their professional skills and so maintain and enhance the quality of their work." (715,716)

In its conclusion the Report does not attempt to draw up a list of recommen tions preferring rather to leave these to be interpreted in the context in which t arise. Instead the Committee identify six agencies

"whose active response ... [is] essential if the changes in mathematical education [recommended] are to be brought about." (809)

These are: teachers, local education authorities, examination boards, central government, training institutions and those who fund and carry our curriculum development and educational research. To these is added a seventh, namely the public at large whose support is a *sine qua non* of progress. There is a widespread belief among the general population that every boy and girl at school needs to develop an understanding of mathematics and confidence in its use.

"In our view this can only come about as the result of good mathematics teaching by teachers who have been trained for their work and who receive continuing in-service support. It must therefore be the task of all who share this belief to support and encourage the implementation of the changes which we believe to be necessary and to make it clear that, as part of the education which our children receive, mathematics counts." (810)

Reference:

Mathematics Counts, Report of the Committee of Inquiry into the Teaching of Mathematics in Schools, HMSO (1982) $US15.75, 311 pages.

P. Fitzpatrick

DINEEN ON HOLOMORPHY - A REVIEW

COMPLEX ANALYSIS IN LOCALLY CONVEX SPACES, by Sean Dineen, North-Holland, 1981, 492 pp.

"The main purpose of this book ..... was to provide an introduction to modern infinite dimensional complex analysis, or infinite dimensional holomorphy as it is commonly called, for the graduate student and research mathematician. Since we were more interested in communicating the nature rather than the scope of infinite dimensional complex analysis ..... we chose to develop a single theme which has made much progress in recent years and which exemplifies the intrinsic nature of the subject, namely the study of locally convex topologies on spaces of holomorphic functions in infinitely many variables." Thus begins the author's foreword. There follows a comprehensive introduction to infinite dimensional holomorphy from the topological viewpoint, complete with exercises for the reader, a helpful historical commentary and an extensive bibliography.

Infinite dimensional holomorphy can trace its origins at least as far back as Hilbert, but in the last 18 years a great explosion of research has taken place, and most of the material of the book has come from this period. The unifying theme is the problem of how best to topologise the space of holomorphic functions. Consider the case of one complex variable. $H(U)$ will denote the set of holomorphic functions on the open subset U of the complex plane. Having formed this set, one's immediate instinct is to equip it with some structures. For example, $H(U)$ is a complex vector space. To see how naturally the question of topology arises, consider the convergence of the Taylor series.

Suppose for simplicity that U is a disc with centre a, so that for every f $H(U)$, the Taylor series at a converges to f at every point of U. Let s_n be the n-th partial sum of this series. In what precise sense does the sequence s_n converge to f in the space $H(U)$? If U is not a disc, the Taylor series at one point will not, in general, represent f throughout U. However, Runge's Theorem tells us that it may be possible to approximate f by polynomials, or rational functions; in other words, these functions form a dense subset of $H(U)$ for a certain topology. The "right" topology in this case is the compact open topology, τ_o — a sequence f_n in $(H(U),\tau_o)$ converges to a function f if $f_n(z)$ converges to $f(z)$ uniformly on each compact subset of U. This topology arises naturally in many settings, and has many useful properties; for example, it is compatible with the vector space structure of $H(U)$, it is metrizable and is complete. Thus $(H(U),\tau_o)$ is a Fréchet space. On a deeper level, $(H(U),\tau_o)$ is also Nuclear. These properties open the doors of an armoury of weapons from Functional Analysis which are essential elements in the proofs of many of the classical theorems of complex analysis in one and several variables.

If U is now a domain in an infinite dimensional space, the situation becomes much more complicated. τ_o is defined on $H(U)$ in the same way, but in many important cases, one finds that those properties which made it so useful in finite dimensions, such as metrizability and nuclearity, no longer apply. There appears a galaxy of different topologies on $H(U)$, each with its own justification, sometimes agreeing with one another, more often not. The exploration of these topologies has been central to the development of infinite dimensional holomorphy in recent years, and very many of the great advances which have been made bear Dineen's name.

This book is not simply an account of the topology of H(U), rather is it a comprehensive introduction to infinite dimensional holomorphy, the inspiration for the development coming from these fundamental topological problems. The prerequisites for the reader would include, of course, complex variables, but, more importantly, a reasonable knowledge of functional analysis, including the elements of locally convex spaces. A useful appendix provides a summary of definitions and results from several complex variables and functional analysis.

Chapter 1, Polynomials On Locally Convex Topological Vector Spaces, introduces the building blocks of the Taylor series, the homogeneous polynomials. Several types of polynomials, such as continuous, hypocontinuous and nuclear, are met, and various topologies on the spaces of polynomials are studied. The duality theory of polynomials is here, together with the special features of polynomials on nuclear spaces.

Chapter 2, Holomorphic Mappings Between Locally Convex Spaces, introduces the reader to holomorphic mappings on open sets, and holomorphic germs on compact sets, and their elementary properties. The three most important topologies on H(U), τ_0, τ_ω and τ_δ are introduced.

Chapter 3, Holomorphic Functions On Balanced Sets : The balanced set in infinite dimensions replaces the disc in the complex plane - it has the crucial property that the Taylor series at the centre of the set represents the function throughout the set. Thus H(U) is, in some sense, the direct sum of the subspaces of homogeneous polynomials. One can then hope that topological properties of the spaces of polynomials can be pieced together to give results about H(U). This idea is exploited here, the main tools being Schauder decompositions and associated topologies.

Chapter 4, Holomorphic Functions On Banach Spaces, and Chapter 5, Holomorphic Functions On Nuclear Spaces With A Basis continue the study of holomorphic functions on two contrasting types of domains. There are no infinite dimensional spaces which are at the same time Banach and nuclear, and the theories for these two types of spaces develop in different ways. For Banach spaces, the emphasis is on the interplay between the geometry of the space and the holomorphic functions. The Maximum Modulus Theorem, Schwarz's Lemma and their applications are here, together with bounding sets, and the equality of the topologies τ_ω and τ_δ on Banach spaces with unconditional bases. In nuclear spaces with bases, we have first a coordinate system, and we find that suitable nuclearity conditions on the space allow us to write the Taylor series using monomials, which are simply products of the coordinates. Again, using the basis, one can construct polydiscs and Reinhardt domains. This leads to a very satisfactory duality theory, and the resolution of many of the topological problems.

Chapter 6, Germs, Surjective Limits, ε-Products and Power Series Spaces. The chapter opens with a further study of spaces of holomorphic germs on compact sets, and their relationship to the study of H(U). Surjective limits provide a method for constructing, by a projective process, spaces with good holomorphic properties. The ε-product, which can be viewed as a generalised tensor product, relates the theory of vector-valued functions to that of scalar values. The chapter closes with some recent results on representations of spaces of holomorphic functions on certain sequence spaces.

Each chapter is accompanied by a set of exercises. Some of these are easy, some challenging, and some, in the author's own words, "quite difficult". They should at least be read, as many indicate further areas of research, and introduce topics not covered in the text. Appendix III, Notes on Some Exercises, has hints and explanations, and references to the literature for the interested problem-solver. Appendix II, already mentioned, consists of definitions and results from functional analysis, complex variables and topology. Appendix I, Further Developments in Infinite Dimensional Holomorphy, is a survey of current research which emphasises areas not treated in the book.

At the end of each chapter is a section entitled Notes and Remarks, comprising a fascinating historical account of the subject matter of the chapter, together with many illuminating insights, suggestions for further research, and a guide to the literature. The Bibliography is enormous, containing some 725 entries in all, ranging from papers by Volterra and Von Koch in the 1880's up to the present. This is the first complete listing of papers in holomorphy and will be of great value to workers in the field, and indeed, to interested spectators.

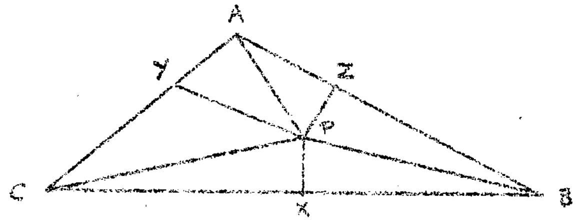
It is the reviewer's opinion that this book is a major contribution to infinite dimensional holomorphy. It succeeds admirably in its stated aims, and while giving a complete account of the theory from the topological viewpoint, is in no way closed or static - one is always led on to think of the next step, the right generalisation, the open problem. This will surely be the holomorphist's bible for many years to come. May it gain many converts!

R. Ryan

Problems Page

New or old, solved or unsolved, published or unpublished, this page will discuss any problem which has that certain something. Please send problems, solutions and references to the Editor.

1. Let P be an arbitrary point in a scalene triangle ABC, and let PX, PY, PZ be the internal bisectors of ∢BPC, ∢CPA, ∢APB respectively.



Prove that

$$|PA| + |PB| + |PC| \geq 2(|PX| + |PY| + |PZ|).$$

As far as I know this is Barrow's inequality, but I have no reference. The weaker inequality, in which PX, PY, PZ are perpendicular to BC, CA, AB respectively, is due to Erdos.

2. This problem came from Tom Laffey. Is

$$\inf \{|n \sin n| : n \in \mathbb{N}\} = 0?$$

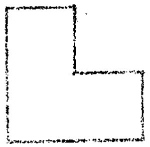
Finbarr Holland, working with his ZX81, found the approximation $\pi \approx 355/113$ which gives $355 \sin 355 \approx 0.0107$. Later Jo Manning checked (on the IBM) that this is the smallest value of $|n \sin n|$ for $1 \leq n \leq 10^6$. The problem would be answered in the affirmative if π were approximable by rationals to order $2 + \varepsilon, \varepsilon > 0$, and so presumably this is also open. Bill Bruce found an article by Chudnovsky (Springer Lecture Notes in Math, 751) which gives negative results on the approximation of π by rationals.

3. If $m,n \in \mathbb{N}$ and $t = \min[m,n]$ then

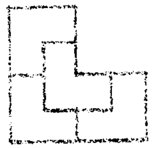
$$\sum_{d=1}^{t} 2^d \binom{m}{d} \binom{n}{d} = \sum_{d=1}^{t} \frac{(m+n-d)!}{(m-d)!\,(n-d)!} .$$

These sums arose as different people's solutions of the same problem, which was mentioned by David Singmaster.

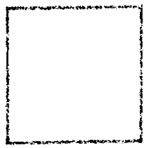
4. Some dissection problems. After getting your victim to dissect



into 4 congruent sets



follow up by asking him to dissect the square

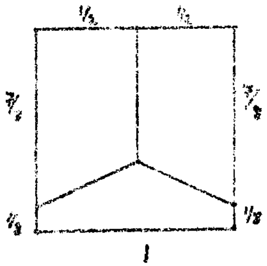


into 5. Time taken is inversely proportional to ingenuity!

A genuine problem here is to show that if p is any odd prime then there is only one dissection (apart from rotations) of the square into p congruent, connected sets.

Another hard problem is to decide whether the square can be dissected into an odd number of triangles each having the same area.

Finally, an easier one. If the unit square is dissected into 3 sets, at least one of them has diameter $\geq d = \sqrt{65/64}$. This number is the best possible.



Phil Rippon

CONFERENCE REPORTS

REPORT ON THE NASECODE II CONFERENCE

(Communicated by JJH Miller of the Numerical Analysis Group, Dublin)

The second international conference on the Numerical Analysis of Semiconductor Devices and Integrated Circuits, NASECODE II, was held in Trinity College, University of Dublin, Dublin, Ireland from June 17th to 19th, 1981, under the auspices of the Numerical Analysis Group. It was attended by over 100 delegates from 22 countries. The aim of this series of conferences is the fostering of a fruitful exchange of ideas between electronic engineers and numerical analysts.

The industrial sector was strongly represented at this conference, as was the case also at the NASECODE I Conference. Therefore, the problems of computation and numerical analysis discussed are of great practical importance as well as being intellectually challenging.

The application of numerical methods to semiconductor device modeling began about 15 years ago, and since then it has developed and broadened in scope very rapidly. To date relatively few professional numerical analysists have worked in this area, and consequently it is still a fertile source of stimulating unsolved problems of widely varying degrees of difficulty.

The models of technological importance are mainly in two space-dimensions and they may also be time dependent. Typically, two or three nonlinear differential equations have to be solved on complicated domains with a variety of boundary conditions. Computational experience indicates that the systems are often very stiff.

For the numerical analyst there is a wealth of problems. Frequently, underflow occur and special tricks have to be used to allow the computation to proceed. Convergence of the iterative method for solving the discrete nonlinear system is usually a problem. The very fine meshes generally used in certain parts of the domain give rise to large discrete systems, and consequently the systems to be solved after linearisation are large. Many standard linear equation solvers, both direct and iterative, are impractical or simply fail for these problems. The development of practical and efficient techniques for solving extensions of these problems to three space dimensions and to the non-stationary case are also needed.

For a representative collection of papers on the subject the reader may consult the three publications (1), (2) and (4) associated with the NASECODE conferences. The first two monographs on the subject are Kurata (3) and Mock (5). The main journals covering engineering aspects are (6) and (7), while the more computational and mathematical aspects will be discussed in the new journal (8). The third conference in the series, NASECODE III, will be held in Galway, Ireland from 15th to 17th June, 1983; it is cosponsored by the Electron Devices Society of the IEEE and the Irish Mathematical Society.

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GROUPS IN GALWAY

This year, the Group Theory Conference was again held in Galway on May 14 and 15. It was a highly successful event with participants (24) and speakers from Ireland, Britain and Australia. The programme consisted of hour long talks by :

R.A. Bryce (Q.M.C./A.N.U.), A. Christofides (E.C.G.), P. Fitzpatrick (U.C.C.), and B. Siefert (U.C.D.) together with shorter talks by N. Barry (Carysfort), T.C. Hurley (U.C.G.), J.D. Key (Birmingham), J. McDermott (U.C.G.), J. Siemons (U.C.C.), and A. Williamson (U.C.G.).

Originally Jan Saxl (Cambridge) was scheduled to talk but due to a bereavement was unable to travel.

The talks, formal and informal, produced much discussion, some new results were announced and many ideas were exchanged. As usual, the social events stimulated the Mathematics and contributed to the enjoyment of the occasion. As usual, also, it never rains and the sun always shines on I.M.S. Groups.

We are grateful to Ray Ryan for this years' group(!) photograph. Unfortunately, two of our Dublin colleagues were omitted as their car broke down. They eventually arrived but too late for the photograph. We hope to insert their photographs so they will not be lost to posterity. This event is too important for people not to have their cars serviced the week before!

Martin Newell and his colleagues are to be congratulated for organising this, by now, annual get-together of Algebraists.

Go mbeirimíd beo ar an am seo arís.

TED HURLEY

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