CONFERENCE ANNOUNCEMENTS

LINEAR MATHEMATICS 1982.

A two-day Algebra Conference will be held at Mary Immaculate College, Limerick on November 12th and 13th, 1982. The main speakers are Prof. J.R. Thompson (Cambridge), Prof. T.J. Laffey (U.C.D.), Prof. W.I. Howell (U.C.G.) and Dr. D. MacHale (U.C.C.). Persons seeking further details or willing to give short talks of up to 20 minutes duration are invited to contact the organiser Dr. G.M. Hartigan, Mathematics Department, Mary Immaculate College, South Circular Road, Limerick.

Phone (061)44583.

PROCEED III.

The Third International Conference on the Numerical Analysis of Semiconductor Devices and Integrated Circuits will be held in Galway from 15th to 17th June 1983, under the auspices of the Numerical Analysis Group and sponsored by the Electron Devices Society of the IEEE, the Institute for Numerical Computation and Analysis and the Irish Mathematical Society.

Contributed papers are solicited from engineers, physicists and mathematicians on any topic relevant to the numerical analysis, modelling and optimization of electronic, opto-electronic and quantum electronic semiconductor devices and integrated circuits. The deadline for the receipt of abstracts and preliminary versions of 20-minute contributed papers is 18th February, 1983. All correspondence should be addressed to: NUMANO Conference, 39 Trinity College, Dublin 2. Phone (01) 772361 ext. 1809/1499.

I.C.T.E. - V

The Fifth International Congress on Mathematics Education will be held in Adelaide, South Australia, in August 1984. The chairman of the International Program Committee is Dr. M.P. Newman, Department of Mathematics, Australian National University, Canberra.

INFINITE EXPONENTIALS

P.J. Rippon

A question of the following type appeared during 1981 in a Regional Math. Contest for high school students in the U.S.A. and caused some difficulties for the referees present.

"Find all the real numbers a such that
\[ a^{a^{a^{\ldots}}} = 2. \]

The expected answer, presumably, was that
\[ a^{a^{a^{\ldots}}} = a \Rightarrow a = 2, \]

and so \[ a = 8^{1/8}. \] Hers (1) has been taken to mean that the sequence
\[ a, a^a, a^{a^a}, \ldots, \]
which we shall denote throughout by
\[ a_1 = a, a_{n+1} = a^{a_n}, \quad (n = 1, 2, \ldots) \]
converges to 8. Now it is not immediately obvious that the sequence \( a_n \) will be convergent when \( a = 8^{1/8} \approx 1.3 \) and this difficulty occurred to the referees at the Math. Contest. In fact it turns out that the sequence is convergent. Unfortunately however it does not converge to 8 and the question was very nearly scrubbed!

In this article I shall attempt to explain the somewhat surprising facts outlined above and survey a number of the known results about the convergence of such 'infinite exponentials'. Both the real and complex cases will be discussed and some new results given. I am greatly in debt to the survey by Knobloch [4] which appeared coincidentally at about the same time as the Math. Contest and which contains a huge bibliography on this topic. I am also grateful to Leon Greenberg, whose ingenious approach to the convergence of \( a_n \) (for real \( a \)) first kindled my interest in this problem, and to many colleagues and students at U.C.C. for advice and encouragement, particularly with the computer.

To see why \( a = 8^{1/8} \) is not a solution of (1) we consider the graph of
\[ y = x^{1/x}, \quad x > 0. \]
If $b$ is the unique number less than $e$ satisfying
\[
 x = \left( \frac{1}{\ln b} \right) e^x,
\]
then $1 < a < b$. Since $a^b = b^a$, we deduce by induction that the increasing sequence $a_n$ is bounded above by $b$ and so converges to some number $c$, $1 < c < b$. Because $a^c = c$, we must have $c = b$. A little work with a calculator shows that this limit $b$ is approximately 1.46.

It turns out that the behavior of the sequence $a_n$ for $a > 0$ was studied as long ago as 1778 by Euler [3]. He, it seems, was already aware of the fact (since rediscovered many times) that the sequence is convergent if and only if
\[
 1/a < a < e^{1/e}.
\]

The proof of convergence when $1/a < a < e^{1/e}$ uses the arguments given above for $b^{1/b}$.

On the other hand if $a_n$ converges to $b$ then $a = b^{1/b}$ and no convergence is possible if $a > e^{1/e}$.

The case $0 < a < 1$ is less straightforward since the sequence $a_n$ is no longer monotonic. In fact
\[
 0 < a_1 < a_2 < a_3 < \ldots < a_0 < a_1 < a_2 < 1,
\]
that is, $a_n$ lies strictly between $a_0$ and $a_1$ for each $n = 2, 3, \ldots$. The proof is by induction, starting from $0 < a < 1$, using logarithmic representation. It is natural then to study the graph of $y = a^x$, $x > 0$. As we have seen, the (unique) solution of $a^x = x$ is the graph of $y = a^x$, $a = 0.53$.

In this case it is clear from the graph that the sequence $a_n$ does not converge.

The problem becomes much harder if we allow $a$ to assume complex values. In this case we put $a_1 = a$ and
\[
 a_n = \exp(a \log a) = a^n, \quad (n = 1, 2, \ldots)
\]
with equality only at the single point of inflection of $y = a^x$, which occurs when $a = (\log e)^{-1}$. If $(1/e)^a < a < 1$, therefore, the graph of $y = a^x$ crosses the graph of $y = x$ exactly once, at $x = b$.
where the principal value of \( \log a \) is taken. This excludes numbers on the negative real axis from consideration, though we note that \( a = -1 \) is an exceptional special case. Quite a lot is known about the convergence problem for complex \( a \) but there are still a number of interesting open questions.

The main problem concerns the set

\[
\mathbb{C}_a = \{ e^{i\theta} : |\theta| \leq \frac{\pi}{2} \}.
\]

which is illustrated below.

This set meets the real axis precisely in the interval \( \mathbb{R}_a = \{(1/a)^n, \pi/(1/a)\} \) on which each sequence \( a_n \) is convergent. Computing evidence suggests that \( a_n \) may in fact be convergent throughout \( \mathbb{R}_a \) and we shall survey a number of partial results towards this.

The set \( \mathbb{R}_a \) was first identified by Carson in his thesis [2] of 1904, who showed that if \( a_n \rightarrow w \) as \( n \rightarrow +\infty \) and if \( a_n \rightarrow w, \ n = 1, 2, \ldots \), then \( a \in \mathbb{R}_a \). In fact, by (3), \( w = \exp(w \log a) \) and on putting \( \xi = w^{1/2} \), we obtain \( w = e^\xi \) and \( a = e^{i\xi} \). It only remains to show that \( |\xi| \leq 1 \). To do this we let

\[
b_n = a_n - 1, \quad (n = 1, 2, \ldots)
\]

so that, by (3),

\[
(1+b_n+1) = \exp(1+b_n) \log a
\]

and hence

\[
b_n+1 = \exp(\xi b_n) - 1.
\]

Since \( b_n \rightarrow 0 \), \( n = 1, 2, \ldots \), and \( b_n \rightarrow 0 \) as \( n \rightarrow +\infty \) we have

\[
b_n \rightarrow 0, \ n \rightarrow +\infty.
\]

Thus

\[
|\xi| = \lim_{n \rightarrow +\infty} \left| \frac{b_n+1}{b_n} \right| \leq 1,
\]

as required. We also observe from (4) and (5) that if \( \arg a \neq 0 \) then

\[
\arg \left( \frac{b_n+1}{b_n} \right) = \arg \left( \frac{b_n+1}{b_n} \right) = \arg \xi, \ n \rightarrow +\infty.
\]

This shows that the sequence \( a_n \) converges to \( w \) in a spiral-like manner. As an example we illustrate the convergence of the sequence \( 1, i, i^2, \ldots \), which will be proved later.

We shall deduce positive results from the following

lemma. Let \( \Omega \) be any domain bounded by a simple closed curve \( \Gamma \) and let \( f: \Omega \rightarrow \Omega \) be a continuous function, analytic in \( \Omega \), which does not map \( \Omega \) conformally onto itself. If \( f \) has no fixed point on \( \Gamma \) then \( f \) has a unique fixed point \( z^* \) in \( \Omega \) and for any \( z \) in \( \Omega \) the sequence

\[
x_{n+1} = f(x_n), \quad (n = 1, 2, \ldots)
\]

converges to \( z^* \).

I am grateful to Peter Walker for pointing out how this lemma can be used to prove the convergence of \( 1, i, i^2, \ldots \). It also serves to unify and simplify many of the approaches which have been made to the general problem. For other results of this type we refer to Burchel [3].

To prove the lemma first note that \( f \) must have at least one fixed point \( z^* \) in \( \Omega \) by Brouwer's theorem (in any particular application the existence of an interior fixed point can often be demonstrated more directly). We assume, as we may by the Riemann mapping theorem, that \( \Omega \) is the unit disc and that \( z^* = 0 \). By
Schwarz's lemma
\[ |f(z)| < |z|, \quad (0 < |z| < 1) \]
the inequality being strict because \( f \) is not a rotation of \( \Lambda \). Thus \( |z_n| \) is decreasing and so
\[ r = \lim_{n \to \infty} |z_n| \]
exists. Some subsequence of \( z_n \) is convergent, to \( z \), say, and by the continuity of \( f \) we have
\[ |f(z)| = |z| = |z| \]
According to (6), therefore, we must have \( z = 0 \) and so \( r = 0 \), as required.

As a first application we show that
\[ R_r = \{ z : |\log a| \leq e^{-\delta} \} \]
is a set of convergence for \( a_n \). This is due to Thron [7].

Letting
\[ \Lambda = \{ z : |\log a| < 1 \} \]
we find that
\[ |\log(a^z)| = |z \log a| < 2 \quad (a \in \mathbb{C}, a \notin R_r) \]
since \( |e| = e \) when \( e \in \mathbb{R} \). The inequalities here are strict, except when \( z = e \), and so \( a^z \) maps \( \mathbb{C} \) properly into itself. The only possible fixed point of \( a^z \) on \( \mathbb{C} \)
is \( z = e \) and this can only occur when \( |\log a| = e^{-\delta} \). This would mean that \( a = e^{1/a} \)
and we already know that \( a_n \) is convergent for this value of \( a \). For any other \( a \) in
\( R_r \) the lemma gives the desired convergence of \( a_n, a_n, \ldots \).

Another major contribution towards the solution of this problem was made
by Schell in his thesis of 1953, part of which was published in [6]. We shall prove
here one result of his, that \( a_n \) is convergent when \( a < \log 2 \).

For any \( a = e^{u-e^{-\delta}} \), \( |z| \leq 2 \), consider the disc
\[ \Omega_{e^z} = \{ z : |z - e^z| < |e^z| \} \]
If \( z \in \Omega_{e^z} \) then \( |z - e^z| \leq 2 \) and so (8)
\[ |a^z - e^z| = |e^{u} - 1| \leq |e^z| \]
\[ |\exp[|z| \log a|] - e^z| \]
\[ \leq |e^z| |\exp[|z| \log a|] - 1| \]
\[ < 2 \cdot |e^z| \]
\[ \leq |z| \]
It follows that \( a^z \) maps \( \Omega_{e^z} \) into itself and evidently has \( z = e^z \) as a fixed point.

This mapping is not a rotation (for instance, its derivative at \( e^z \) is equal to \( e^z \))
and it is easy to check that \( z = 1 \) lies in \( \Omega_{e^z} \) except when \( z = -\log 2 \) (corresponding
to \( a = \frac{1}{2} \), where the convergence of \( a_n \) is already known). According to the proof of
the lemma the sequence \( 1, a, a^2, \ldots \) converges to \( e^z \), as required.

(1) The inequality \( |e^z - 1| < |z| - 1 \), which we use here, follows immediately from the
Taylor series for \( e^z \).
A more careful study of (9) reveals that if \( \|z\| < 1 \) then there is a number \( \delta(z) > 0 \) (which depends continuously on \( z \)) such that \( \delta \) maps the disc \( \Delta_{\delta} \) with centre \( e^z \) and radius \( \delta(z) \) into itself. Thus if \( a_n \in \Delta_{\delta} \) for any \( n \) we must have \( a_n \rightarrow e^z \) as \( n \rightarrow \infty \). By continuity we deduce that the set of interior points of \( \mathbb{C} \) for which the sequence \( a_n \) is convergent forms an open set.

By now the reader will appreciate that the approach being used is to find a 'domain of invariance' for the mapping \( \delta \) which is large enough to include the point \( z \) itself. To give some idea of the types of domains of invariance which might be encountered we illustrate the sequence \( a_n \) in two particular cases where its convergence has not been established.

An equivalent way to look at this problem is to make the substitution used earlier in (4), namely,

\[
b_n = a_n^{\frac{1}{n}} - 1, \quad (n = 1, 2, \ldots)
\]

where \( a = e^{\delta(z)} \). We then have

\[
b_1 = a^{\frac{1}{n}} - 1 = \exp[\delta(a^{\frac{1}{n}} - 1)] - 1, \quad \text{and}
\]

\[
b_{n+1} = \exp[b_n] - 1, \quad (n = 1, 2, \ldots)
\]

and it is natural to also put \( b_0 = a^{\frac{1}{n}} - 1 \) and even \( b_{-1} = -1 \). The problem then is to show that \( b_n \to 0 \) whenever \( |\delta| \leq 1 \). Shil's argument showed that this was true when \( |\delta| \leq \log 2 \) by observing that the unit disc is invariant under the mapping \( e^{\delta(z)} - 1 \) for such \( \delta \).

To provide some food for thought we illustrate the spiral-like behaviour of the sequence \( b_n \) when \( \delta = 1 \) (which corresponds to \( a = 1.99 + 1.19i \)).

It seems clear that this sequence converges to 0, albeit slowly. For instance, calculations on a computer indicate that when \( n = 200,000 \) the modulus of \( b_n \) is about 0.1. In this case however there would be no hope of finding a suitable domain of invariance \( \Omega \) for \( f(z) = e^{iz} - 1 \) since \( |f'(0)| = 1 \), which would force \( f \) to map \( \Omega \) conformally onto itself.

Neither of the positive results proved earlier covers the case \( a = 1 \), corresponding to \( \delta = 0.64071 \), which is naturally of particular interest. Apparently a proof of the convergence when \( a = 1 \) was given in Shil's thesis, but this does not appear in his subsequent paper. However the following stunningly simple proof was published by Macintyre [5] in 1966. Let

\[
\Omega = \{z: 0 < x, y < 1\}.
\]

Then \( f(z) = e^{iz} \) maps \( \Omega \) properly into itself, since

\[
i = e^{i\pi/2} = e^{-i\pi/2} \quad \text{and} \quad 1 = e^0.
\]

Then \( f(z) = e^{iz} \) maps \( \Omega \) properly into itself, since

\[
i = e^{i\pi/2} = e^{-i\pi/2} \quad \text{and} \quad 1 = e^0.
\]
By inspection $f$ has no fixed points on $\mathbb{R}$, and so we can apply the lemma with $z_1 = 1$, which is a fixed point of $f$. The same argument shows that the sequence $z_n$ converges whenever $z$ is of the form $e^{i\theta}$, $0 \leq \theta \leq \pi/2$. By continuity, therefore, we have convergence throughout $R_N = \{e^{i\theta} : 0 \leq \theta \leq \pi/2\}$.

To get an impression of the results so far we illustrate the sets $R_C, R_{34}, R_2$, and $R_N$ together and remind the reader that the set of convergence is open in the interior of $R_C$.

![Diagram of convergence sets]

It is interesting that there is no relationship of containment among $R_C, R_{34}, R_2$, and $R_N$.

We finish with an intriguing fact that does not seem to have been mentioned in the literature. The sequence $a_n$ is in fact convergent for many numbers lying outside $R_C$. An obvious example is $a = -1$ but there are also, for instance, unbounded sequences of numbers $a$ in the first and fourth quadrants having the property that $a^n = a$. For such numbers we clearly have $a_1 = a_2 = a_3 = \ldots$.

To demonstrate their existence we show that, for each positive integer $k$, there is a number $a$ lying in

$$S = \{z : z > 0, 0 < \arg z < \pi/2\},$$

such that

$$e^{i\theta} - 1 = 2\pi ki.$$  \hspace{1cm} (10)

In that case $a = e^{i\theta}$ lies in the first quadrant and

$$a^n = \exp[z\log(a)] = \exp[z e^{i\theta}] = \exp[z + 2\pi ki] = a.$$  

Moreover the solution $z$ of (10) will be unbounded with $\theta$. One way to show that these solutions exist is to rewrite (10) as

$$z = \log(1 + 2\pi ki)\frac{1}{a}.$$  

The function on the right-hand side of this equation maps the first quadrant into $S$ and has no fixed points on the boundary. Apart from proving that (10) has a unique solution $z$ in $S$ for each $k$, the lemmas allow us to compute approximate values for these solutions and for the corresponding numbers $a$. The first few of these are illustrated below and it is clear that they all lie outside $R_C$, a fact which can also be proved analytically.

![Graph illustrating solutions]

Further work along these lines (details of which will appear elsewhere) leads to the existence of other numbers such that $a_2 = a_3 = a_5 = \ldots$. We give some examples, which the reader is invited to check (preferably with the help of a computer): Due allowance should be made for round-off errors.

<table>
<thead>
<tr>
<th>$a_1 = a_2 = \ldots$</th>
<th>$a_2 = a_3 = \ldots$</th>
<th>$a_3 = a_4 = \ldots$</th>
<th>$a_4 = a_5 = \ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.8629 + 3.2231i</td>
<td>2.4279 + 0.55465i</td>
<td>1.9813 + 0.16031i</td>
<td>1.78295 + 0.082164i</td>
</tr>
<tr>
<td>3.7273 + 5.3180i</td>
<td>2.6921 + 0.58735i</td>
<td>2.0599 + 0.16702i</td>
<td>1.81660 + 0.071851i</td>
</tr>
<tr>
<td>4.4332 + 7.1930i</td>
<td>2.8513 + 0.60067i</td>
<td>2.1024 + 0.15843i</td>
<td>1.83403 + 0.066124i</td>
</tr>
</tbody>
</table>

Finally I should be grateful if someone would disprove (or, better still, prove) the following (rather wild) conjecture which has been bothering me for some time now. Could it be that the set of numbers $a$ such that the sequence $a_n$ converges is actually dense in the complex plane?
Majorization and Schur Functions

Philip J. Boland
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The concepts of majorization and Schur functions lay the basis for a rich and elegant theory in which many classical and applicable inequalities may be viewed. In this expository paper the basic definitions and properties of majorization and Schur functions are presented, together with a variety of applications emphasizing in particular some in reliability theory. For a thorough and recent account of majorization and Schur functions, the interested reader should consult the excellent inequalities: Theory of Majorization and its Applications by Marshall and Olkin (1979).

1. Majorization

Given a vector \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), let \( x_{(1)} \preceq \ldots \preceq x_{(n)} \) denote a decreasing rearrangement of \( x_1, \ldots, x_n \).

**Definition 1.1** If \( x, y \in \mathbb{R}^n \), then \( x \prec y \) if

\[
\sum_{1}^{j} x_{(1)} \leq \sum_{1}^{j} y_{(1)} \quad \text{for } j = 1, \ldots, n-1
\]

and

\[
\sum_{1}^{n} x_{(1)} = \sum_{1}^{n} y_{(1)}.
\]

If \( x \prec y \), we say that \( x \) is majorized by \( y \). Note that if \( x \prec y \), then the components of \( y \) are more "spread out" than those of \( x \). For example \( \frac{1}{n}, \ldots, \frac{1}{n} \prec \left( \frac{1}{n-1}, \ldots, \frac{1}{n-1}, 0 \right) \prec (1,0,\ldots,0) \), and \((3,\bar{x},\ldots,\bar{x}) \prec (x_1,\ldots,x_n)\) where \( \bar{x} = \frac{\sum x_i}{n} \).