IRISH MATHEMATICAL SOCIETY

NEWSLETTER

No. 2 1979

The Irish Mathematical Society was founded at a meeting in T.C.D. in December 1976. Its aims are to foster and encourage mathematical development in Ireland.

The aim of this Newsletter is to inform members of the Society of the activities of the Society and also of items of general mathematical interest. The success of the Newsletter depends very much on the co-operation of the members of the Society with the compilers. Information on activities of interest to members are sought, as well as survey articles, problems and solutions.

The present address for correspondence relating to the Newsletter is:- I.M.S. Newsletter, Department of Mathematics, U.C.D. Belfield, Dublin 4.

TABLE OF CONTENTS

REPORT ON THE ACTIVITIES OF THE SOCIETY

T.C. Hurley

Since our last Newsletter the Society put most of its effort into two main areas, viz. conferences and the Mathematical Olympiad in schools.

The Group Theory Conference in Galway was a tremendous success with over 25 participants, including a small number from abroad. Great credit is due to the main speakers for their very informative presentation, tending to review some of the main areas of the theory and not getting bogged down in technical detail. Martin Newell, the organiser, has plans well in hand for another conference in May this year and informs us that he hopes to make it an annual event. Group Theory is alive, well and flourishing in Ireland.

The History of Mathematics Conference in Cork attracted a great deal of interest with about 25 participants. The nature of the material made it very amenable, and all the participants seemed to enjoy it very much. The article "Calculating Prodigies" by J. Callagy, which appears in this Newsletter had its origin at this Conference. Many thanks are due to Des MacHale, the organiser. (It didn't turn out to be another Kerryman joke as some believed!).

The Conference on Matrix Theory and its Applications held recently attracted not only Mathematicians, but also Statisticians, Economists, Business Admists., Engineers and Educationalists. Personally, I had not realised the extent of the applications of the theory in all areas, and the interplay between the various disciplines was very stimulating.

The Mathematics Contest for schools took place on 6th March with about 700 Irish students taking part. The contest was organised by the Society with the co-operation of the Irish Mathematics Teachers Association. The organising subcommittee consisted of Finbarr Holland, Tom Laffey and Fred Holland. I don't know how they did it, but they managed, despite the postal strike, to get the papers delivered to places as far away as Letterkenny and Bunclody. Results are slowly trickling in but the postal strike is holding things up. The contest is being sponsored by the Educational Company of Ireland Ltd., and we are grateful for their support.

At a meeting in Helsinki, agreement was not reached on the formal creation of a European Federation of Mathematical Societies. The Oberwolfach Institute for Mathematicians has, in the meanwhile, placed their offices at Freiburg at the disposal of the interested parties and is, among other things, collecting, collating and disseminating information of Mathematical interest. It is from there that we receive the list of forthcoming Mathematical Meetings which you will have seen.

At the instigation of the Society, in particular Finbarr Holland and Martin Newell, the National Council for Educational Awards (N.C.E.A.) conducted a numeracy-type test in the Regional Colleges. The N.C.E.A. decided not to reveal the results.

We have about 120 members in the Society but a number of these have fallen behind in their subscription - still only £2.00. If you are one of them, our treasurer Martin Newell (U.C.G.) would be most pleased to hear from you. A list of the Committee appears on page 36, and you may also hand your subscription to any one of these. We would welcome contributions to the Newsletter - short surveys, problems, solutions to previous problems, news and information, jokes, etc. - even criticism. The address is:- The Editor, I.M.S. Newsletter, Mathematics Department, University College, Belfield, Dublin 4.

NEWS AND NOTES

IRISH NATIONAL MATHEMATICS CONTEST

The first National Mathematics Contest was held on March 6, 1979, and the results are now being compiled. The chief organiser of the contest is Dr. Finbarr Holland, who as first president of the I.M.S. made the institution of this contest one of the aims of the I.M.S. Through his trojan efforts, the contest was held despite the handicap of the postal dispute and over 700 students participated. The Society is very grateful to the M.A.A. Committee on High School Contests and the Executive Director of the American High School Mathematics Examination who gave permission to use their test. (This test is used in the United Kingdom and Canada as well as the U.S. and it is held simultaneously in participating countries). The contest could not have been held here without the co-operation of the teachers and we wish to thank them individually and as members of the I.M.T.A., for carrying out the test and for encouraging their students to participate. The Society also wishes to express its gratitude to the Educational Company of Ireland Ltd. who have kindly agreed to sponsor the contest.

A full account of this year's contest and the results will appear in the next issue of the Newsletter.

SHORT CONFERENCES

One of the main activities of the Society has been the organisation of short, mainly instructional, conferences. The next conference of

this type will be held in University College, Galway, on May 11-12, and will be on Group Theory. The organiser is Professor M.L. Newell. A similar conference on Function Theory will be held in University College, Cork, in the Autumn.

These conferences are a good way of encouraging research and of building up a mutual awareness of the work of one's colleagues. They nicely supplement the D.I.A.S. symposia in that they are each devoted to one particular topic, while the symposia cater for the mathematical community in a more general context. For this reason, the Society strongly encourages its members to undertake the task of organising more conferences of this type.

CONTRIBUTIONS TO THE NEWSLETTER

As was stated in the first issue of the Newsletter, it is hoped to include in each issue some short surveys of research topics or topics in the history of Mathematics. Contributions of this type are sought from readers for consideration for inclusion in the Newsletter. From the discussion on Mr. Con O'Caoimh's talk at the I.M.S. conference on Matrix Theory and its Applications, it is clear that many members have very strong (and divergent) views on what type of Mathematics should be taught in second-level schools. We would also welcome contributions on this topic for the Newsletter.

FOR YOUR DIARY

I.M.S. Group Theory Conference. U.C.G., May 11-12, 1979. (Details from Professor M. Newell, U.C.G.).

Conference on the Numerical Analysis of Semiconductor Devices. Trinity College Dublin. June 27-29, 1979. (Details from Secretary, NASECODE 1, 39 T.C.D., Dublin 2).

Conference on Current Problems in General Relativity. Dublin Institute for Advanced Studies, July 2-6, 1979. (Details from The Director, School of Theoretical Physics (Working Seminar 1979), D.I.A.S., 10 Burlington Road, Dublin 4).

Dundee Biennial Conference on Numerical Analysis, Dundee, June 26-29, 1979.

L.M.S. Conference on Aspects of Contemporary Complex Analysis. Collingwood College, Durham University, July 1-20, 1979.

L.M.S. Conference on Progress in Analytic Number Theory, University of Durham, July 22-August 1, 1979.

L.M.S. Conference on Noetherian Rings and Rings with Polynomial Identity. University of Durham, July 22-August 1, 1979. (Details from Dept. Math., University of Durham, Durham DH1 3LE).

A.M.S. Summer Research Institute Finite Group Theory. University of California, Santa Cruz. June 25-July 20, 1979 . (Details from Dr. W.J. Le Veque, A.M.S., P.O. Box 6248, Providence, R.I. 02940).

CALCULATING PRODIGIES

James Callagy *(U.C.G.)*

During the eighteenth and nineteenth centuries some remarkable individuals appeared from time to time whose phenomenal powers of rapid mental calculation excited the interest of mathematicians and psychologists. Those who possessed such powers from an early age and were mostly self-taught are the most interesting:

> Buxton (1707 - 1772), an English farmhand; Fuller (1710 - 1790), a negro slave; Whateley (1757 - 1863), later Protestant Archbishop of Dublin; Colburn (1804 - 1840), a farmer's son of Vermont, U.S.A.; Bidder (1806 - 1878), son of an English stonemason; Dase (1824 - 1861), of Hamburg; Safford (1836 - 1901), of Harvard; and three shepherds, Mondeux, Mangiamele and Inaudi, about 1867 - 1900.

It is significant that all but two have been completely ignored by historians of mathematics who always tell us of the extraordinary calculating powers possessed by great mathematicians like Euler and Gauss at an early age. Apparently, there exists a consensus of expert opinion that ability to do mental arithmetic rapidly has very little, if any, correlation with mathematical insight and creativity. Whatever about the converse, this view probably derives from a tradition originating with classical Greek mathematicians who clearly defined 'arithmetica' or number theory, as a liberal pursuit, and 'logistica', or practical computation, as unworthy of inclusion in the 'mathemata'.

We find this view reflected in Renaissance writers like Leonardo Bruni (1536) who expressed nothing but contempt for the medieval 'calculatores' and in our own times the historian of science, George Sarton, protesting that press reporters and other ignorant people attribute mathematical genius to those lightning calculators who can perform fantastic computations.

When one studies the methodology of some of those mentioned in achieving amazing feats of numerical skill, however, the classical argument appears insufficient nor does it justify the attitude adopted by historiographers towards them. Sarton's statement that it represents mathematical ability of a very low order loses much of its credibility too, when one considers the unique analytical processes (often devised ad hoc) of calculators like Bidder, Colburn and Inaudi, apart altogether from the time factor. We are fortunate in this respect to have complete explanations, notably from Bidder and Colburn on their own methods and from Binet and

Darboux on those of Inaudi. They indicate mathematical ability of no mean order and at times achieve the quality of elegance which we may define as the direct attainment of a foreseen end. Modern mathematics, to be sure, owes a supreme debt to the Greeks for having consistently substituted ideas for calculations, but one feels that those calculating prodigies belong to a more ancient tradition, the wealth of which has only recently been tapped by archeological research. This tradition goes back over four millenia to the indefatigable Babylonian calculators who compiled those admirable tables of reciprocals, powers and 'Pythagorean triples', using a cumbrous cuneiform notation to achieve amazingly accurate results. Since their sexagesimal system was not readily adaptable to abacal computation, and for that matter being positional it does not require it, the art of rapid mental calculation must have been highly cultivated a good thousand years before the Greeks appeared on the scene.

Perhaps the best lightning calculator of recent times was Professor A.C. Aitken of Edinburgh who described what goes on in the mind of a rapid mental calculator. After a lecture in 1954 he referred to the discouraging influence of modern machines on people with talents like his, remarking that "mental calculaters may, like the Tasmanians or the Maori, be doomed to extinction, but you may be able to feel an almost anthropological interest in having seen one".

The humanists amongst us, however, still continue to feel a lively mathematical interest in their careers and methodology and at our History conference in U.C.C. we looked at the achievements of some of those heroic individuals of the eighteenth and nineteenth centuries already mentioned. The following discussion necessarily excludes some relevant complementary material which was projected on transparencies at the conference (30/9/75).

JEDEDIAH BUXTON (1707 - 1772) born at Elmton, Derbyshire, was the son of the village schoolmaster who completely neglected his education. He never learned to read or to write figures and his mental faculties were of a low order. Numbers however had such a fascination for him that if any object was stated he began at once to compute how many inches it measured or what its area or volume was. His only practical accomplishment was to estimate by inspection the acreage of an irregular field very accurately but his calculations were slow. He could find the time for sound to travel 5 miles at 1142 ft. per second mentally in 15 minutes, but in 1751 he calculated the product of three numbers of seven, eight and five digits respectively carrying on the computation over several days in his mind.

In 1754, he was examined by some members of the Royal Society while in London, where some of his friends brought him to Drury Lane to see a play of Garrick's. Curious to study his reaction, they asked his opinion which was negative except for the fact that he could tell them the exact number of words uttered by the various actors and the number of steps in the dances. It was found also that he worked on bases 60 and 15 and had invented a notation of his own using terms like 'tribe' for 10^{18} and 'cramp' for 10^{36} when recording large numbers. A curious and perhaps unique feature in his case was his ability to handle two or more different calculations simultaneously.

THOMAS FULLER (1710 - 1790) was a Negro, born in Africa. He was captured in 1724 and exported as a slave to Virginia, U.S.A. Like Buxton, he never learned to read or write and his abilities were confined to mental arithmetic. He could multiply more rapidly than Buxton two numbers of up to nine digits each and give the number of grains of corn in a given mass or the number of seconds in a given period of time. Problems involving proportion, or the 'rule of three' as it was known, were his speciality. He was a slow calculator compared with others who follow.

RICHARD WHATELY (1757 - 1863) was remarkable for his ability to calculate rapidly at the age of five or six years and the fact that he later became Archbishop of Dublin. We shall allow him to speak for himself:- "Soon" he wrote, "I got to do the most difficult sums always in my head, for I knew nothing of figures beyond numeration, nor had I any names for the different processes I used. But I believe my sums were chiefly in multiplication, division and the rule of three. I did these much quicker than anyone could do on paper and I never remember having committed an error. I was engaged either in calculation or 'castle-building' morning, noon and night but when I went to school, at which time the passion wore off, I became a perfect dunce at ciphering, and so have continued to be ever since".

ZERAH COLBURN (1864 - 1840) was perhaps the most celebrated of the lightning calculators who could claim some recognition from historiographers because of his paramount influence on the career of William Rowan Hamilton at the age of fifteen. To quote Sir Edmund Whittaker (May 1954) "Young Hamilton loved poetry and the classics but his interests and the whole course of his life were completely changed when he met one Zerah Colburn, an American youngster, who gave an exhibition in Dublin of his powers as a lightning calculator in 1820". Referring to the occasion in later life, Hamilton recalled: "For a long time afterwards, I liked to perform long operations in arithmetic in my mind, extracting square and cube roots and everything that related to the properties of numbers".

Colburn was the son of a small farmer of Cabut, Vermont, U.S.A., and at the age of six showed extraordinary powers of mental calculation which he exhibited on tours of America. In 1812, he performed in London where he was repeatedly examined by competent observers. It was clear that the child of eight years operated by certain rules and while doing his calculations his lips moved as if he was expressing the process in words. He was able to explain his method in some cases but his speed in finding factors of large numbers, square and cube roots almost instantaneously was amazing. When asked once for the square of 4395, however, he hesitated but on the repetition of the question he gave the correct answer 19,316,025. Questioned about his hesitation, he replied that he disliked multiplying two four-figure numbers but he had thought of another way. In his own words: "I got the factors of 4395 as 293 x 15. I multiplied 293 by 293 and then the product twice by 15".

In 1814 he was in Paris but his exhibition fell flat amid the political turmoil of the time. Some English friends and Americans like Washington Irving raised a fund for his education and he was admitted to the Lycée Napoleon and later to the Westminster School in London. With education, however, his powers of rapid calculation declined and he lost the boyish frankness which had charmed his audiences. Subsequently, he commenced on the stage, tried his hand at teaching school, then became an itinerant preacher and, finally, a 'professor of languages'. It would seem that he was destined to have saved William Rowan Hamilton from a similar fate. Colburn did however find time off to write his autobiography containing an account of his methods before he died at the age of 36. His longest time for any calculation seems to have been about three seconds, otherwise he gave correct answers instantly to such problems as 8^{16}; $n^{10}\ \forall n \leq 9,\ n \in N$; $106929^{\frac{1}{2}}$; $268{,}336{,}125^{\frac{1}{3}}$ so rapidly that the gentleman who was taking them down had to ask him to repeat them. But, although unable to explain his method of factoring, his ability to deal with primes was marvellous - as Cajori records - in readily showing that $2^{2^5}+1 = 4{,}294{,}967{,}297$ had prime factors $6{,}700{,}417 \times 641$. This was the example given by Euler as invalidating Fermat's Prime Number formula $2^{2^n}+1$, of which Colburn as a boy was unaware.

GEORGE PARKER BIDDER (1806 - 1878) a contemporary of Colburn, however, is by far the most interesting of all self-taught calculators, because he subsequently received a University education, retained his lightning power of calculation to the end of his life, and gave us a full analysis of the methods he invented and used. Born at Moreton, in Devonshire, at the age of six he was taught by his father to count up to 100. Although he knew nothing of the symbols nor the meaning of arithmetical terms, using the counting process only he taught himself the results of addition, subtraction and multiplication of numbers up to 100 by arranging and re-arranging marbles, buttons or pebbles in patterns. In later life he attached great importance to this experience and believed that his powers of calculation were strengthened by the fact that, ignorant of the written symbols, he had to rely on concrete vocal representation of numbers only. When he was nine years old his father took him on tours about the country to exhibit his extraordinary powers. In less than a minute he gave the time it would take sound to travel a distance of 123,256 miles at 4 miles per minute, in days, hours and minutes. By the age of 10 he could find the square root of 119,550,669,121 in 30 seconds. In 1817, two distinguished Cambridge graduates, Jephson and Herschel, were so impressed by his general intelligence as well as his calculating abilities that they raised a fund for his education and persuaded the father to abandon the role of showman. But in a few months the father had changed his mind and insisted upon his son's return. In 1818 he was matched against Colburn who was then two years older than Bidder, and proved to be the better calculator. Finally, the father and son came to Edinburgh where some members of the University succeeded at last in persuading the father to leave the boy in their care. In due course, Bidder graduated with distinction in Civil Engineering, became a successful railroad engineer and designed, and also supervised, the construction of the Victoria docks in London. He retained his amazing powers of rapid calculation to the end of his life, which proved a valuable asset to him as a frequent parliamentary witness in engineering matters. Just before his death, at the age of 72, he gave an illustration of them to a friend who, in connection with the then recent discoveries in physics, remarked that if 36,918 waves of red light, occupying only one inch, are required to give the impression of red to the eye, how immense must be the number of rays striking the eye in one second, if light travels at 190,000 mls per second; "You needn't work it out", said Bidder, "the number will be 444,433,651,200,000".

Bidder had two elder brothers, one an actuary, the other in religion, and also a son who later became a distinguished barrister - all of them had similar powers of rapid mental calculation but none of them developed their powers to the same extent. Even in the third generation, a grandson and a granddaughter inherited the same talents.

HENRI MONDEUX and VITO MANGIAMELE, both of whom were born in 1826, are an interesting pair about whom there was a suspicion that they had been exploited by others who taught them rules enabling them to simulate powers they didn't possess. Both were shepherds, in very poor circumstances and after short careers as exhibitioners returned to their sheep once more. In 1839 and 1840 they were brought to Paris and tested by Arago, Cauchy and others. Mondeux's performances were striking. One question put to him was the solution of the equation $x^3 + 84 = 37x$, to which he answered 3 and 4, not detecting the third root -7; another was to find integral solutions for $x^2 - y^2 = 133$, to which he replied at once (67, 66) and when asked for a simpler solution he said instantly (13, 6). As children they were indeed remarkable, but Mondeux if he were really self-taught like the other prodigies, would have to be credited with the discovery of algebraic theorems making him a mathematical genius. In that case, he would certainly have achieved far more than he did.

JOHANN MARTIN ZACCHARIAS DASE (1824 - 1861) of Hamburg, on the other hand, was the calculating prodigy who made some special contributions to the history of mathematics. Having had a fair education and every opportunity to develop his powers he made little progress beyond reckoning and numerical calculation. He was dull-witted, knew only German, and remained ignorant of geometry to the end of his life. He held various small official posts in Germany from time to time and also gave exhibitions of his skill in Germany, Austria and England. While in Vienna he met Strasznicky who urged him to apply his powers to scientific purposes and was introduced to Gauss, Schumacher and Petersen. As a lightning calculator he holds the unbeaten record for having found the square root of a 100-digit number in 52 minutes. Like the others he had a phenomenal memory and could repeat all the numbers mentioned in a performance one hour afterwards. His peculiar gift, somewhat like Buxton's, was ability to calculate at a glance the number of sheep in a flock, books on a shelf or the number of letters in a line of print chosen at random. His calculations on paper were incredibly rapid but always correct, and at the age of 16, Strasznicky taught him the formula :

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{5} + \arctan\frac{1}{8}$$

and asked him to calculate π approximately. In two months he carried the approximation to 205 places of which the first 200 were correct - a result which was published in Crelle's Journal 1844. From 1844 to 1847 he was employed in the Prussian Survey and in his leisure time calculated the natural logarithms of the first 1,005,000 numbers to seven decimal places. On the recommendation of Gauss, the Hamburg Academy of Sciences gave him a special grant for the purpose of making tables of factors for all numbers from 7×10^6 to 10^7 but he lived only long enough to finish about half the work - he was aged 37 when he died.

TRUMAN HENRY SAFFORD (1836 - 1901) of Royalton, Vermont, U.S.A., although always a rapid calculator in his youth, gradually lost his exceptional powers with education at Harvard where he became a professional astronomer. At the age of 10 he was examined by experts answering readily such questions as: 'What is the surface area of a regular pyramid of slant height 17' and whose base is a regular pentagon of side 33½' — which he answered correctly as 3354.5558 sq. ft. in two minutes. Like Colburn, he factorized large numbers with ease but could not say how.

JACQUES INAUDI (1867 -) who started life as a shepherd spent the long hours of his watch pondering on numbers but, unlike Bidder, used no concrete representation like marbles etc. - except perhaps 'variants' such as sheep. At the age of six his brother, an organ-grinder, took him on tours through Provence, during which he earned a few sous in street exhibitions. His extraordinary ability in calculation attracted the attention of some showmen who took him to Paris in 1880. Still ignorant of reading and writing, he was very impressive in his performances and his powers steadily improved until he could multiply two numbers of ten digits rapidly. In Paris, Binet and Darboux posed him many problems such as the square root of $\frac{1}{6}$ the difference between the square of 4801 and unity which he found immediately. He could find integral roots of equations and integral solutions of problems by a method of trial and error, but his most remarkable feat was the expression of numbers less than 10^5 in the form of a sum of four squares, taking no more than a minute or two. This power was unique and most mental calculators found considerable strain attempting it - as indeed have most mathematicians attempting to prove Bachet de Méziriac's (1612) theorem that any number may be expressed in this form, which was first proved by Lagrange in 1770.

We are indeed indebted to Binet and Darboux for detailed descriptions of a typical performance of Inaudi's, published in the Comptes Rendus and the Revue des deux Mondes of 1892, and a complete analysis of his own methods by Bidder, given in a lecture to the Institute of Civil Engineers in 1856. It was generally thought that as well as having phenomenal memories most lightning calculators visualized numbers, but both Bidder and Inaudi relied altogether on articulation and hearing. Inaudi always repeated numbers proposed to him slowly to his assistant who wrote on a blackboard, and relied on his speech muscles and his ear during calculation; he never glanced at the written symbols which would only confuse him, he said. Bidder said it would have taken him four times longer if numbers were written for him as they would not be so vividly impressed on his imagination, and never used symbols at all in his mental procedure. He thought of a number, like 984 for instance, in a concrete way as a collection which could be arranged in 24 groups of 41. Dase on the other hand appears to have visualized the numerals as on paper. Another feature common to many of them was the ability to repeat the numbers occurring in a performance hours afterwards, and also to give the correct sequence of digits forwards or backwards from any selected point. Bidder once did this, one hour afterwards, in the case of a number of 43 digits. He had developed an associative principle for multiplication at the age of eight, by which he could multiply two numbers of six digits each in about seven seconds - a facility he acquired by practice at the village blacksmith's forge where the locals congregated in those days to match feats of skill. He multiplied from left to right - contrary to school practice - adding his partial products as he proceeded so that he never had more than two numbers to add at a time. It is remarkable that Inaudi also multiplied in this way and sometimes used negative quantities as in $27 \times 729 = 27(730 - 1)$. In division Bidder used a process peculiar to himself which he termed a digital process, where a 'digital' is defined as 'the sum of the digits (mod 9)'. That the digital of a number is equal to the product of the digitals of its factors (mod 9) is a theorem which can be applied to find if 73, for instance, is a factor of 23,141.

A curious question has been raised as to whether a law can be found for the rapidity of mental working of calculating prodigies. Bidder stated that in multiplying a number of n digits by itself he believed the strain on his mind varied as n^1 (assuming strain proportional to time). In the case of Dase it seems to have been n^3, but more detailed information and observation

would be needed to support any theory on this subject. Perhaps, some day, computer scientists, interested in cybernetics, and behavioural psychologists, interested in numerics, may be able to enlighten us when they come up with the calculating prodigy par excellence - an automatic self-programming digital computer.

PERFECT NUMBERS

John Cosgrave
(Carysfort)

A natural number m is said to be perfect if the sum of all the divisors of m equals $2m$. Thus, 6 and 28 are perfect, because 1,2,3 and 6 are the factors of 6; 1,2,4,7,14 and 28 are those of 28 and $1+2+3+6 = 12 = 2 \times 6$.

$$1+2+4+7+14+28 = 56 = 2 \times 28.$$

One could say that m is perfect if the sum of all the factors of m, less than m, equals m. Thus

$$1+2+3 = 6$$

and $1+2+4+7+14 = 28$.

Anyone who did the calculations required to find which integers m, with $m \leqslant 1000$ (say), were perfect, would find only three examples, namely 6,28 and 496. These numbers have the following "prime factorization"

$$6 = 2^1 \times 3^1$$
$$28 = 2^2 \times 7^1$$
$$496 = 2^4 \times 31^1$$

3,7,31 are the only proper odd factors of 6,28 and 496 respectively, and they are also primes - not only that, for it is also striking that they are each one less than a power of 2, and also these powers of two are in each case one more than the corresponding power of 2 occurring in the prime decompositions of 6,28 and 496 (namely $2^1, 2^2, 2^3$) so one is led to guess that:

<u>Theorem 1</u> If the number (2^n-1) is prime, then the number $2^{n-1}(2^n-1)$ is perfect.

<u>Proof</u> Let $p = (2^n-1)$ and suppose p is prime. Then the factors of $2^{n-1}.p$ are as follows:

$$1,2,2^2,\ldots,2^{n-1} \qquad \text{(their sum is } (2^n-1))$$

and $p, 2p, 2^2p, \ldots, 2^{n-1}p$ (their sum is $(2^n-1)p$).

Thus the sum of all the factors of $2^{n-1}.(2^n-1)$ is:

$$(2^n-1) + (2^n-1)p, \quad \text{i.e. } (2^n-1)(1+p). \quad \text{This equals } (2^n-1)2^n,$$

which is $2.(2^{n-1}.p)$. So $2^{n-1}(2^n-1)$ is perfect.

Now it <u>could</u> be that $2^4 p$ is perfect for some prime p other than 31. Lets see what p would have to be like. Suppose 2^4p <u>was perfect</u>, then since the factors of 2^4p are: $1,2,2^2,2^3,2^4$ (whose sum is 31), and $p,2p,2^2p,2^3p,2^4p$ (whose sum is $31p$), it would follow that

$$31 + 31p = 2 \times (2^4p) = 32p.$$

i.e. $31 + 31p = 32p$

i.e. $31 = 32p - 31p = p.$

so p would <u>have to</u> equal 31 after all!

You are now ready for

<u>Theorem 2</u> If the integer $2^{n-1}.p$ is perfect, where p is an odd prime, then $p = (2^n-1)$.

<u>Proof</u> The factors of $2^{n-1}.p$ are:

$1, 2, 2^2, \ldots, 2^{n-1}$ (whose sum is (2^n-1))

and $p, 2p, 2^2p, \ldots, 2^{n-1}p$ (whose sum is $(2^n-1)p$),

and it would follow that:

$$(2^n-1) + (2^n-1)p = 2 \times (2^{n-1}p) = 2^n p.$$

$$\text{i.e.} \quad (2^n-1) + (2^n-1)p = 2^n p.$$

$$\text{i.e.} \quad (2^n-1) = 2^n p - (2^n-1)p = p.$$

So we have $(2^n-1) = p.$

Theorem 1 and 2 were known to Euclid and tell us that the only even perfect numbers of the form $2^{n-1}p$ are those for which $p = (2^n-1)$. These perfect numbers are said to be of Euclid type. The theorem of Euler (18th century) states that the only even perfect numbers are those of Euclid type (so there are none of the form, say, $2^4p^2q^3$, where p, q are odd primes).

What we now wish to ask is this, when is the number (2^n-1) a prime number. These numbers are called the Mersenne numbers $(M_n, M_n=(2^n-1))$ after the 17th century French mathematician Mersenne, (see for example, Volume One of Dicksons Theory of Numbers).

M_2, M_3, M_5 and M_7 (i.e. 3,7,31 and 127) are prime, M_4, M_6, M_8, M_9 and M_{10} are composite. It is easy to show that if n is composite then M_n is composite. It was once (wrongly) thought that if n is prime then M_n is prime and this is seen to be false for the next case after M_7 for $M_{11} = (2^n-1) = 2047 = 23 \times 89$.

Theorem 3 If n is composite so also is M_n.

Proof Since n is composite, then $n = a \times b$ for some integers a and

b with $a, b \geq 2$. Now $M_n = (2^n-1) = (2^{ab}-1)$

$$= (2^a-1)(2^{a(b-1)}+\ldots+2^b+1).$$

Now $a \geq 2 \Rightarrow (2^a-1) \geq (2^2-1) = 3$,

and $b \geq 2 \Rightarrow (2^{a(b-1)}+\ldots+2^b+1) \geq (\text{——} +2^2+1) = 5$.

Thus M_n is the product of two integers (2^4-1) and $(2^{a(b-1)}+\ldots+2^b+1)$ neither of which is 1, and so M_n is composite.

So we are now interested in the numbers $M_2, M_3, M_5, M_7, M_{11}, M_{13}, M_{17}, M_{19}, M_{23}, M_{29}, M_{31}, \ldots$ i.e. the Mersenne numbers with prime suffix. M_2, M_3, M_5 and M_7 were known to be prime by A.D.100, Regius (1536) noted that M_{11} is composite. Cataldi (1603) showed that M_{13}, M_{17} and M_{19} are primes (by checking for prime divisors a la Eratosthenes), but erred in claiming that so also are M_{23} and M_{29} and M_{37} (why did he not say anything about M_{31}?). These last three were put right by Fermat (1640) who showed that $(2^{23}-1)$ is divisible by 47, and that $(2^{37}-1)$ is divisible by 223; and Euler (1732) who showed that $(2^{29}-1)$ is divisible by 1103.

In 1644, Mersenne claimed that the first eleven perfect numbers are given by $2^{p-1}(2^p-1)$ for $p = 2,3,5,7,13,17,19,31,67,127,257$; but he erred at least in including 67 and excluding 61,89 and 107. Laczo (1867) proved that $(2^{67}-1)$ is composite (without actually exhibiting a factorization). Cole (1903) gave an explicit factorization of M_{67} which was

$$(2^{67}-1) = 193{,}707{,}721 \times 761{,}838{,}257{,}287.$$

Euler (1750) showed that $(2^{31}-1)$ is prime;

Lucas (1876) showed that $(2^{127}-1)$ is prime;

Pervouchini (1883) showed that $(2^{61}-1)$ is prime;

Powers (1911) showed that $(2^{89}-1)$ is prime;

Powers (1914) showed that $(2^{107}-1)$ is prime.

So all the Mersenne primes (in order) up to M_{127} are:

$$M_{2,3,5,7,13,17,19,31,61,89,107,127}.$$

$$M_{127} = 170,141,183,460,469,231,731,687,303,715,884,105,727.$$

D. Lehiner (1951-52) showed that $(2^{521}-1)$, $(2^{607}-1)$, $(2^{1279}-1)$, $(2^{2203}-1)$ and $(2^{2281}-1)$ are primes. Riesel (1958) showed that $(2^{3217}-1)$ is prime. Harwitz (1960?) showed that $(2^{4253}-1)$ and $(2^{4423}-1)$ are primes. Gollies (1964) showed that $(2^{9689}-1)$, $(2^{9941}-1)$ and $(2^{112,3}-1)$ are primes. Tuckermann (1971) showed that $(2^{19937}-1)$ is prime. Lastly, the 25th Mersenne prime was discovered on the 30 October 1978 by two 18 year old students (at the California State University), Laura Nickel and Kurt Noll:

$$M_{21,701} = (2^{21,701}-1) \text{ is prime,}$$

and so,

$$2^{21,700}(2^{21,701}-1) \text{ is the 25th even perfect number.}$$

__Theorem__ (Euler): Let p be an odd prime, then any prime factor of $M_p(=(2^p-1))$ must be of the form $(2np+1)$ for some integer n. This, together with the observation that if a number N is composite it has a prime factor $\leq\sqrt{N}$, enabled Euler to verify the primality of Mp for small values of p. To take an example: to check if $(2^{13}-1)$ is prime, $(2^{13}-1) = 8191$. According to Euler, any prime factor of $(2^{13}-1)$ must be of the form $(2n.13)+1$, i.e. of the form

$(26n+1)$. According to Eratosthenes we need only search for prime divisors up to $\sqrt{8191}$, which is between 90 and 91. Now when

Now when $n=1$, $26n+1 = 27$ is not prime;
$n=2$, $26n+1 = 53$ is prime;
$n=3$, $26n+1 = 79$ is prime;
$n\geq 4$, $26n+1 \geq 105 > 91$; so no need to check.

Now it only remains to check if 53 or 79 divide 8191. If neither does, we know 8191 is prime. You should check to see.

The Euler test involves so much calculation for large p as to make his test of no practical value. The following test (the only one used - the only other one, in fact) is the one which has been used since 1876.

__Lucas-Lehmer Theorem__: Let $p = 3,5,7,11,13,17,19,\ldots$ (odd prime). Define the sequence $\{a_n\}$ as follows:

$$a_1 = 14, \quad a_2 = a_1^2-2, \quad a_3 = a_2^2-2 \quad \text{etc.}$$

Then (i) if M_p divides a_{p-2}, M_p is prime.
(ii) if M_p does not divide a_{p-2}, M_p is composite.

__Examples__ __When $p=3$__. $M_p = 2^3-1 = 7$. $a_{p-2} = a_{3-2} = a_1 = 14$ and here M_p divides a_{p-2}, and M_p (i.e. 7) is prime.

__When $p=5$__. $M_p = (2^5-1) = 31$. $a_{p-2} = a_{5-2} = a_3$. Now $a_1 = 14$. $a_2 = 14^2-2 = 196-2 = 194$. $a_3 = 194^2-2 = 37636-2 = 37634$, therefore M_5 divides a_3, and 31 divides 37634, therefore M_5 is prime.

Shorter Solution (using congruences)

$a_1 = 14$, $a_2 = 14^2-1 = 196-2 = 194$.

Now $a_2 = 194 \equiv 8 \pmod{31}$ (since $\underline{194} = (6.31)+8$)

Therefore $a_2 \equiv 8^2 \pmod{31}$

$a_2^2-2 \equiv 8^2-2 \equiv 64-2 \equiv 62 \equiv 0 \pmod{31}$

$a_3 \equiv 0 \pmod{31}$. Therefore M_5 divides a_3, and M_5 is prime.

SECONDARY SCHOOL MATHEMATICS

T.J. Laffey

There has been quite an amount of discussion for some time now on the content of mathematics courses in secondary schools. Many people feel that there has been a general decline in the level of computational skill and ability to solve problems in students who leave school. Various attempts are being made to identify the reasons for this and to find remedies. Among the reasons suggested are the following:-

(1) The "New Math",

(2) Over-emphasis on teaching "concepts" and a feeling that as long as the "concepts" are O.K., the answer doesn't matter,

(3) The more abstract presentation of material, particularly geometry, with the result that the students are not taught to relate mathematics to commonsense and experience,

(4) An unwillingness on the part of teachers and pupils to spend large amounts of time going through, perhaps somewhat dull and repetitive routines, in order that students get to know these techniques thoroughly.

At present, there is a great deal of debate going on throughout the world on the value of the so-called New Math. (i.e. sets, relations nd functions done abstractly, geometry based on axioms about mappings, etc) and there appears to be a growing feeling against its being appropriate material for secondary school curricula. While very many mathematicians would not lament if students never heard the word set or group in a mathematical context before coming to Third Level courses, completely reversing curricula to what they were twenty years ago would not necessarily solve the problem. In the U.K. similar difficulties in relation to computational ability have been encountered, despite the fact that relatively little of the New Math. has been introduced in curricula there.

In relation to (2),(3) above, many remedies are being suggested, such as that pupils (particularly pass level students) should only be subjected to Mathematics which is immediately applicable in non-mathematical areas (e.g. calculation of area, volume, compound interest, etc.). However, to concentrate on teaching students formulae and results (without proofs) and applying them in the most obvious way, is very far from the ideal of mathematics as a precise logical structure. Also, the type of mathematics which is applicable depends very much on the area of application and also on the sophistication of the applier. For example, the "non-applicable" binomial theorem becomes important in later applications of probability. The teaching of geometry in schools has always created problems. Weaker students have often just succeeded in learning off proofs by heart. The new geometry (using mappings) was to some extent intended to remove the constructive element in Euclidean geometry and make the subject more amenable to algebraic techniques. However, it appears to be generally felt that it has not succeeded and that the weaker students now know even less geometry than they did before.

Many people think that (4) is symptomatic of a general feeling in society that things should always be "easy" and "not boring". There is also an emphasis on "discovery" as a method of teaching rather than teacher stating the facts. While many mathematicians view the teachings of educational theorists and sociologists with cynicism if not outright amusement, these theorists have quite an influence on the design of curricula etc., particularly at primary level and their views should be considered and challenged if that is felt appropriate.

In Great Britain, a core syllabus for A Level Mathematics has been proposed recently and it is reasonable to expect that in the next few years here, many proposals to change the existing structures will also be put forward. I feel that mathematicians should express their views on this matter, since the future mathematical life of the country depends very much on fostering and developing an interest in, and knowledge of, mathematics in schools.

PROBLEM SECTION

It is hoped to include in each issue of the Newsletter a set of problems of general mathematical interest. Readers are invited to submit problems for inclusion in this section. It is envisaged that the problems posed should be intelligible to (though not necessary soluble by) people with a degree in Mathematics. It is hoped that most of the problems posed should be soluble and it is intended to publish solutions to those in subsequent issues of the Newsletter. Readers are invited to submit solutions to the problems posed for consideration for inclusion in the Newsletter. Correct solutions will be acknowledged in the Newsletter.

Problem Set # 2

(2.1) Let x be a fixed real number. Find

$$\sup\left\{ \frac{\sin nx}{n} \,\middle|\, n = 1,2,3,\ldots \right\}$$

(2.2) Prove that if c is a real number such that n^c is a natural number for every natural number n, then c is a non-negative integer.

(2.3) Prove that a commutative Noetherian ring has only finitely many idempotents.

(2.4) Let $a_1,\ldots,a_n$ be complex numbers such that $a_1^k+a_2^k+\ldots+a_m^k$ is real and non-negative for all natural numbers k. Prove that $\max_{1\leq i\leq n} |a_i| = a_j$ for some j.

(2.5) Evaluate $\displaystyle\int_0^1 \frac{\log y\,dy}{\sqrt{1-y^2}}$

(2.6) Is it possible to make more six letter words out of MUHAMMAD ALI than CASSIUS CLAY?

(2.7) Let ϕ be Euler's function, i.e. $\phi(k)$ is the number of natural numbers r with $1\leq r\leq k$ and h.c.f.$(r,k) = 1$. Let $a\geq 3$ be odd. Prove that $4n^2$ divides $\phi(a^{2n}-1)$.

(2.8) Let $f(x),g(x)$ be monic integral polynomials and let α,β be roots of $f(x),g(x)$, respectively (in the complex field). Suppose α,β can both be expressed as integral linear combinations of square roots of integers (e.g. $\alpha = 3\sqrt{-5} - 7\sqrt{17} + 3\sqrt{2}$). Prove that there exist integral polynomials $h(x),k(x)$ such that the highest common factor of $f(h(x)),g(k(x))$ has degree greater than one. Generalize!

(2.9) Let n be the least natural number such that there exists a simple group of order n^2. Prove or disprove that $n = 11760$. Let m be the second smallest such natural number. Is $m = 81{,}898{,}320$.

(2.10) Let A be the companion matrix of a polynomial $c(x)$ of degree n (with the coefficients of $c(x)$ on the bottom row) and let B be the matrix with 1 in the $(n,1)$ position, zeros elsewhere. Prove that $c(A+xB) = xI$.

Solutions to Problems

In this section, we give outline solutions or references to solutions to problems posed in the Problem Sections of previous Newsletters. In this issue, we discuss three of the problems contained in Problem Set #1.

1. *Given a sequence of n^2+1 integers, show that it is possible to find a subsequence of $n+1$ integers which is either increasing or decreasing.*

While a direct proof of this is possible, we think that the following generalization is of some interest.

Let P be a partially ordered set. A <u>chain</u> of P is a <u>totally-ordered</u> subset of P. An <u>anti-chain</u> of P is a subset S of P such that no two elements of P are comparable (i.e. if $s,t \in S$, the statements $s<t, t<s$ are both false).

<u>DILWORTH'S THEOREM</u> Let P be a partially ordered set. Then the number of disjoint chains which together contain all the elements of P is equal to the maximal size of an anti-chain of P.

An elementary (but not easy) proof of this appears on pages 62-64 of M. Hall: Combinatorial Theory (Blaisdell Publ. Co., 1967).

To apply this in the question under discussion, let $x_1,\ldots,x_m$ where $m = n^2+1$ be the given sequence. Let $P = \{(i,x_i) \mid i=1,2,\ldots,m\}$.

Define a partial order on P by $(i,x_i) < (j,x_j)$ if $i<j$ and $x_i < x_j$. Let C be a chain in P, say $C = \{(i_1,u_1),\ldots,(i_s,u_s)\}$, where $(i_k,u_k) < (i_{k+1},u_{k+1})$, $k = 1,2,\ldots,s-1$. Then $u_1,\ldots,u_s$ is an increasing subsequence of $x_1,\ldots,x_m$. So if for some chain C, $s \geq n+1$, the question is solved. Suppose for all chains C, $s \leq n$. Since P has $m>n^2$ elements Dilworth's Theorem now implies that there exist (at least) n+1 elements $(j_1,v_1),\ldots,(j_{n+1},v_{n+1})$ of P no two of which are comparable. We may assume $j_1 < j_2 < \ldots < j_{n+1}$. But then $v_1 > v_2 > v_3 > \ldots > v_{n+1}$ (since $v_q < v_r$ for $q<r$ implies $(j_q,v_q) < (j_r,v_r)$ which is false). Hence we have a decreasing subsequence in this situation.

In connection with this problem, we pose here a related problem (M. Hall ibid. p.65): A set of white mice is being studied experimentally. If there are mn+1 mice, show that either (a) there is a sequence of (m+1) mice, each a descendant of the next, or (b) there are (n+1) mice, no one of which is a descendant of another.

A generalization of Dilworth's Theorem (to a statement about Young diagrams) has been fond recently by Greene and was discussed by Gian-Carlo Rota at the International Congress in Helsinki.

6. *Let* A,B *be* $n{\times}n$ *(complex) matrices such that* $AB-BA$ *has rank one. Prove that* A,B *have a common eigenvector (i.e. there exists* $v \neq 0$ *with* $Av = \lambda v$, $Bv = \mu v$ *for some* λ,μ*).*

This result was first proved as Thoerem (1.4) of Laffey, "Simultaneous triangularization of matrices - low rank cases and the non-derogatory case" (Linear & Multil. Algebra, to appear). The proof depends on a long series of trace calculations. More recently, a different proof has been

found by R.M. Guralnick, "A note on matrix commutators of rank one" (ibid., to appear). However, the proofs are too long for inclusion here.

There is one easy special case whose proof we include.

<u>Defn</u> A is called <u>non-derogatory</u> if each eigenspace of A has dimension one.

Assume that in the Question, A is non-derogatory. Let $C = AB-BA$. For each polynomial $f(A)$, $\operatorname{tr} f(A)C = 0$. However, since C has rank one, $Cf(A)C = (\operatorname{tr} f(A)C)C$. So $Cf(A)C = 0$. Let w be a vector with $Cw \neq 0$. Let $p(x)$ be the non-zero polynomial of least degree such that $p(A)Cw = 0$ and let λ be a root of $p(x)$. Let $p(x) = (x-\lambda)p_1(x)$ and let $v = p_1(A)Cw$. Then $(A-\lambda I)v = 0$, $Cv = 0$.

Now $Cv = 0$ gives $A(Bv) = B(Av) = \lambda(Bv)$ so, since the λ-eigenspace of A has dimension one, $Bv = \mu v$ for some μ.

Guralnick's proof of the general result depends in part on showing that in the Zariski topology, a neighbourhood of A contains non-derogator matrices A_1 with $C = A_1B-BA_1$.

It may perhaps be worth mentioning that the result fails (for every $n>1$) if AB-BA has rank greater than one.

7. *S is an infinite set of points in the plane such that the distance between any pair of points of S is an integer. Prove that all the points of S are collinear (i.e. lie on one straight line).*

This problem and its solution are due to P. Erdős.

Recall that if A,B are fixed points, the locus of the point P which moves so that the difference PA-PB of the distances of P from A,B, respectively, is a hyperbola having AB as axis.

Suppose that not all the points of S are collinear. Let A,B be two points of S such that there are an infinite number of points of S not collinear with AB. (Such points A,B must exist if the result fails). Suppose the distance AB = k. Let $P \in S$. Then looking at the triangle PAB, we get

$$AP+AB \geq PB, \quad AP+AB \geq AP.$$

So $-k \leq AP-BP \leq k$. However, AP,BP are integers and hence there exist only finitely many possibilities for AP-BP. Since S has an infinite number of points not collinear with AB, we thus find that there exists an infinite number of such points P with AP-BP = r for some integer r with $-k<r<k$. So all these points P lie on a hyperbola with AB as axis. Let S_o be the set of such points P and let C be an element of S_o. Applying the same argument with B replaced by C and S by S_o, we see that there exists an infinite subset of the points of S_o which lie on a (fixed) hyperbola with AC as axis. However A,B,C are not collinear, so the two hyperbolas are distinct, so their intersection has only a finite number of points. This is a contradiction.

I wish to acknowledge the assistance of F.J. Gaines and R. Gow in the compiling and solution of the above problems.

COMMITTEE OF THE IRISH MATHEMATICAL SOCIETY, 1979

President	:	Dr. T.J. Laffey, U.C.D.
Vice-President	:	Dr. F. Hodnett, N.I.H.E.
Secretary	:	Dr. P.J. Boland, U.C.D.
Treasurer	:	Professor M.L. Newell, U.C.G.

Dr. R. Bates, Met. Service
Dr. F. Holland, U.C.C.
Dr. T.C. Hurley, U.C.D.
Dr. P. McGill, N.U.U.
Professor J.J.H. Miller, T.C.D.
Mr. C. O'Caoimh, Dept. Education
Professor A.G. O'Farrell, Maynooth.