(2.9) Let n be the least natural number such that there exists a simple group of order n^2. Prove or disprove that $n = 11760$. Let m be the second smallest such natural number. Is $m = 81,898,320$.

(2.10) Let A be the companion matrix of a polynomial $c(x)$ of degree n (with the coefficients of $c(x)$ on the bottom row) and let B be the matrix with 1 in the $(n,1)$ position, zeros elsewhere. Prove that $c(A+xB) = xI$.

Solutions to Problems

In this section, we give outline solutions or references to solutions to problems posed in the Problem Sections of previous Newsletters. In this issue, we discuss three of the problems contained in Problem Set #1.

1. *Given a sequence of n^2+1 integers, show that it is possible to find a subsequence of $n+1$ integers which is either increasing or decreasing.*

While a direct proof of this is possible, we think that the following generalization is of some interest.

Let P be a partially ordered set. A chain of P is a totally-ordered subset of P. An anti-chain of P is a subset S of P such that no two elements of P are comparable (i.e. if $s,t \in S$, the statements $s<t, t<s$ are both false).

DILWORTH'S THEOREM Let P be a partially ordered set. Then the number of disjoint chains which together contain all the elements of P is equal to the maximal size of an anti-chain of P.

An elementary (but not easy) proof of this appears on pages 62-64 of M. Hall: Combinatorial Theory (Blaisdell Publ. Co., 1967).

To apply this in the question under discussion, let $x_1,\dots,x_m$ where $m = n^2+1$ be the given sequence. Let $P = \{(i,x_i) | i=1,2,\dots,m\}$.

Define a partial order on P by $(i,x_i) < (j,x_j)$ if $i<j$ and $x_i < x_j$. Let C be a chain in P, say $C = \{(i_1,u_1),\dots,(i_s,u_s)\}$, where $(i_k,u_k) < (i_{k+1},u_{k+1})$, $k = 1,2,\dots,s-1$. Then $u_1,\dots,u_s$ is an increasing subsequence of $x_1,\dots,x_m$. So if for some chain C, $s \geq n+1$, the question is solved. Suppose for all chains C, $s \leq n$. Since P has $m>n^2$ elements Dilworth's Theorem now implies that there exist (at least) n+1 elements $(j_1,v_1),\dots,(j_{n+1},v_{n+1})$ of P no two of which are comparable. We may assume $j_1 < j_2 < \dots < j_{n+1}$. But then $v_1 > v_2 > v_3 > \dots > v_{n+1}$ (since $v_q < v_r$ for $q<r$ implies $(j_q,v_q) < (j_r,v_r)$ which is false). Hence we have a decreasing subsequence in this situation.

In connection with this problem, we pose here a related problem (M. Hall ibid. p.65): A set of white mice is being studied experimentally. If there are mn+1 mice, show that either (a) there is a sequence of (m+1) mice, each a descendant of the next, or (b) there are (n+1) mice, no one of which is a descendant of another.

A generalization of Dilworth's Theorem (to a statement about Young diagrams) has been fond recently by Greene and was discussed by Gian-Carlo Rota at the International Congress in Helsinki.

6. *Let* A,B *be* $n{\times}n$ *(complex) matrices such that* $AB-BA$ *has rank one. Prove that* A,B *have a common eigenvector (i.e. there exists* $v \neq 0$ *with* $Av = \lambda v$, $Bv = \mu v$ *for some* λ,μ*).*

This result was first proved as Thoerem (1.4) of Laffey, "Simultaneous triangularization of matrices - low rank cases and the non-derogatory case" (Linear & Multil. Algebra, to appear). The proof depends on a long series of trace calculations. More recently, a different proof has been found by R.M. Guralnick, "A note on matrix commutators of rank one" (ibid., to appear). However, the proofs are too long for inclusion here.

There is one easy special case whose proof we include.

<u>Defn</u> A is called <u>non-derogatory</u> if each eigenspace of A has dimension one.

Assume that in the Question, A is non-derogatory. Let $C = AB-BA$. For each polynomial $f(A)$, $\operatorname{tr} f(A)C = 0$. However, since C has rank one, $Cf(A)C = (\operatorname{tr} f(A)C)C$. So $Cf(A)C = 0$. Let w be a vector with $Cw \neq 0$. Let $p(x)$ be the non-zero polynomial of least degree such that $p(A)Cw = 0$ and let λ be a root of $p(x)$. Let $p(x) = (x-\lambda)p_1(x)$ and let $v = p_1(A)Cw$. Then $(A-\lambda I)v = 0$, $Cv = 0$.

Now $Cv = 0$ gives $A(Bv) = B(Av) = \lambda(Bv)$ so, since the λ-eigenspace of A has dimension one, $Bv = \mu v$ for some μ.

Guralnick's proof of the general result depends in part on showing that in the Zariski topology, a neighbourhood of A contains non-derogator matrices A_1 with $C = A_1B-BA_1$.

It may perhaps be worth mentioning that the result fails (for every $n>1$) if AB-BA has rank greater than one.

7. *S is an infinite set of points in the plane such that the distance between any pair of points of S is an integer. Prove that all the points of S are collinear (i.e. lie on one straight line).*

This problem and its solution are due to P. Erdős.

Recall that if A,B are fixed points, the locus of the point P which moves so that the difference $PA-PB$ of the distances of P from A,B, respectively, is a hyperbola having AB as axis.

Suppose that not all the points of S are collinear. Let A,B be two points of S such that there are an infinite number of points of S not collinear with AB. (Such points A,B must exist if the result fails). Suppose the distance $AB = k$. Let $P \in S$. Then looking at the triangle PAB, we get

$$AP+AB \geq PB, \qquad AP+AB \geq AP.$$

So $-k \leq AP-BP \leq k$. However, AP,BP are integers and hence there exist only finitely many possibilities for $AP-BP$. Since S has an infinite number of points not collinear with AB, we thus find that there exists an infinite number of such points P with $AP-BP = r$ for some integer r with $-k<r<k$. So all these points P lie on a hyperbola with AB as axis. Let S_o be the set of such points P and let C be an element of S_o. Applying the same argument with B replaced by C and S by S_o, we see that there exists an infinite subset of the points of S_o which lie on a (fixed) hyperbola with AC as axis. However A,B,C are not collinear, so the two hyperbolas are distinct, so their intersection has only a finite number of points. This is a contradiction.

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