PROBLEM SECTION

It is hoped to include in each issue of the Newsletter a set of problems of general mathematical interest. Readers are invited to submit problems for inclusion in this section. It is envisaged that the problems posed should be intelligible to (though not necessary soluble by) people with a degree in Mathematics. It is hoped that most of the problems posed should be soluble and it is intended to publish solutions to those in subsequent issues of the Newsletter. Readers are invited to submit solutions to the problems posed for consideration for inclusion in the Newsletter. Correct solutions will be acknowledged in the Newsletter.

Problem Set # 2

(2.1) Let x be a fixed real number. Find

$$\sup\left\{ \frac{\sin nx}{n} \;\middle|\; n = 1,2,3,\ldots \right\}$$

(2.2) Prove that if c is a real number such that n^c is a natural number for every natural number n, then c is a non-negative integer.

(2.3) Prove that a commutative Noetherian ring has only finitely many idempotents.

(2.4) Let $a_1,\ldots,a_n$ be complex numbers such that $a_1^k+a_2^k+\ldots+a_m^k$ is real and non-negative for all natural numbers k. Prove that $\max_{1\le i\le n} |a_i| = a_j$ for some j.

(2.5) Evaluate $\displaystyle\int_0^1 \frac{\log y\,dy}{\sqrt{1-y^2}}$

(2.6) Is it possible to make more six letter words out of MUHAMMAD ALI than CASSIUS CLAY?

(2.7) Let ϕ be Euler's function, i.e. $\phi(k)$ is the number of natural numbers r with $1\le r\le k$ and h.c.f.$(r,k) = 1$. Let $a\ge 3$ be odd. Prove that $4n^2$ divides $\phi(a^{2n}-1)$.

(2.8) Let $f(x),g(x)$ be monic integral polynomials and let α,β be roots of $f(x),g(x)$, respectively (in the complex field). Suppose α,β can both be expressed as integral linear combinations of square roots of integers (e.g. $\alpha = 3\sqrt{-5} - 7\sqrt{17} + 3\sqrt{2}$). Prove that there exist integral polynomials $h(x),k(x)$ such that the highest common factor of $f(h(x)),g(k(x))$ has degree greater than one. Generalize!

(2.9) Let n be the least natural number such that there exists a simple group of order n^2. Prove or disprove that $n = 11760$. Let m be the second smallest such natural number. Is $m = 81,898,320$.

(2.10) Let A be the companion matrix of a polynomial $c(x)$ of degree n (with the coefficients of $c(x)$ on the bottom row) and let B be the matrix with 1 in the $(n,1)$ position, zeros elsewhere. Prove that $c(A+xB) = xI$.

Solutions to Problems

In this section, we give outline solutions or references to solutions to problems posed in the Problem Sections of previous Newsletters. In this issue, we discuss three of the problems contained in Problem Set #1.

1. *Given a sequence of n^2+1 integers, show that it is possible to find a subsequence of $n+1$ integers which is either increasing or decreasing.*

While a direct proof of this is possible, we think that the following generalization is of some interest.

Let P be a partially ordered set. A chain of P is a totally-ordered subset of P. An anti-chain of P is a subset S of P such that no two elements of P are comparable (i.e. if $s,t \in S$, the statements $s<t, t<s$ are both false).

DILWORTH'S THEOREM Let P be a partially ordered set. Then the number of disjoint chains which together contain all the elements of P is equal to the maximal size of an anti-chain of P.

An elementary (but not easy) proof of this appears on pages 62-64 of M. Hall: Combinatorial Theory (Blaisdell Publ. Co., 1967).

To apply this in the question under discussion, let $x_1, \ldots, x_m$ where $m = n^2+1$ be the given sequence. Let $P = \{(i,x_i) | i=1,2,\ldots,m\}$.