IRISH MATHEMATICAL SOCIETY

NEWSLETTER

No. 1 1978

The Irish Mathematical Society was founded at a meeting in T.C.D. in December 1976. Its aims are to foster and encourage mathematical development in Ireland.

The aim of this Newsletter is to inform members of the Society of the activities of the Society and also of items of general mathematical interest. The success of the Newsletter depends very much on the co-operation of the members of the Society with the compilers. Information on activities of interest to members are sought, as well as survey articles, problems and solutions.

The present address for correspondence relating to the Newsletter is: I.M.S. Newsletter, Department of Mathematics, U.C.D. Belfield, Dublin 4.

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Report of the First Year's Activities

T.C. Hurley

The Irish Mathematical Society has been in existence now for just over a year. The membership is approximately 100 but there is potentially much scope for enlargement. For our first year we have had to rely for our financing on membership fees only (£2 and 50p for students) and this has obviously placed a great obstacle on what we can accomplish. However, we have had a very hard-working Committee and even with the limited resources and the problems of becoming established, we can report some success in our first year.

Our main speakers for the year have been Professor Jean A. Dieudonne, Professor Michael F. Atiyah and Dr. J.D. Murray. Professor Dieudonne's 9-day visit to Ireland was a tremendous success and all of us were very impressed by the great man's still youthful enthusiasm for the subject. His visit was arranged by the Society but most of the financing was obtained through the National Science Council - CNRS (Paris) agreement. Professor Atiyah was invited to Ireland by the D.I.A.S. and very kindly agreed to talk to the Society. Part of his message is that quite a number of branches of mathematics seem, at last, to be converging - I would like to see more evidence! Dr. Murray spoke in Galway on Mathematical Biology. This appears to be an area that deserves more attention from Mathematicians (after all, most of our younger doctors have A in Leaving Certificate Honours Maths!).

This last year has also seen the first steps towards the formation of a European Federation of Mathematical Societies and our President, Dr. Holland, represented us very ably in Strasbourg during the initial stages. The Federation is to be formally created at the International Congress in Helsinki in August.

Also concerning Helsinki, there are still a few places available on the tour being arranged by the Society to coincide with the Congress. The saving is approximately £95 on the excursion air fare. Details have been sent out but further information may be obtained from the Secretary.

To encourage promising young mathematicians, most countries now have Mathematical Olympiads for students in Secondary Schools and we have been endeavouring to organise one here in Ireland. Sample problems have been drawn up, information on other Olympiads have been compiled and the teachers have also been contacted - without the enthusiastic support of the teachers such a scheme would quickly fall apart. However, our approach for sponsorship has so far been unsuccessful. If anybody has any ideas please let us have them.

Much concern has been expressed to us on the lack of numeracy and manipulative skill amongst students leaving Secondary Schools. Not much factual information for Ireland is available so far but with the cooperation of the National Council for Education Awards a test is being compiled. We have also contacted the N.C.E.A. regarding organising meetings for Regional Colleges Mathematics Teachers. We hope to have further information on

these two matters in our next Newsletter and again we would like your views and suggestions.

The Irish Mechanics Group have pioneered the idea of weekend meetings and we have adopted this idea for our initial Conferences. The first Conference was held in Galway 12-13 May, and further details are given below. Volunteers are required to organise such Conferences at their institutions and suggest they be of an instructional nature.

Other Mathematical Societies have been contacted with a view to reciprocal arrangements and the Canadian, French, Italian, Australian and London Mathematical Societies have agreed to send us copies of their Newsletters and Seminar and Conference announcements. The American Mathematical Society are also sending a copy of their Notices by airmail. Information of a general nature obtained this way is being dispersed to the members but samples etc. may be obtained from the Secretary.

The aims of the Society are to promote the teaching and research of Mathematics, particularly in Ireland. Being a relatively small Society we need the active support of a good percentage of our members, especially in the initial formative stage. So let us hear from you!

NEWS SECTION

"ONE-DAY" INSTRUCTIONAL CONFERENCES

The Irish Mathematical Society intends to hold a series of short instructional conferences. It is intended to choose a particular theme for each conference and to have a number of survey-type lectures on this theme. The aim is to survey a particular area for the benefit of non-specialists. A small number of short more specialized contributions may also be given for the benefit of experts in the area.

The first of these conferences was held in U.C.G. May 12-13. The organiser was Martin L. Newell and the topic was Group Theory. About twenty-five people participated. The following lectures were given:

Some aspects of permutation groups.	J.P. McDermott (U.C.G.)
Finite simple groups.	T.J. Laffey (U.C.D.)
The representations of GL(n,q).	G. Thomas (U.C.C.)
Augmentation quotients of some finite nonabelian groups.	N. Losey (Manitoba & Warwick).

The next mini-conference will be held at U.C.C. in late September. The topic will be in the area of the history of Mathematics. Details about this conference will be issued later.

IRISH MECHANICS GROUP

The National Science Council has approved a scheme submitted by the Irish Mechanics Group to organise instructi-nal seminars on three topics:

(a) Seismic data analysis;

(b) Ocean wave generation and modelling of sea and swell;

(c) Environmental fluid mechanics;

and will fund the seminars to a maximum of £1,000 each.

SURVEY ARTICLES

The Newsletter is not, of course, a research journal and the Irish Mathematical Society cannot produce such a journal because of the obvious financial constraints. However, it is hoped to include in each issue some short surveys of research topics or topics in the history of Mathematics. Again, contributions of this type are invited from readers for consideration for inclusion in the Newsletter. Clearly, such articles must be fairly short. It is hoped to include some book reviews in future issues.

PERSONAL ITEMS

We wish to congratulate the following members of the Society: Professors P.M. Quinlan of U.C.C. and D.J. Simms of T.C.D. on their election to the Royal Irish Academy, Dr. P.K. Currie of U.C.D. on his appointment to a research position with Shell in Rijswijk in The Netherlands, Dr. P. O'Leary of U.C.C. on his appointment to a (statutory) lectureship in Mathematical Physics at U.C.G., Doctors R. Dark of U.C.G. and C. Walter of U.C.D. on being appointed to permanent positions in their respective colleges.

FOR YOUR DIARY

L.M.S. Research Symposium on Finite Simple Groups. Durham July 31 - August 10. (For details contact M. Collins, Math. Inst., St. Giles, Oxford OX1 3LE).

A.S.I. Conference on Rings with Polynomial Identities. Antwerp. August 1-13. (For details, contact Professor Dr. F. van Oystaeyen, University of Antwerp, U.I.A., Dept. of Mathematics, Universiteitsplein 1, 2610 Wilrijk, Belgium).

International Congress of Mathematicians. Helsinki. August 15-23. (For details of group travel from Ireland to the congress, contact T.C. Hurley, U.C.D.).

I.M.S. Conference on the history of Mathematics, U.C.C., late September. (Full details of this conference wi-l be issued later).

Dublin Institute for Advanced Studies Christmas Symposium, December 20-21, 1978.

Polynomial identities and central identities for matrices

Thomas J. Laffey

Let A,B,C be 2×2 matrices. It is easy to verify that $(AB-BA)^2$ is a scalar matrix and hence

$$(AB-BA)^2C - C(AB-BA)^2 = 0$$

(This is called Wagner's identity). This may be expressed by saying that $(xy-yx)^2z - z(xy-yx)^2$ is a polynomial identity for 2×2 matrices. In general, a nonzero polynomial $f(x_1,\dots,x_m)$ in the noncommuting indeterminates (or symbols) $x_1,\dots,x_m$ is called a polynomial identity for $n \times n$ matrices if $f(A_1,A_2,\dots,A_m) = 0$ for all $n \times n$ matrices $A_1,A_2,\dots,A_m$.

For example, it is easy to check that Wagner's identity is not a polynomial identity for 3×3 matrices. Another example of a polynomial identity for 2×2 matrices is $(x_1^2x_2-x_2x_1^2)(x_1x_2-x_2x_1)-(x_1x_2-x_2x_1)(x_1^2x_2-x_2x_1^2$

We now give an example of a polynomial identity for $n \times n$ matrices. First, we need a definition.

Definition The standard polynomial $s_m(x_1,\dots,x_m)$ of degree m is defined by

$$s_m(x_1,\ldots,x_m) = \sum_{\sigma\in S_m} (\text{sign } \sigma)x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(m)}$$

where S_m denotes the symmetric group of degree m and order $m!$, and sign $(\sigma) = +1$ or -1 depending on whether the permutation σ is even or odd.

Thus, for example, $s_2(x_1,x_2) = x_1x_2-x_2x_1$,
$s_3(x_1,x_2,x_3) = x_1x_2x_3 - x_1x_3x_2 - x_3x_2x_1 - x_2x_1x_3 + x_2x_3x_1 + x_3x_1x_2$, etc. $s_m(x_1,\ldots,x_m)$ is a homogeneous polynomial of degree m.

We now state the famous theorem of Amitsur-Levitski (Proc. Amer. Math. Soc. 1, 449-463, (1950)).

Theorem $s_{2n}(x_1,\ldots,x_{2n})$ is a polynomial identity for $n\times n$ matrices. There is no polynomial identity of degree $<2n$ for $n\times n$ matrices.

Up till recently, no simple proof was known for this result. However, a simple proof was discovered a few years ago by S. Rosset (Israel J. Math. 23, (1976), 187-188). Rosset's proof also appears in Cohn's book Algebra II (Wiley 1977), p.457.

Remark The polynomial $s_m(x_1,\ldots,x_m)$ is multi-linear i.e.

$$s_m(x_1,\ldots,x_{i-1}, x_i+x_i', x_{i+1},\ldots,x_m) = s_m(x_1,\ldots,x_{i-1}, x_i, x_{i+1},\ldots,x_m) + s_m(x_1,\ldots,x_{i-1}, x_i', x_{i+1},\ldots,x_m)$$

and vanishes if two of the x's are equal, i.e. $s_m(x_1,\ldots,x_i,\ldots,x_i,\ldots,x_m) = 0$.

Hence, to prove the Amitsur-Levitski theorem, it suffices to show that $s_{2n}(B_1,\ldots,B_{2n}) = 0$ for all distinct $B_1,\ldots,B_{2n}$ chosen from a fixed vector space basis of the algebra of $n\times n$ matrices, for example, from the set E_{ij} $i,j = 1,2,\ldots,n$ where E_{ij} is the matrix with 1 in the (i,j) position, zeros elsewhere. This enables one to verify the theorem easily for $n = 2$ since these reduce the theorem to verifying that $s_4(E_{11},E_{12},E_{21},E_{22}) = 0$ in that case. Several proofs of the Amitsur-Levitski theorem are based on this idea, but Rosset's one depends on a particularly ingenious application of the Hamilton-Cayley theorem.

As mentioned at the outset, if A,B are 2×2 matrices, then $(AB-BA)^2$ is scalar, (though not necessarily zero), and thus lies in the centre of the algebra of 2×2 matrices. This is expressed by saying that $(x_1x_2-x_2x_1)^2$ is a central identity for 2×2 matrices. In general, a nonzero polynomial $f(x_1,\ldots,x_m)$ (with constant term zero) in the non-commuting indeterminates $x_1,\ldots,x_m$ is called a central identity for $n\times n$ matrices if $f(B_1,\ldots,B_m)$ is a scalar matrix for all $n\times n$ matrices $B_1,\ldots,B_m$ and $f(B_1,\ldots,B_m)\neq 0$ for some choice of $n\times n$ matrices $B_1,\ldots,B_m$. Kaplansky posed the problem of deciding whether central polynomials exist for $n\times n$ matrices, $n>2$. This problem was solved in 1972 by Formanek and independently by Razmyslov, both of whom explicitly constructed (different) central polynomials for every $n>2$. Razmyslov's construction is described in Cohn's book Algebra II (Wiley 1977), p.462. We now describe Formanek's construction.

Let $z_1,\ldots,z_{n+1}$ be distinct commuting indeterminates and let

$$g(z_1,\dots,z_{n+1}) = \left[\prod_{i=2}^{n}(z_1-z_i)(z_{n+1}-z_i)\right]\left[\prod_{\substack{i\ j=2\\ i\neq j}}^{n}(z_i-z_j)^2\right]$$

Then

$$g(z_1,\dots,z_{n+1}) = \sum_{(\nu)} a_{(\nu)} z_1^{\nu_1} z_2^{\nu_2} \dots z_{n+1}^{\nu_{n+1}}$$

for some integers $a_{(\nu)}$.

Let $x, y_1, \dots, y_n$ be distinct non-commuting indeterminates.
Put

$$f(x;\ y_1,\dots,y_n) = \sum_{(\nu)} a_{(\nu)} x^{\nu_1} y_1 x^{\nu_2} y_2 \dots x^{\nu_n} y_n x^{n+1}$$

(Thus f is constructed from g by first sticking in y's between the distinct z's and then replacing all the z's by x's).

Finally put

$$F(x;y_1,\dots,y_n) = f(x;y_1,\dots,y_n) + f(x;y_2,y_3,\dots,y_n,y_1)$$
$$+ f(x;y_3,y_4,\dots,y_n,y_1,y_2) + \dots$$
$$+ f(x;y_n,y_1,\dots,y_{n-1}).$$

Then F is a homogeneous polynomial of degree n^2 and Formanek (J. Algebra 23 (1972) 129-133) proved that F is a central polynomial for $n \times n$ matrices. It is clear that F is multilinear in $y_1,\dots,y_n$ and thus in proving the result one can assume that the y_i are replaced by matrices of the form $E_{s_i t_i}$. It can also be assumed that x is

replaced by diagonal matrix. Now the "discriminant-like" form of g makes the proof succeed.

If $X, Y_1, \dots, Y_n$ are $n \times n$ matrices of the form $\begin{pmatrix} & & 0 \\ & U & \vdots \\ 0 & \dots 0 & 0 \end{pmatrix}$ (i.e. the last row and column are zero), $F(X;Y_1,\dots,Y_n)$ being scalar can only be the zero-matrix. Thus $F(x;y_1,\dots,y_n)$ is a polynomial identity for $(n-1)\times(n-1)$ matrices but not for $n \times n$ matrices.

Several other examples are known now of polynomial identities and central identities for $n \times n$ matrices. Accounts can be found in Procesi's book "Rings with polynomial identities", Marcel-Dekker, 1973, Jacobson's notes "P.I. algebras" (Springer Lecture Notes in Mathematics # 441), and the recent papers of Amitsur and Rowen.

The existence of polynomial identities for matrix algebras has led to the development of a major theory of algebras satisfying a polynomial identity. In recent years the existence of central identities has led to very considerable simplifications and improvements in this theory and also a theory of forming central quotients (for details, see for example the books of Procesi and Jacobson above and the papers of Formanek and Rowen). In addition, they play a role in elucidating the structure of the generic division algebras constructed by Amitsur (Israel J. Math. 12 (1972), 408-420) which he has used to solve many outstanding problems - in particular, he has constructed finite dimensional division algebras which are not crossed products (i.e. they have no maximal subfields normal over their centres).

We conclude with an amusing identity. Let $a_1,\dots,a_t$ $b_1,\dots,b_t$ be integers and let S_t be the symmetric group on t letters. For $\sigma \varepsilon S_t$, let

$$\phi(\sigma) = a_{\sigma(2)}b_{\sigma(1)} + a_{\sigma(3)}(b_{\sigma(1)} + b_{\sigma(2)}) + \dots$$

$$+ a_{\sigma(t)}(b_{\sigma(1)} + b_{\sigma(2)} + \dots + b_{\sigma(t-1)}).$$

For each $k > 1$, let ω be a primitive kth root of one and let

$$u(a_1,\dots,a_t;\, b_1,\dots,b_t) = \sum_{\sigma\varepsilon S_t} (\text{sign } \sigma)\omega^{\phi(\sigma)}.$$

Then we have

$u(a_1,\dots,a_t;\, b_1,\dots,b_t) = 0$ <u>for all choices of</u> $a_1,\dots,a_t,\, b_1,\dots,b_t$ <u>if and only if</u> $t \geq 2k$.

It can be shown that this identity is equivalent to the Amitsur-Levitski theorem, so it would be interesting to have an elementary proof of it, e.g. by expressing the right-hand-side as a determinant.

<u>Mathematics in Europe in the 13th and 14th Centuries</u>

F.J. Gaines

1. <u>Fibonacci</u>

The most celebrated mathematician of the Middle Ages in Europe was undoubtedly Leonardo of Pisa (*alias* Leonardo Bigollo, *alias* Fibonacci). His well-known problem on the breeding of rabbits, which leads to the Fibonacci numbers is contained in his *Liber Abaci* which first appeared in 1202. This rather boring book, in its 15 chapters deals with positional numerals and the basic arithmetical operations. It also discusses such matters as factorization into primes, fractions, numerical problems in geometry and problems in commercial arithmetic. The *Liber Abaci* is sometimes credited with introducing the Hindu-Arab system of numerals into Europe, but this is too facile an explanation of a complex historical problem. For example, Gerbert (Pope Sylvester II), who died in 1003, knew symbols for 1 to 9 and is credited with introducing these on markers on the abacus (apices) to help speed up calculations, but he did not know zero.

It may be suggested that the *Liber Abaci* was not a popular work - there is no evidence of its use in any of the Universities and it is perhaps significant that it did not appear in print until the 19th century. Possibly the two most popular works for spreading the new Hindu-Arab arithmetic were the *Carmen de Algorismo* of Alexandre Villedieu (c.1220) and John Sacrobosco's *Algorismus Vulgaris*. Another important 13th century work was the *Arithmeticus* of Jordanus de Nemore (d.1260). In this work he attempted to set up number theory in the axiomatic style of Euclid's geometry. The Arithmeticus was the basis of popular commentaries in the University of Paris up to the 16th century.

Fibonacci wrote five mathematical works that we know of. Part of his Practica Geometriae has been shown to be basically a work of Euclid. In his Flos ... ("The flower of solutions to certain problems concerning numbers and geometry") he shows that the cubic $x^3+2x^2+10x = 20$ cannot have a solution which is either rational or can be obtained by combinations of square or cube roots. He then gives (without explanation) the approximation 1.3688081075 (correct to 9 places) to the only real root. It is possible he may have used Horner's method, which was known to the Chinese at that time. In his Liber Quadratorum (c.1225) Fibonacci finds a rational solution to the pair of equations $x^2+5 = y^2$, $x^2-5 = z^2$. His solution is $x = 3\frac{5}{12}$, $y = 4\frac{1}{12}$, and $z = 2\frac{7}{12}$.

There is no evidence to connect Fibonacci with any of the universities, nor is there any evidence that any of his books were ever used as texts.

2. The 14th Century

Probably the two most capable mathematicians of the 14th century were Thomas Bradwardine at Oxford and Nicole Oresme at Paris.

It is said of Oxford in the 14th century that the principal subjects in the Faculty of Arts were logic and Mathematical Physics. Aristotle had claimed that if a body moved under a force F against a resistance R then its speed v would be proportional to F/R. Bradwardine showed this was incorrect and argued that αv is proportional to $(F/R)^\alpha$, where $\alpha = 1, 2^{\pm 1}, 3^{\pm 1}, \ldots$. This relationship, called "Bradwardine's function", although itself incorrect, was the subject of much study. Oresme at Paris in his discussion of Bradwardine's function was led to the law of exponents $x^m x^n = x^{m+n}$, with m,n rational. At Bradwardine's college, Merton, was produced the "Merton College Rule": the distance s covered by a body travelling in a straight line from rest with uniform acceleration a in time T is given by $s = \frac{1}{2}aT^2$ (modern notation!).

The Parisian scholars viewed their Oxford colleagues' work with some envious interest. Walter Burleigh at Oxford said: "The Parisian masters wrap up their doctrines in unskilled discourse and are losing all propriety of logic, except that our English subtleties, which they denounce in public, are the subject of their furtive vigils". Jean Buridan (1300-1360) developed a theory of impetus (= momentum). Oresme, as well as developing Bradwardine's ideas on motion developed a pictorial representation of motion with latitude" = time and "longitude" = speed. These pictures led some people to call Oresme the father of co-ordinate geometry. Oresme wrote on probability, infinite series (he showed the harmonic series diverges), philosophy and theology. He ended his days as Archbishop of Lisieux. Oresme was well-known in his day for his condemnation of astrology (using a probabilistic argument!). He was responsible for bringing some 200 terms into the Old French language.

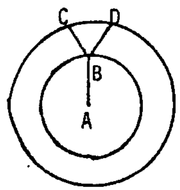
3. Mathematics in Philosophy and Theology

Mathematical arguments were used in the 14th century in discussing the problems of atomism and also of infinity, since Aristotle (whose complete works had not been available in Latin until this time) had shocked the medieval mind with his notion that the world had always existed. Bradwardine in his Tractatus de Continuo wrote (in the style of Euclid) the most complete refutation of 14th century atomism that we have.

A typical argument against the atomists is one due to John Duns Scotus (1266-1308):

If atoms exist they are identified with geometrical points.

Take two circles with centre A. Let C,D be adjacent atoms on the outer circle. Join AC,AD. If these lines always meet the inner circle at different points, then the outer and inner circles have the same number of atoms and hence have the same size. This contradiction implies AC and AD for some C,D meet the inner circle at B. But a tangent to the inner circle at B is perpendicular to both BC and BD. Thus we have two right angles, one greater than the other. Hence a circle is not composed of indivisibles.

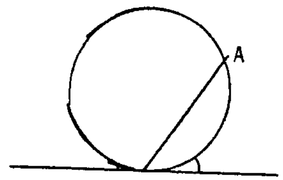


Henry of Harclay (1275-1313) was an atomist who also discussed the problem of infinities. If, as Aristotle says, the world always existed, time up to today can be viewed as a semi-infinite line segment. But this line segment can be superimposed on the line segment representing time up to yesterday. Thus the part equals the whole, contradicting a basic Euclidean

notion. Nicole Oresme and Albert of Saxony argued that the rules which apply to finite quantities do not necessarily apply to infinite ones. We note that arguments on parts, wholes and infinities also occur at times in Greek, Roman and Arab writings.

4. The Horn Angle

The authoritative medieval translation of Euclid's Elements into Latin was made from the Arabic by Johannes Campanus of Novara (c.1260). This was in fact the first edition of Euclid to be printed in 1482. In his edition of Euclid Campanus (and other writers elsewhere) discussed the concept of the horn angle or angle of contact, i.e. the angle between the tangent to a circle and the curve of the circle. No matter how finely the rectlinear angle ABC is subdivided, we never obtain an angle smaller than the horn angle.



This contradicts Euclid X, Prop.1, but Campanus realised that Euclid's proposition must apply only to magnitudes of the same type. Of course, the horn angle can be dealt with in the context of curvature.

5. Conclusion

It is hoped that in this note we have shown that in the 13th and 14th centuries, mathematics was not a complete wasteland, but that a number of significant ideas were discussed, even if their complete explanation had to wait for later times.

Bibliography

1. T. Bradwardine, Tractatus de proportionibus, translated by H.L. Crosby, Jr. (Madison, 1955).

2. E. Grant (Ed.), A Source Book in Medieval Science (Harvard University Press, 1974).

3. N. Oresme, De proportionibus proportionum and Ad pauca respicientes, translated by E. Grant (Madison, 1966).

PROBLEM SECTION

It is hoped to include in each issue of the Newsletter a set of problems of general mathematical interest. Readers are invited to submit problems for inclusion in this section. It is envisaged that the problems posed should be intelligible to (though not necessary soluble by) people with a degree in Mathematics. It is hoped that most of the problems posed should be soluble and it is intended to publish solutions to those in subsequent issues of the Newsletter. Readers are invited to submit solutions to the problems posed for consideration for inclusion in the Newsletter. Correct solutions will be acknowledged in the Newsletter.

PROBLEM SET # 1

1. Given a sequence of n^2+1 distinct integers, show that it is possible to find a subsequence of $n+1$ integers which is either increasing or decreasing.

2. Let $f(x)$ be a monic polynomial in $\mathbb{Z}[x]$ which divides x^n-1 and suppose that a is a natural number which divides all the coefficients of $f'(x)$. Prove that $f(x) = g(x^a)$ for some monic integral polynomial $g(x)$.

3. Let $A = (a_{ij})$ be the $n \times n$ matrix with $a_{ij} = (1+|i-j|)^{-1}$ where $|\cdot|$ denotes absolute value. Prove that $\det A > 0$.

4. Necklaces of n beads are to be made out of an infinite supply of beads in k different colours. How many distinctly different necklaces can be made.

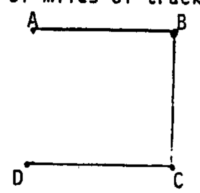
5. Let Q be the point $(2,0)$ in the plane and O the origin. Let P be a point in the first quadrant which lies on the curve $x^4+xy^2+y^2 = 3x^2$. Let θ be the angle in the interval $0 < \theta < \pi$ which the line PQ makes with the positive x-axis and let ϕ be the acute angle which the line OP makes with the positive x-axis. Prove that $\theta = 3\phi$ (i.e. OP trisects θ).

6. Let A,B be $n \times n$ (complex) matrices such that $AB-BA$ has rank one. Prove that A,B have a common eigenvector (i.e. there exists $v \neq 0$ with $Av = \lambda v$, $Bv = \mu v$ for some λ,μ).

7. S is an <u>infinite</u> set of points in the plane such that the distance between any pair of points in S is an integer. Prove that all the points in S are collinear (i.e. lie on one straight line).

8. Four towns A,B,C,D lie at the vertices of a square of side 100 miles. An engineer wishes to construct a railway line so that it is possible to travel from any of the towns to any other by rail. What is the minimum number of miles of track required.

<u>Note</u>



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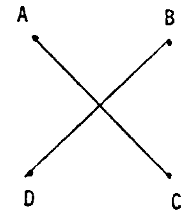


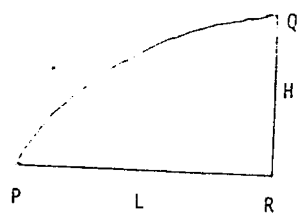
are good solutions, but not the best.

9. Prove that

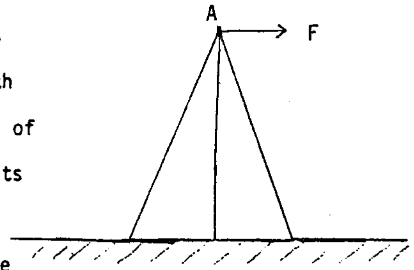
$$\frac{1}{2^2} + \frac{1^2.3}{2^2.4^2} + \frac{1^2.3^2.5}{2^2.4^2.6^2} + \frac{1^2.3^2.5^2.7}{2^2.4^2.6^2.8^2} + \ldots = 1 - \frac{2}{\pi}.$$

10. Let G be a finite abelian group of order n and let $g_1,\ldots,g_{2n-1}$ be elements of G. Prove that there exists a subsequence $g_{r_1}, g_{r_2}, \ldots, g_{r_n}$ of exactly n g's with $g_{r_1} g_{r_2} \cdots g_{r_n} = 1$.

11. Neglecting air resistence a particle projected under gravity travels on a parabola which has the property that at any point A on the path, the directrix is horizontal and is at a height $v^2/2g$ above A where v is the velocity of the particle at A. Q is at a height H above the horizontal plane containing P and PR has length L. What is the minimum speed of projection at P so that the particle reaches Q.



12. A right circular cone of weight W is placed on a rough horizontal table with coefficient of friction $\frac{1}{2}$. The radius of the base of the cone is one unit and its height is three units. A horizontal force is applied at the apex A. Prove that the cone begins to topple before its base begins to move. A heavy particle is attached to the cone, so that it begins to move before it begins to topple. What is the minimum weight of the particle required and where should it be attached to the cone.



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