

Jean-Pierre Serre: Finite Groups: an introduction,
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In the preface Serre tells us that he based *Finite Groups* on handwritten notes from a course he taught in the late 1970's, available at arXiv:math/0503154 (in French), recently translated, revised and expanded. As one might expect from the author, the book is beautifully written. Results are either proved in full, or the gaps highlighted and referenced. However it is certainly not an introduction to group theory and might be more accurately titled 'Finite Groups: an apprenticeship'. The text proceeds at break-neck speed and important results and methods are relegated to the exercises. As well as being of interest to graduate students, non-specialists can dip into the book to learn about particular topics. Specialists will appreciate the efficient development of the theory and the elegance of the proofs.

Serre does not define what a group is. Instead, he begins with the notion of a G -set, which reflects the fact that every group is the group of symmetries of some set. By page 6 he has defined normal and characteristic subgroup, simple group (a key topic of interest throughout the book) and proved the Jordan-Hölder theorem on the uniqueness of composition factors. His choice of results of Goursat and Ribet in Section 1.4 is a bit idiosyncratic. The exercises indicate the sophistication expected of the reader: they cover the n -transitivity of G -sets and showing that the alternating groups A_n are simple, for $n \geq 5$.

Chapter 2 gives a conventional but efficient treatment of Sylow's three theorems. In addition to Burnside's fusion theorem, Serre proves Alperin fusion theorem: that the conjugation of p -elements is p -locally controlled (i.e. in the normalizers of non-trivial p -subgroups).

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This anticipates the current theory of Fusion Systems. Generalizations of Sylow subgroups are described, including Hall subgroups of solvable groups (the subject of a later chapter), tori of compact Lie groups and Borel subgroups of linear algebraic groups. The exercises include results on normal p -complements, and determining the groups of order pq and the Sylow subgroups of symmetric groups.

The next chapter deals with solvable and nilpotent groups. It is disappointing that the structure theorem for finite abelian groups, although stated, is not proved. Applications of solvable groups to ruler and compass constructions, Galois theory and proving that \mathbb{C} is algebraically closed are described. Hall-Burnside's theorem on p' -groups of automorphisms of p -groups is also covered. J. Thompson proved that a finite group is solvable if and only if all its two generator subgroups are solvable. The proof (omitted!) reduces to checking 'minimal' simple groups. This prefigures the use of the classification of finite simple groups to prove results about finite groups. Some of the exercises are strenuous: the reader is asked to prove Iwasawa's simplicity theorem and use it to prove the simplicity of (most) finite projective special linear groups. They also cover such varied topics as supersolvable groups, towers of quadratic field extensions and torsion in nilpotent groups.

Serre covers group cohomology and its application to the existence and uniqueness of group extensions (both abelian and non-abelian) with commendable clarity and completeness in Chapter 4. In particular he proves Zassenhaus theorem on the existence and uniqueness of complements. In one section he shows that every representation of a finite group over the field with p -elements lifts to a representation over the field of p -adic numbers. The exercises in this chapter include a useful exploration of the group-theoretic interpretations of low degree cohomology groups.

P. Hall proved that if G is a finite solvable group and π is a set of primes, then G has a subgroup of order $|G|_{\pi}$. Moreover all Hall π -subgroups of G are G -conjugate and every π -subgroup of G is contained in a Hall π -subgroup. Proving this is the main task of Chapter 5.

A finite group is said to be Frobenius if it acts transitively on a set so that every non-trivial group element fixes at most one element of the set. For a Frobenius group G , the stabilizer H of a point is called a Frobenius Complement. G. Frobenius used character

induction to show that the identity and the set of elements of G which lie in no conjugate of H forms a normal complement N to H , now called a Frobenius Kernel. Serre defers proving this to a later chapter, and instead focuses in Chapter 6 on proving interesting properties of Frobenius groups. He gives elementary proofs that N is nilpotent if it possesses a fixed-point-free automorphism of order 2 or 3 and discusses Thompson's theorem that N is always nilpotent. Frobenius groups are a rich source of interesting examples, and this is reflected in the exercises.

Given a group G and a finite index subgroup H , Transfer is a homomorphism from G to the abelianization of H . One important application is Hall's Focal Subgroup Theorem. This characterises the group generated by all commutators in a finite group which lie in a fixed Sylow subgroup of the group (Exercise 10 in chapter 7). Transfer also provides the 'correct' framework to prove Gauss's lemma in quadratic reciprocity. Serre describes other applications of transfer to number theory and topology and demonstrates its usefulness in classifying 'small' finite simple groups.

Each complex representation of a finite or compact group has an associated character. This is a complex valued function on the group which is constant on conjugacy classes. The theory of characters was used in the Odd Order paper of Feit-Thompson, and in conjunction with modular characters, in the classification of the simple groups with small 2-rank. Therefore it was instrumental in kick-starting the classification of all finite simple groups. Chapter 8 runs briskly through the existence of invariant Hermitian forms and Maschke's theorem, orthogonality relations, Wedderburn's theorem on the structure of the group algebra and applications of integrality properties such as Burnside's $p^a q^b$ theorem. But there is much more: induction, the existence of Frobenius Kernels, Adams operations on characters, Galois conjugation of characters and fields of values of characters. I particularly liked Serre's thorough treatment of Frobenius-Schur indicators, which includes using the theory of positive definite hermitian forms. The 34 exercises at the end of the chapter indicate Serre's deep interest in characters.

The penultimate Chapter 9 deals with the problem of bounding the order of a finite matrix group. This is not standard textbook material and the approach strays well beyond mere algebra. Minkowski demonstrated that the order of a finite group of rational matrices is

bounded by an explicit function in the size n of the matrices. Later Jordan proved that a finite group of complex matrices contains an abelian normal subgroup whose index is bounded by a function of n . The chapter displays the breadth of Serre's interests in algebra and geometry.

The final chapter is called 'Small Groups'. This walks the reader through an array of interesting examples and outlines many of the exceptional isomorphisms between low order almost-simple groups, including $\mathcal{S}_4 \cong \mathrm{PGL}_2(3)$, $\mathcal{A}_5 \cong \mathrm{SL}_2(4)$, $\mathcal{S}_5 \cong \mathrm{PGL}_2(5)$, $2.\mathcal{A}_5 \cong \mathrm{SL}_2(5)$, $\mathrm{SL}_3(2) \cong \mathrm{PSL}_2(7)$, $\mathcal{A}_6 \cong \mathrm{PSL}_2(9)$, $\mathcal{S}_6 \cong \mathrm{Sp}_4(2)$ and (of course) $\mathcal{A}_8 \cong \mathrm{SO}_6(2) \cong \mathrm{SL}_4(2)$. There is also a section on embeddings of \mathcal{A}_4 , \mathcal{S}_4 and \mathcal{A}_5 in $\mathrm{PGL}_2(q)$, anticipating modular representation theory.

Serre includes an extensive bibliography of 41 books and 40 academic papers on group theory, representation theory, number theory and homological algebra. In addition, there is a list of books dealing with relevant mathematical 'Topics' and a handy index of names. This slim volume will sit handsomely on any mathematician's bookshelf.

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