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# Bulletin

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# Irish Mathematical Society Bulletin

The aim of the *Bulletin* is to inform Society members, and the mathematical community at large, about the activities of the Society and about items of general mathematical interest. It appears twice each year. The *Bulletin* is supplied free of charge to members; it is sent abroad by surface mail. Libraries may subscribe to the *Bulletin* for 30 euro per annum.

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# EDITORIAL

Brendan Goldsmith is preparing an obituary of the late Eoin Coleman (Oren Kolman) for the next edition of the Bulletin and would be pleased to hear from anyone with information that might be relevant to such an obituary. His address is brendan.goldsmith@dit.ie

Colm Mulcahy's Archive of Irish Mathematics and Mathematicians is now hosted at `http://mathsireland.ie/`.

Readers of the Newsletter of the Irish Mathematics Teachers Association will be glad to know that digital copies of many IMTA Newsletters are now available on the IMTA website `www.imta.ie`. This archive contains Newsletters No. 1 to No. 38 (38 Newsletters) No. 53 to No. 80 inclusive (28 Newsletters) and No. 104 to No. 114 (12 Newsletters).

The IMS Committee has adopted revised guidelines for conference organisers who wish to apply for support. These may be found at the IMS website. Organisers are reminded that reports should be submitted to the Bulletin by December, in good time for the Winter issue.

The present issue has a mixture of research articles and classroom notes. We had to pull a substantial survey article at a late stage when it transpired that the content was apparently copied from another source. Members may be interested to note that the editors at ArXiv.org have begun using the algorithm described in the paper archived at the link `http://www.pnas.org/content/112/1/25.full#ref-6`. This algorithm flags papers that re-use substantial amounts of text from other papers on ArXiv.

One of the classroom notes in this issue is a paper by Jill Tysse that uses Irish set-dancing patterns to illustrate some simple group theory. We note that there are strong views out there about the order in which the product (composition) of two permutations should be written. See, for instance, `http://mathoverflow.net/questions/151258/how-do-most-people-write-permutations`
and `http://tinyurl.com/z2wmdbh`. The Bulletin does not impose any standard on its authors in this matter, so *caveat lector*.

Linik's online essay on the state of mathematics is worth reading: `http://www.math.u-psud.fr/~limic/som/stateofmath.html`.

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The remaining schools with Ph.D. programmes in Mathematics are invited to send their preferred link to the editor, a url that works. All links are live, and hence may be accessed by a click, in the electronic edition of this Bulletin[1].

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(2) The current subscription fees are given below:

| | |
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| Institutional member | €160 |
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| DMV, I.M.T.A., NZMS or RSME reciprocity member | €12.50 |
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The subscription fees listed above should be paid in euro by means of a cheque drawn on a bank in the Irish Republic, a Eurocheque, or an international money-order.

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The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

(4) Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

(5) Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

(6) Subscriptions normally fall due on 1 February each year.

(7) Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.

(8) Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

(9) Please send the completed application form with one year's subscription to:

The Treasurer, IMS
School of Mathematics, Statistics and Applied Mathematics
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**Deceased Member**

It is with regret that we report that our former member Eoin Coleman (more recently known as Oren Kolman) of Hughes Hall, Cambridge, died on the fourth of December, 2015.

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*E-mail address*: `subscriptions.ims@gmail.com`

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# A Contribution to Metric Diophantine Approximation : the Lebesgue and Hausdorff Theories

FAUSTIN ADICEAM

This is an abstract of the PhD thesis *A Contribution to Metric Diophantine Approximation : the Lebesgue and Hausdorff Theories* written by F. Adiceam under the supervision of Dr. Detta Dickinson at the School of Mathematics and Statistics, Maynooth University, and defended in March 2015.

This thesis is concerned with the theory of Diophantine approximation from the point of view of measure theory. After the prolegomena which conclude with a number of conjectures set to understand better the distribution of rational points on algebraic planar curves, Chapter 1 provides an extension of the celebrated Theorem of Duffin and Schaeffer. This enables one to state a generalized version of the Duffin–Schaeffer conjecture. Chapter 2 deals with the topic of simultaneous approximation on manifolds, more precisely on polynomial curves. The aim is to develop a theory of approximation in the so far unstudied case when such curves are *not* defined by *integer* polynomials. A new concept of so–called "liminf sets" is then introduced in Chapters 3 and 4 in the framework of simultaneous approximation of independent quantities. In short, in this type of problem, one prescribes the set of integers which the denominators of all the possible rational approximants of a given vector have to belong to. Finally, a reasonably complete theory of the approximation of an irrational by rational fractions whose numerators and denominators lie in prescribed arithmetic progressions is developed

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in Chapter 5. This provides the first example of a Khintchine type result in the context of so–called uniform approximation.

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# The Structure of Spectrally Bounded Operators on Banach Algebras

MATTHEW YOUNG

This is an abstract of the PhD thesis *The Structure of Spectrally Bounded Operators on Banach Algebras* written by Dr. Matthew Young under the supervision of Dr. Martin Mathieu at the School of Mathematics and Physics, Queen's University Belfast, and submitted in May 2016.

Let $A$ and $B$ be unital Banach algebras and $T\colon A \to B$ a linear mapping. We say $T$ is spectrally bounded if there is $M > 0$ such that $r(Tx) \leq Mr(x)$ for all $x \in A$, where $r(\cdot)$ denotes the spectral radius. If $r(Tx) = r(x)$ for all $x \in A$, then $T$ is called a spectral isometry. My thesis is concerned with further understanding these operators.

Although the terms were introduced by Mathieu in [1], the history of these operators originates in Banach algebra and automatic continuity theory in the 1970s. In the latter, they are important as they provide the link between the analytic and algebraic structure of the Banach algebra [2].

Spectrally bounded operators are closely interconnected with the non-commutative Singer–Wermer conjecture, which is equivalent to asking the following question: if the commutator $[x, \delta x]$ belongs to the centre modulo the radical for all $x$ in a Banach algebra $A$ does this imply that the derivation $\delta$ is spectrally bounded? This is likely the deepest and most important open problem on spectrally bounded operators.

In [3], Mathieu proposed the following conjecture: every unital surjective spectral isometry between unital $C^*$-algebras is a Jordan isomorphism. See [4] and [5] for more details. We focus on furthering our understanding of this conjecture by placing ourselves in the setting of unital semisimple Banach algebras and consider a particular

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class of operators between them; the class of elementary operators. We establish that for arbitrary length elementary operators with given conditions, surjectivity of the operator is equivalent to the operator being an inner automorphism. Additionally, we show that a unital surjective spectrally bounded elementary operator of length three on a unital arbitrary $C^*$-algebra is a Jordan automorphism.

We also close some gaps in the literature for the situation for commutative Banach algebras. We examine the case where our spectral isometries are neither unital nor surjective. In general we lose multiplicativity, however using Gelfand theory and techniques from function space theory, we recover a multiplicative induced mapping into $C(X)$ where $X$ is a particular subset of the Gelfand space of the codomain.

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# A Class of Weakly Nil-Clean Rings

PETER V. DANCHEV

Abstract. We completely describe the structure of weakly nil-clean rings with the strong property. The main characterization theorem somewhat improves on results of Diesl in J. Algebra (2013) concerning strongly nil-clean rings and Breaz-Danchev-Zhou in J. Algebra Appl. (2016) concerning abelian weakly nil-clean rings.

## 1. Introduction and Background

Throughout the present paper all rings $R$ considered shall be assumed to be associative and unital with identity element 1. As usual, $U(R)$ denotes the set of all invertible elements of $R$, $Id(R)$ the set of all idempotents of $R$ and $Nil(R)$ the set of all nilpotents of $R$. Traditionally, $J(R)$ will always denote the Jacobson radical of $R$. All other notions and notations, not explicitly stated herein, are standard and may be found in [10].

The following concept appeared in [11].

**Definition 1.1.** A ring $R$ is called *clean* if each $r \in R$ can be expressed as $r = u + e$, where $u \in U(R)$ and $e \in Id(R)$.

If, in addition, the existing idempotent is unique, then $R$ is said to be *uniquely clean*. A clean ring $R$ with $ue = eu$ is said to be *strongly clean*. If again the existing idempotent is unique, the ring is called *uniquely strongly clean* (see [4]).

It is well known that uniquely clean rings, being abelian clean rings, are strongly clean. The converse, however, does not hold in general. Nevertheless, uniquely clean rings are uniquely strongly clean, which containment cannot be reversed. However, [4, Example 4] demonstrates that uniquely clean rings are exactly the abelian uniquely strongly clean rings.

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In particular, in [9] the following concept was introduced:

**Definition 1.2.** A ring $R$ is called *nil-clean* if each $r \in R$ can be written as $r = q + e$, where $q \in Nil(R)$ and $e \in Id(R)$.

If, in addition, the existing idempotent is unique, then $R$ is said to be *uniquely nil-clean.* A nil-clean ring $R$ with $qe = eq$ is said to be *strongly nil-clean.* If again the existing idempotent is unique, the ring is called *uniquely strongly nil-clean.*

It is well known that uniquely nil-clean rings, being abelian nil-clean rings, are strongly nil-clean. This implication is not reversible, however. Nevertheless, it follows from [3, Theorem 4.5] and [7] that uniquely nil-clean rings are precisely the abelian nil-clean rings (see [9, Theorem 5.9] as well). Also, commutative nil-clean rings are always uniquely nil-clean (compare with [8, Proposition 1.6]), and even it was proved in [9, Corollary 3.8] that strongly nil-clean rings and uniquely strongly nil-clean rings do coincide in general.

On the other hand, the latter concept of nil-cleanness was extended in [8] and [2], respectively, by defining the notion of *weak nil-cleanness* as follows:

**Definition 1.3.** A ring $R$ is called *weakly nil-clean* if every $r \in R$ can be presented as either $r = q + e$ or $r = q - e$, where $q \in Nil(R)$ and $e \in Id(R)$.

If, in addition, the existing idempotent is unique, then $R$ is said to be *uniquely weakly nil-clean.* A weakly nil-clean ring with $qe = eq$ is said to be *weakly nil-clean with the strong property.* If again the existing idempotent is unique, the ring is called *uniquely weakly nil-clean with the strong property.*

It was established in [2] and [8] that weakly nil-clean rings are themselves clean. Likewise, in [2] was established a complete characterization of abelian weakly nil-clean rings as those abelian rings $R$ for which $J(R)$ is nil and $R/J(R)$ is isomorphic to a Boolean ring $B$, or to $\mathbb{Z}_3$, or to $B \times \mathbb{Z}_3$. We notice also that uniquely weakly nil-clean rings were classified in [5] as abelian weakly nil-clean rings.

The objective of this article is to continue these two explorations by giving a complete description of the structure of the above-defined weakly nil-clean rings having the strong property. As an application, we will characterize the class of rings equipped with the strong nil-involution property, defined as follows:

**Definition 1.4.** We will say that a ring $R$ has the *nil-involution property* if, for any $r \in R$, we have either $r = v+1+w$ or $r = v-1+w$, where $v \in Nil(R)$ and $w^2 = 1$.

If, in addition, $vw = wv$, then we call that $R$ has the *strong nil-involution property.*

The motivation for considering and exploring the sum of an invertible plus an idempotent, as well as other such variations of elements as above, is illustrated in details via concrete examples in [11], [9] and [2], respectively.

## 2. The Main Result

We come now to our main result in which we give a comprehensive characterization of weakly nil-clean rings enabled with the strong property.

**Theorem 2.1.** *A ring $R$ is weakly nil-clean with the strong property if, and only if, $R$ is either a strongly nil-clean ring, or $R/J(R)$ is isomorphic to $\mathbb{Z}_3$ with $J(R)$ nil, or $R$ is a direct product of two such rings.*

*Proof.* We show that for any such weakly nil-clean ring $R$ will follow that $R \cong R_1 \times R_2$, where $R_1/J(R_1)$ is Boolean with $J(R_1)$ nil, and $R_2$ is either $\{0\}$ or $R_2/J(R_2) \cong \mathbb{Z}_3$ with $J(R_2)$ nil. In fact, in accordance with [2], we write that $R \cong R_1 \times R_2$, where $R_1$ is a nil-clean ring and $R_2$ is a weakly nil-clean ring for which 2 is invertible. Since $R$ has the "strong" property, it follows at once that $R_1$ is strongly nil-clean. Thus, appealing to [7], $R_1/J(R_1)$ must be Boolean with nil $J(R_1)$.

As for the second direct factor, we claim that all units of $R_2$ are the sum or the difference of a nilpotent and 1, that is, they belong to $Nil(R_2) \pm 1$. In fact, if $u$ is an arbitrary unit in $R_2$, then being an element in $R$, one can write that $u = q + e$ or $u = q - e$, where $q$ is a nilpotent of $R_2$ and $e$ is an idempotent of $R_2$ which commute, i.e., $qe = eq$. Thus, $uq = (q \pm e)q = q^2 \pm eq = q^2 \pm qe = q(q \pm e) = qu$. Therefore, $u - q = e$ and $q - u = e$ are again units and simultaneously idempotents. This means that $e = 1$ in both cases, hence $u = q \pm 1$ as claimed. Moreover, in view of [2], 6 being a central nilpotent in $R_2$ lies in $J(R_2)$ which is nil. Since 2 inverts in $R_2$, it follows immediately that 3 lies in $J(R_2)$, so that $R/J(R_2)$ is of characteristic 3. Furthermore, for any idempotent $e \in R_2$, the element $1 - 2e$ is an involution because $(1-2e)^2 = 1$ and thus, by what we have already shown, either $1-2e = q+1$ or $1-2e = q-1$ for some $q \in Nil(R)$. Hence one of the following equalities $2e = -q$ or $2(1-e) = q$ holds, so that $e = -\frac{q}{2}$ or $1-e = \frac{q}{2}$. Since these are both idempotents and nilpotents, it follows at once that $e = 0$ or $e = 1$. Consequently, $R_2$ being indecomposable is abelian and, by [2], we deduce that $R_2/J(R_2)$ has to be isomorphic to $\mathbb{Z}_3$, as stated. □

As a consequence to the above fact, we have the following generalization of the corresponding fact from [9].

**Corollary 2.2.** *If $R$ is a weakly nil-clean ring with the strong property, then for any idempotent $e$ of $R$ the corner ring $eRe$ is a weakly nil-clean ring with the strong property. In particular, if $\mathbb{M}_n(R)$ is a weakly nil-clean ring having the strong property, then so is the ring $R$.*

*Proof.* First of all, we apply Theorem 2.1. Next, one observes that $Id(R) = \{0,1\}$, provided $R/J(R)$ is isomorphic to $\mathbb{Z}_3$. To that goal, if $r \in Id(R)$, then $r + J(R) \in Id(R/J(R)) = \{J(R), 1 + J(R)\}$. So, either $r \in J(R)$ or $r \in 1 + J(R) \leq U(R)$ which ensures that $r = 0$ or $r = 1$, respectively. This substantiates our assertion. Henceforth, the proof goes on utilizing the fact that, if $R$ has all unipotent units, then $eRe$ also has only unipotent units (see [7]).

For the second part, one sees that $R \cong E_{1n}\mathbb{M}_n(R)E_{1n}$ for the idempotent matrix $E_{1n}$ with $(1,n)$-entry 1 and the other entries 0, so we are finished. □

Dropping off the "strong" condition, it is unknown at this stage whether or not if $R$ is weakly nil-clean, then so does $eRe$ for any idempotent $e$ of $R$. Adapting some results from [2], we can conclude that the validity of the converse implication for the corner problem cannot be happen.

We are now ready to proceed by proving of the following.

**Theorem 2.3.** *Suppose that $R$ is a ring equipped with the strong nil-involution property. Then $R/J(R) \cong \mathbb{Z}_3$ and $J(R)$ is nil. The converse is also true.*

*Proof.* We shall show that such a ring $R$ is weakly nil-clean with the strong property, for which $3 \in J(R)$. To that purpose, given $r \in R$, we write $r = q + 1 + v$ or $r = q - 1 + v$ for some existing nilpotent $q$ and involution $v$ which commute. Thus $q + v$ is again a unit, say $u$, and $r = u + 1$ or $r = u - 1$. Since 1 cannot be written as $u + 1$, it follows that $1 = u - 1$. So, $2 = u$ is a unit whence 2 inverts in $R$. Moreover, it is elementary to check that both $\frac{1+v}{2}$ and $\frac{1-v}{2}$ are idempotents. Since $2r \in R$, one sees that $2r = q + 1 + v$ or $2r = q - 1 + v = q - (1 - v)$ which implies that $r = \frac{q}{2} + \frac{1+v}{2}$ or that $r = \frac{q}{2} - \frac{1-v}{2}$. Since $\frac{q}{2}$ remains a nilpotent, it is now clear that $R$ is weakly nil-clean having the strong property, as asserted. In conjunction with [2], it follows that 6 belongs to $J(R)$ and hence 3 lies in $J(R)$. We furthermore need apply the idea for proof in Theorem 2.1 to get the wanted claim.

Reciprocally, if $r \in R$, then $r+J(R)$ can be written as one of $J(R)$, $1+J(R)$ or $-1+J(R)$. Since $J(R) \subseteq Nil(R)$, one derives that either $r = q$ or $r = q+1$ or $r = q-1$, for some nilpotent $q$. Since $3 \in J(R)$ is a nilpotent, we infer that either $r = q + 1 + (-1) = q - 1 + 1$ or $r = (q + 3) - 1 + (-1)$ or $r = (q - 3) + 1 + 1$. But $q \pm 3$ remains a nilpotent and $(-1)^2 = 1^2 = 1$, so we are set. □

**Remark**. It is worthwhile noticing that it follows from the proof of Theorem 2.1 above in a combination with [2] that a ring satisfies the (strong) nil-involution property if and only if it is a weakly nil-clean ring (with the strong property) for which 2 is invertible.

We are now in a position to obtain an element-wise characterization of weakly nil-clean elements with the strong property. To that aim, similarly to above, an element $a$ of a ring $R$ is called *clean* if $a = u + e$ where $u \in U(R)$ and $e \in Id(R)$. If $a = q + e$ with $q \in Nil(R)$ and $e \in Id(R)$, $a$ is said to be *nil-clean*, while $a$ is said to be *weakly nil-clean* provided $a = q + e$ or $a = q - e$. In addition, if $q$ and $e$ commutes, we will say that $a$ is either strongly nil-clean or weakly nil-clean with the strong property.

It is in principle known and easy to prove that $a \in R$ is strongly nil-clean $\iff$ $a^2 - a \in Nil(R)$. This can be substantially extended to the following:

**Proposition 2.4.** *An element $a \in R$ is weakly nil-clean having the strong property if, and only if, either $a^2 - a \in Nil(R)$ or $a^2 + a \in Nil(R)$.*

*Proof.* "⇒". Writing $a = q + e$ or $a = q - e$ with $qe = eq$, we have $a^2 - a = q^2 - q + 2qe$ or $a^2 + a = q^2 + q - 2qe$. In both cases, these are nilpotents, as expected.

"$\Leftarrow$". First, suppose that $a^2 + a \in Nil(R)$ whence $(a^2+a)^n = 0$ for some $n \in \mathbb{N}$. Setting $e = (1-(1+a)^n)^n$, by the Newton binomial formula we deduce that $e = ka^n = 1 - m(1+a)^n$ for some $k, m \in R$ depending only on $a$, and thus $1-e = m(1+a)^n$. It is immediate that $ae = ea$ because $ak = ka$ as well as $am = ma$ whence we observe that $e \in Id(R)$ since $e(1-e) = ka^n \cdot m(1+a)^n = km(a+a^2)^n = 0$. Furthermore, $a + e = a(1-e) + e(1+a)$ and hence $(a+e)^n = a^n(1-e) + e(a+1)^n = a^n m(1+a)^n + ka^n(1+a)^n = m(a+a^2)^n + k(a+a^2)^n = 0$. Finally, this means that $a + e \in Nil(R)$ and, therefore, $a = (a+e) - e \in Nil(R) - Id(R)$ with $(a+e)e = e(a+e)$. We can process in a similar way letting $a^2 - a \in Nil(R)$ (see cf. [12] too) to conclude that $a = (a-e) + e \in Nil(R) + Id(R)$ with $(a-e)e = e(a-e)$, so that in both cases $a$ is a weakly nil-clean element with the strong property, as asserted. $\square$

**Remark**. This can also be directly deduced, because $a$ is a weakly nil-clean element with the strong property $\iff$ either $a$ or $-a$ is a strongly nil-clean element. In fact, $a^2 - a \in Nil(R)$ or $(-a)^2 - (-a) = a^2 + a \in Nil(R)$.

Recollect that a ring $R$ is called *weakly Boolean* if each element of $R$ is idempotent or minus idempotent. Since it is self-evident that the element $a$ or $-a$ is nil-clean $\iff$ $a$ is weakly nil-clean $\iff$ $-a$ is weakly nil-clean, adapting the idea from [12, Proposition 3.9] along with [6], one can infer an other confirmation of Theorem 2.1 in a more convenient form. Namely, $R$ is a weakly nil-clean ring with the strong property $\iff$ $J(R)$ is nil and $R/J(R)$ is weakly Boolean $\iff$ $J(R)$ is nil and either $R/J(R) \cong B$, or $R/J(R) \cong \mathbb{Z}_3$, or $R/J(R) \cong B \times \mathbb{Z}_3$, where $B$ is a Boolean ring.

Generally, if $R/J(R)$ is a reduced weakly nil-clean ring having the strong property and $J(R)$ is nil, then $R$ is a weakly nil-clean ring having the strong property. In fact, by what we have proved above in Proposition 2.4, the relation $(a+J(R))^2 + (a+J(R)) = a^2 + a + J(R) \in Nil(R/J(R)) = J(R)$ tells us that $a^2 + a \in J(R) \subseteq Nil(R)$, as required. Same for $a^2 - a \in Nil(R)$, and so again with the aid of Proposition 2.4 we are finished.

On the other side, in [1] was proven that there is a nil-clean element which is not clean. However, every strongly nil-clean element has to be clean. In fact, even much more is true:

**Proposition 2.5.** *Each weakly nil-clean element having the strong property is clean.*

*Proof.* Writing $b = n+e$ or $b = n-e$ with $ne = en$ for some nilpotent $n$ and idempotent $e$, we have either $b = (n + 2e - 1) + (1 - e)$ or $b = (n - 1) + (1 - e)$. In both cases, $b$ is a clean element because $(2e - 1)^2 = 1$ and $2e - 1$ commutes with $n$, so that $n + 2e - 1$ is a unit as well as so is $n - 1$, whereas $1 - e$ is an idempotent. □

## 3. Concluding Discussion

We close the work with the following challenging problem.

**Conjecture 1**. A ring $R$ is weakly nil-clean if, and only if, $R$ is either a nil-clean ring or $R/J(R)$ is isomorphic to $\mathbb{Z}_3$ with $J(R)$ nil or $R$ is a direct product of these two rings.

We notice that this question will be resolved in the affirmative provided that the following holds:

**Conjecture 2**. A ring $R$ satisfies the nil-involution property if, and only if, $R/J(R) \cong \mathbb{Z}_3$ and $J(R)$ is nil.

Indeed, to show that the "and only if" part of Conjecture 1 is true, we decompose $R$ as the direct product of a nil-clean ring and a ring with the nil-involution property. In fact, since by [2] we know that $6^n = 0$ for some $n \in \mathbb{N}$ and since $(2^n, 3^n) = 1$, i.e., there exist non-zero integers $u, v$ such that $2^n u + 3^n v = 1$, it plainly follows that $R = 2^n R \oplus 3^n R$ because $2^n R \cap 3^n R = \{0\}$. In fact, to show that this intersection is really zero, given $x = 2^n a = 3^n b$ for some $a, b \in R$, we then have $2^n au = 3^n bu$. However, $a(1 - 3^n v) = 3^n bu$ whence $3^n(av+bu) = a$. Multiplying both sides by $2^n$, we derive that $0 = 2^n a = x$, as required. So, with the Chinese Reminder Theorem at hand, or directly by the above-given direct decomposition of $R$ into the sum of two ideals, we deduce that $R \cong L \times P$, where $L \cong R/2^n R \cong 3^n R$ and $P \cong R/3^n R \cong 2^n R$. Utilizing [2], it follows that both $L$ and $P$ are weakly nil-clean as epimorphic images of $R$. But it is obvious that $2 \in J(L)$, so appealing once again to [2], we conclude that $L$ is nil-clean, as claimed.

As for $P$, we may assume that $P \neq 0$. Thus $3 \in J(P)$ and, in addition, $2 \in U(P)$. Applying [2] and [9], we infer that $P$ is indecomposable and not nil-clean. Moreover, a new application of [2] implies that $J(P)$ is nil. Letting now $a \in P$, there exist $b \in Nil(P)$ and $e \in Id(P)$ such that $a = b + e$ or $a = b - e$. In the first case,

$a = ((b+3e)-1)+(1-2e)$ with $(1-2e)^2 = 1$. Moreover, as $b^m = 0$ for some integer $m > 0$ and $3 \in J(P)$, it readily follows that $(b+3e)^m \in J(P)$, whence $b+3e$ is a nilpotent because $J(P)$ is nil. Next, if $a = b-e$, then $a = ((b-3e)+1)+(-1+2e)$ with $(-1+2e)^2 = 1$. As above, since $b^m = 0$ and $3 \in J(P)$, it easily follows that $(b-3e)^m \in J(P)$, so that $b-3e$ is a nilpotent as $J(P)$ is nil. This finally enables us that $P$ satisfies the nil-involution property, as claimed. We furthermore apply Conjecture 2 to get the desired claim.

To demonstrate now that the "if" part of Conjecture 1 is valid, exploiting [2] and Conjecture 2, it is enough to prove that any ring equipped with the nil-involution property is weakly nil-clean. In fact, one easy sees that $3 \in J(R)$ and hence $2 \in U(R)$. Let now $a \in R$. Then $-2a = v + w$, where $v \in Nil(R) \pm 1$ and $w^2 = 1$. If $v = b+1$ with $b \in Nil(R)$, then $a = (-\frac{b}{2}) - \frac{1+w}{2}$ with $-\frac{b}{2} \in Nil(R)$ and $\frac{1+w}{2}$ an idempotent. If now $v = b-1$ with $b \in Nil(R)$, then $a = (-\frac{b}{2}) + \frac{(1-w)}{2}$ with $-\frac{b}{2} \in Nil(R)$ and $\frac{1-w}{2}$ an idempotent. So, $R$ is weakly nil-clean, as needed.

This completes the proof.

Note also that it is not too hard to verify that the sufficiency in Conjecture 2 is always fulfilled, so that it suffices to establish only the necessity.

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# The chain rule for $\mathcal{F}$-differentiation

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Abstract. Let $X$ be a perfect, compact subset of the complex plane, and let $D^{(1)}(X)$ denote the (complex) algebra of continuously complex-differentiable functions on $X$. Then $D^{(1)}(X)$ is a normed algebra of functions but, in some cases, fails to be a Banach function algebra. Bland and the second author ([3]) investigated the completion of the algebra $D^{(1)}(X)$, for certain sets $X$ and collections $\mathcal{F}$ of paths in $X$, by considering $\mathcal{F}$-differentiable functions on $X$.

In this paper, we investigate composition, the chain rule, and the quotient rule for this notion of differentiability. We give an example where the chain rule fails, and give a number of sufficient conditions for the chain rule to hold. Where the chain rule holds, we observe that the Faá di Bruno formula for higher derivatives is valid, and this allows us to give some results on homomorphisms between certain algebras of $\mathcal{F}$-differentiable functions.

Throughout this paper, we use the term *compact plane set* to mean a non-empty, compact subset of the complex plane, $\mathbb{C}$. We denote the set of all positive integers by $\mathbb{N}$ and the set of all non-negative integers by $\mathbb{N}_0$. Let $X$ be a compact Hausdorff space. We denote the algebra of all continuous, complex-valued functions on $X$ by $C(X)$ and we give $C(X)$ the uniform norm $|\cdot|_X$, defined by

$$|f|_X = \sup_{x\in X}|f(x)| \qquad (f \in C(X)).$$

This makes $C(X)$ into a commutative, unital Banach algebra. A subset $S$ of $C(X)$ *separates the points of* $X$ if, for each $x, y \in X$ with $x \neq y$, there exists $f \in S$ such that $f(x) \neq f(y)$. A *normed*

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*function algebra* on $X$ a normed algebra $(A, \|\cdot\|)$ such that $A$ is a subalgebra of $C(X)$, $A$ contains all constant functions and separates the points of $X$, and, for each $f \in A$, $\|f\| \geq |f|_X$. A *Banach function algebra* on $X$ is a normed function algebra on $X$ which is complete. We say that such a Banach function algebra $A$ is *natural* (on $X$) if every character on $A$ is given by evaluation at some point of $X$. We refer the reader to [5] (especially Chapter 4) for further information on Banach algebras and Banach function algebras.

Let $D^{(1)}(X)$ denote the normed algebra of all continuously (complex) differentiable, complex-valued functions on $X$, as discussed in [6] and [7]. Furthermore, let $D^{(n)}(X)$ denote the normed algebra of all continuously $n$-times (complex) differentiable, complex-valued functions on $X$, and let $D^{(\infty)}(X)$ denote the algebra of continuous functions which have continuous (complex) derivatives of all orders. Dales and Davie ([6]) also introduced the algebras

$$D(X, M) := \left\{ f \in D^{(\infty)}(X) : \sum_{j=0}^{\infty} \frac{|f^{(j)}|_X}{M_j} < \infty \right\},$$

where $M = (M_n)_{n=0}^{\infty}$ is a suitable sequence of positive real numbers. The algebras $D(X, M)$ are called *Dales-Davie* algebras.

The usual norms on the algebras $D^{(n)}(X)$ ($n \in \mathbb{N}$) and $D(X, M)$ above need not be complete, so we often investigate the completion of these algebras. One approach to this was introduced by Bland and Feinstein [3], where they discussed algebras of $\mathcal{F}$-differentiable functions (see Section 2), and these algebras were investigated further in [7] and [12].

Kamowitz and Feinstein investigated the conditions under which composition with an infinitely differentiable map induces an endomorphism ([13, 9, 10]) or a homomorphism ([11]) between Dales-Davie algebras.

In this paper, we investigate composition, the chain rule, and the quotient rule for $\mathcal{F}$-differentiation. We give an example where the chain rule for $\mathcal{F}$-differentiation fails, and give a number of sufficient conditions for the chain rule to hold. We also prove a version of the quotient rule for $\mathcal{F}$-differentiable functions.

Where the chain rule holds, we observe that the Faá di Bruno formula for higher derivatives is valid, and this allows us to give some sufficient conditions, similar to those in [11], for composition with an infinitely $\mathcal{F}$-differentiable function to induce a homomorphism between the $\mathcal{F}$-differentiability versions of Dales-Davie algebras.

## 1. Paths in the complex plane

We begin with a discussion of collections of paths in the complex plane.

**Definition 1.1.** A *path* in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$, where $a < b$ are real numbers. Let $\gamma : [a, b] \to \mathbb{C}$ be a path. The *parameter interval of* $\gamma$ is the interval $[a, b]$. The *endpoints of* $\gamma$ are the points $\gamma(a)$ and $\gamma(b)$, which we denote by $\gamma^-$ and $\gamma^+$, respectively. We denote by $\gamma^*$ the *image* $\gamma([a, b])$ of $\gamma$. A *subpath* of $\gamma$ is a path obtained by restricting $\gamma$ to a non-degenerate, closed subinterval of $[a, b]$. If $X$ is a subset of $\mathbb{C}$ then we say that $\gamma$ *is a path in* $X$ if $\gamma^* \subseteq X$.

Let $\gamma$ be a path in $\mathbb{C}$. We say that $\gamma$ is a *Jordan path* if $\gamma$ is an injective function.

Let $[a, b]$ be a non-degenerate closed interval. A *partition* $\mathcal{P}$ of $[a, b]$ is a finite set $\{x_0, \ldots, x_n\} \subseteq [a, b]$ such that $x_0 = a$, $x_n = b$ and $x_j < x_{j+1}$ for each $j \in \{0, 1, \ldots, n-1\}$. If $\mathcal{P}$ and $\mathcal{P}'$ are partitions of $[a, b]$ then we say that $\mathcal{P}'$ is *finer* than $\mathcal{P}$ if $\mathcal{P} \subseteq \mathcal{P}'$.

**Definition 1.2.** Let $\gamma : [a, b] \to \mathbb{C}$ be a path and let $c, d \in [a, b]$ with $c < d$. The *total variation* of $\gamma$ over $[c, d]$ is

$$V_c^d(\gamma) := \sup\left\{\sum_{j=0}^{n-1} |\gamma(x_{j+1}) - \gamma(x_j)| : \mathcal{P} = \{x_0, \ldots, x_n\}\right\}$$

where the supremum is taken over all partitions $\mathcal{P}$ of $[c, d]$. We say that $\gamma$ is *rectifiable* if $V_a^b(\gamma) < \infty$, in which case we set $\Lambda(\gamma) := V_a^b(\gamma)$; otherwise it is *non-rectifiable*. The *length* of a rectifiable path $\gamma$ is $\Lambda(\gamma)$.

For a detailed discussion of paths, total variation and path length, see [2, Chapter 6].

We say that a path $\gamma$ is *admissible* if $\gamma$ is rectifiable and contains no constant subpaths. Let $\gamma : [a, b] \to \mathbb{C}$ be a non-constant (but not necessarily admissible) rectifiable path. We define the *path length parametrisation* $\gamma^{\mathrm{pl}} : [0, \Lambda(\gamma)] \to \mathbb{C}$ of $\gamma$ to be the unique path satisfying $\gamma^{\mathrm{pl}}(V_a^t(\gamma)) = \gamma(t)$ $(t \in [a, b])$; see, for example, [8, pp. 109-110] for details. We define the *normalised path length parametrisation*

$\gamma^{\rm no} : [0,1] \to \mathbb{C}$ of $\gamma$ to be the path such that $\gamma^{\rm no}(t) = \gamma^{\rm pl}(t\Lambda(\gamma))$ for each $t \in [0,1]$. It is clear that $\gamma^{\rm pl}$ and $\gamma^{\rm no}$ are necessarily admissible paths and $(\gamma^{\rm pl})^* = (\gamma^{\rm no})^* = \gamma^*$. It is not hard to show, using [8, Theorem 2.4.18], that

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma^{\rm pl}} f(z)\,\mathrm{d}z = \int_{\gamma^{\rm no}} f(z)\,\mathrm{d}z,$$

for all $f \in C(\gamma^*)$. We shall use this fact implicitly throughout.

**Definition 1.3.** Let $X$ be a compact plane set and let $\mathcal{F}$ be a collection of paths in $X$. We define $\mathcal{F}^* := \{\gamma^* : \gamma \in \mathcal{F}\}$. We say that $\mathcal{F}$ is *effective* if $\overline{\bigcup \mathcal{F}^*} = X$, each path in $\mathcal{F}$ is admissible, and every subpath of a path in $\mathcal{F}$ belongs to $\mathcal{F}$. We denote by $\mathcal{F}^{\rm no}$ the collection $\{\gamma^{\rm no} : \gamma \in \mathcal{F}\}$.

Let $X$ be a compact plane set and let $\mathcal{F}$ be a collection of paths in $X$. It is clear that $\mathcal{F}^* = (\mathcal{F}^{\rm no})^*$.

We introduce the following definitions from [3, 6] and [7].

**Definition 1.4.** Let $X$ be a compact plane set. We say that $X$ is *uniformly regular* if there exists a constant $C > 0$ such that, for all $x, y \in X$, there exists a rectifiable path $\gamma$ in $X$ with $\gamma^- = x$ and $\gamma^+ = y$ such that $\Lambda(\gamma) \le C|x - y|$. We say that $X$ is *pointwise regular*, if for each $x \in X$, there exists a constant $C_x > 0$ such that, for each $y \in X$, there exists a path $\gamma$ in $X$ with $\gamma^- = x$ and $\gamma^+ = y$ such that $\Lambda(\gamma) \le C_x|x - y|$. We say that $X$ is *semi-rectifiable* if the union of the images of all rectifiable, Jordan paths in $X$ is dense in $X$.

We also require the following definition from [7].

**Definition 1.5.** Let $X$ be a compact plane set and let $\mathcal{F}$ be an effective collection of paths in $X$. We say that $X$ is *$\mathcal{F}$-regular at* $x \in X$ if there exists a constant $C_x > 0$ such that for each $y \in X$ there exists $\gamma \in \mathcal{F}$ with $\gamma^- = x$, $\gamma^+ = y$ and $\Lambda(\gamma) \le C_x|x - y|$. We say that $X$ is *$\mathcal{F}$-regular* if $X$ is $\mathcal{F}$-regular at each point $x \in X$.

We note that if $X$ is a compact plane set which is $\mathcal{F}$-regular, for some effective collection $\mathcal{F}$ of paths in $X$, then $X$ is pointwise regular.

## 2. Algebras of $\mathcal{F}$-differentiable functions

In this section we discuss algebras of $\mathcal{F}$-differentiable functions as investigated in [3] and [7], along with algebras of $\mathcal{F}$-differentiable functions analogous to the Dales-Davie algebras introduced in [6].

**Definition 2.1.** Let $X$ be a perfect compact plane set, let $\mathcal{F}$ be a collection of rectifiable paths in $X$, and let $f \in C(X)$. A function $g \in C(X)$ is an *$\mathcal{F}$-derivative of $f$* if, for each $\gamma \in \mathcal{F}$, we have

$$\int_\gamma g(z)\,\mathrm{d}z = f(\gamma^+) - f(\gamma^-).$$

If $f$ has an $\mathcal{F}$-derivative on $X$ then we say that $f$ is $\mathcal{F}$-differentiable on $X$.

The following proposition summarises several properties of $\mathcal{F}$-derivatives and $\mathcal{F}$-differentiable functions on certain compact plane sets. Details can be found in [3] and [7].

**Proposition 2.2.** *Let $X$ be a semi-rectifiable compact plane set and let $\mathcal{F}$ be an effective collection of paths in $X$.*

(a) *Let $f, g, h \in C(X)$ be such that $g$ and $h$ are $\mathcal{F}$-derivatives for $f$. Then $g = h$.*
(b) *Let $f \in D^{(1)}(X)$. Then the usual complex derivative of $f$ on $X$, $f'$, is an $\mathcal{F}$-derivative for $f$.*
(c) *Let $f_1, f_2, g_1, g_2 \in C(X)$ be such that $g_1$ is an $\mathcal{F}$-derivative for $f_1$ and $g_2$ is an $\mathcal{F}$-derivative for $f_2$. Then $f_1g_2 + g_1f_2$ is an $\mathcal{F}$-derivative for $f_1f_2$.*
(d) *Let $f_1, f_2, g_1, g_2 \in C(X)$ and $\alpha, \beta \in \mathbb{C}$ be such that $g_1$ is an $\mathcal{F}$-derivative for $f_1$ and $g_2$ is an $\mathcal{F}$-derivative for $f_2$. Then $\alpha g_1 + \beta g_2$ is an $\mathcal{F}$-derivative for $\alpha f_1 + \beta f_2$.*

Let $X$ be a semi-rectifiable compact plane set, and let $\mathcal{F}$ be an effective collection of paths in $X$. In this setting we write $f^{[1]}$ for the unique $\mathcal{F}$-derivative of an $\mathcal{F}$-differentiable function and we will often write $f^{[0]}$ for $f$. We write $D^{(1)}_{\mathcal{F}}(X)$ for the algebra of all $\mathcal{F}$-differentiable functions on $X$. We note that, with the norm $\|f\|_{\mathcal{F},1} := |f|_X + |f^{[1]}|_X$ $(f \in D^{(1)}_{\mathcal{F}}(X))$, the algebra $D^{(1)}_{\mathcal{F}}(X)$ is a Banach function algebra on $X$ ([7, Theorem 5.6]).

For each $n \in \mathbb{N}$, we define (inductively) the algebra

$$D^{(n)}_{\mathcal{F}}(X) := \{f \in D^{(1)}_{\mathcal{F}}(X) : f^{[1]} \in D^{(n-1)}_{\mathcal{F}}(X)\},$$

and, for each $f \in D_{\mathcal{F}}^{(n)}(X)$, we write $f^{[n]}$ for the $n$th $\mathcal{F}$-derivative of $f$. We note that, for each $n \in \mathbb{N}$, $D_{\mathcal{F}}^{(n)}(X)$ is a Banach function algebra on $X$ (see [3]) when given the norm

$$\|f\|_{\mathcal{F},n} := \sum_{k=0}^{n} \frac{|f^{[k]}|_X}{k!} \qquad (f \in D_{\mathcal{F}}^{(n)}(X)).$$

In addition, we define the algebra $D_{\mathcal{F}}^{(\infty)}(X)$ of all functions which have $\mathcal{F}$-derivatives of all orders; that is, $D_{\mathcal{F}}^{(\infty)}(X) = \bigcap_{n=1}^{\infty} D_{\mathcal{F}}^{(n)}(X)$. It is easy to see that, for each $n \in \mathbb{N}$, we have $D^{(n)}(X) \subseteq D_{\mathcal{F}}^{(n)}(X)$ and $D^{(\infty)}(X) \subseteq D_{\mathcal{F}}^{(\infty)}(X)$.

## 3. Maximal collections and compatibility

We aim to prove a chain rule for $\mathcal{F}$-differentiable functions, but first we must investigate collections of paths further. Throughout this section, let $X$ be a semi-rectifiable, compact plane set, and let $\mathcal{A}$ be the collection of all admissible paths in $X$. In this section, we identify $D_{\mathcal{F}}^{(1)}(X)$ with the subset $S_{\mathcal{F}}$ of $C(X) \times C(X)$ consisting of all pairs $(f, g)$ where $f \in D_{\mathcal{F}}^{(1)}(X)$ and $g$ is the $\mathcal{F}$-derivative of $f$. We begin with a definition.

**Definition 3.1.** Let $\gamma$ be an admissible path in $X$. Let $f, g \in C(X)$. We say that $g$ is the $\gamma$-*derivative* of $f$ if, for each subpath $\sigma$ of $\gamma$, we have

$$\int_{\sigma} g(z)\,\mathrm{d}z = f(\sigma^+) - f(\sigma^-).$$

Note that, in the above, if $\mathcal{G}$ denotes the collection of all subpaths of $\gamma$, then $\mathcal{G}$ is effective in $\gamma^*$, so $\mathcal{G}$-derivatives on $\gamma^*$ are unique. Thus, if $f$ has a $\gamma$ derivative $g$ on $X$, then $g|_{\gamma^*}$ is uniquely determined.

**Definition 3.2.** Let $S \subseteq C(X) \times C(X)$. We define

$$\mathfrak{p}(S) := \{\gamma \in \mathcal{A} : \text{for all } (f,g) \in S,\ g \text{ is the } \gamma\text{-derivative of } f\}.$$

Let $\mathcal{F}$ be an effective collection of paths in $X$. Then we write $\mathfrak{m}(\mathcal{F})$ for $\mathfrak{p}(S_{\mathcal{F}})$, where $S_{\mathcal{F}} = \{(f, f^{[1]}) : f \in D_{\mathcal{F}}^{(1)}(X)\}$ as above.

The following lemma follows quickly from the definition of $\mathfrak{m}(\mathcal{F})$. We omit the details.

**Lemma 3.3.** *Let $S, T \subseteq C(X) \times C(X)$. If $S \subseteq T$ then we have $\mathfrak{p}(T) \subseteq \mathfrak{p}(S)$. Let $\mathcal{F}, \mathcal{G}$ be effective collections of paths in $X$. Then we have $D^{(1)}_{\mathfrak{m}(\mathcal{F})}(X) = D^{(1)}_{\mathcal{F}}(X)$. Moreover, $S_{\mathcal{F}} \subseteq S_{\mathcal{G}}$ if and only if $\mathfrak{m}(\mathcal{G}) \subseteq \mathfrak{m}(\mathcal{F})$.*

Note that $S_{\mathcal{F}} \subseteq S_{\mathcal{G}}$ implies that $D^{(1)}_{\mathcal{F}}(X) \subseteq D^{(1)}_{\mathcal{G}}(X)$.

We now investigate some elementary operations on $\mathfrak{m}(\mathcal{F})$. Let $\gamma_1 : [a, b] \to \mathbb{C}, \gamma_2 : [c, d] \to \mathbb{C}$ be paths such that $\gamma_1^+ = \gamma_2^-$. We write $\gamma_1 \dot{+} \gamma_2$ for the path given by

$$(\gamma_1 \dot{+} \gamma_2)(t) = \begin{cases} \gamma_1(a + 2t(b - a)), & t \in [0, 1/2), \\ \gamma_2(c + (2t - 1)(d - c)), & t \in [1/2, 1]. \end{cases}$$

We call the path $\gamma_1 \dot{+} \gamma_2$ the *join* of $\gamma_1$ and $\gamma_2$. Note that, if $\gamma_1$ and $\gamma_2$ are admissible, then $\gamma_1 \dot{+} \gamma_2$ is admissible. The *reverse of* $\gamma_1$, denoted by $-\gamma_1$, is given by $-\gamma_1(t) = \gamma_1(b - t(b - a))$ $(t \in [0, 1])$. Our notation for joining and reversing is not entirely standard and there are many ways to parametrise these paths.

**Lemma 3.4.** *Let $\mathcal{F}$ be an effective collection of paths in $X$. Then $\mathfrak{m}(\mathcal{F})$ has the following properties*:

(a) *if $\gamma \in \mathfrak{m}(\mathcal{F})$ then $-\gamma \in \mathfrak{m}(\mathcal{F})$;*
(b) *if $\gamma \in \mathfrak{m}(\mathcal{F})$ then $\gamma^{\mathrm{pl}}, \gamma^{\mathrm{no}} \in \mathfrak{m}(\mathcal{F})$;*
(c) *if $\gamma_1, \gamma_2 \in \mathfrak{m}(\mathcal{F})$ such that $\gamma_1^+ = \gamma_2^-$ then $\gamma_1 \dot{+} \gamma_2 \in \mathfrak{m}(\mathcal{F})$.*

*Proof.* (a) This is clear from the definitions.

(b) This is clear from the definitions, and the discussions in Section 1.

(c) This is effectively [3, Theorem 4.5], and follows from the definitions. □

We also make the following observation about collections of paths generated by a set in $C(X) \times C(X)$ and its closure in the norm given by $\|(f, g)\|_1 = |f|_X + |g|_X$ for each $(f, g) \in C(X) \times C(X)$.

**Lemma 3.5.** *Let $S \subseteq C(X) \times C(X)$. Then $\mathfrak{p}(\overline{S}) = \mathfrak{p}(S)$ where the closure of $S$ is taken in the norm $\|\cdot\|_1$ on $C(X) \times C(X)$ as above.*

*Proof.* By Lemma 3.3, we have $\mathfrak{p}(\overline{S}) \subseteq \mathfrak{p}(S)$. Let $(f_n, g_n)$ be a sequence in $S$ such that $(f_n, g_n) \to (f, g) \in \overline{S}$ as $n \to \infty$. Let $\gamma \in \mathfrak{p}(S)$. Then we have

$$f_n(\gamma^+) - f_n(\gamma^-) = \int_\gamma g_n(z)\,\mathrm{d}z,$$

for all $n \in \mathbb{N}$. But now $f_n \to f$ and $g_n \to g$ uniformly as $n \to \infty$, so

$$f(\gamma^+) - f(\gamma^-) = \int_\gamma g(z)\, \mathrm{d}z.$$

Thus $\gamma \in \mathfrak{p}(\overline{S})$. It follows that $\mathfrak{p}(S) \subseteq \mathfrak{p}(\overline{S})$ and this completes the proof. □

We also require the following elementary lemma, which is a minor variant of a standard result. We include a proof for the convenience of the reader.

**Lemma 3.6.** *Let $\mathcal{F}$ be an effective collection of paths in $X$. Let $\varphi \in D^{(1)}_{\mathcal{F}}(X)$ and let $\gamma : [a,b] \to X \in \mathcal{F}$. Then $\varphi \circ \gamma$ is rectifiable, and hence, if $\varphi \circ \gamma$ is non-constant, $(\varphi \circ \gamma)^{\mathrm{pl}}$ is admissible.*

*Proof.* Let $\mathcal{P} = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$. Set $\sigma := \varphi \circ \gamma$ and let $\gamma_j$ be the subpath of $\gamma$ obtained by restricting $\gamma$ to $[x_j, x_{j+1}]$ for each $j \in \{0, \ldots, n-1\}$. We have

$$V(\sigma, \mathcal{P}) := \sum_{j=0}^{n-1} |\sigma(x_{j+1}) - \sigma(x_j)| = \sum_{j=0}^{n-1} \left| \int_{\gamma_j} \varphi^{[1]}(z)\, \mathrm{d}z \right| \leq |\varphi^{[1]}|_X \Lambda(\gamma).$$

It follows that $\Lambda(\sigma) = \sup V(\sigma, \mathcal{P}) \leq |\varphi^{[1]}|_X \Lambda(\gamma)$, where the supremum is taken over all partitions $\mathcal{P}$ of $[a,b]$. As noted earlier, if $\sigma$ is non-constant then $\sigma^{\mathrm{pl}}$ is admissible. This completes the proof. □

We now introduce our notion of compatibility.

**Definition 3.7.** Let $Y$ be a semi-rectifiable compact plane set, let $\mathcal{F}$ be an effective collection of paths in $X$ and let $\mathcal{G}$ be an effective collection of paths in $Y$. Let $\varphi \in D^{(1)}_{\mathcal{F}}(X)$ such that $\varphi(X) \subseteq Y$. We say that $\varphi$ is *$\mathcal{F}$-$\mathcal{G}$-compatible* if, for each $\gamma \in \mathcal{F}$, either $\varphi \circ \gamma$ is constant or we have $(\varphi \circ \gamma)^{\mathrm{pl}} \in \mathfrak{m}(\mathcal{G})$.

Let $X, Y$ be semi-rectifiable compact plane sets, let $\mathcal{F}$ be an effective collection of paths in $X$, and $\mathcal{G}$ be an effective collection of paths in $Y$. If $\mathfrak{m}(\mathcal{G})$ is the collection of all admissible paths in $Y$ then, for any $\varphi \in D^{(1)}_{\mathcal{F}}(X)$ with $\varphi(X) \subseteq Y$, $\varphi$ is $\mathcal{F}$-$\mathcal{G}$-compatible.

**Example 3.8.** Let $X := \{x + \mathrm{i}y \in \mathbb{C} : x, y \in [0,1]\}$. Let $\mathcal{F}$ be the collection of all line segment paths in $X$ parallel to the real axis and let $\mathcal{G}$ be the collection of all line segment paths in $X$ parallel to the imaginary axis. Set $\varphi(z) := z$ $(z \in X)$. Then $\varphi : X \to X$ is continuously differentiable on $X$ and $\varphi(X) = X$. Clearly $\varphi \circ \gamma \in \mathcal{F}$

for all $\gamma \in \mathcal{F}$. It is not hard to show that, if $\gamma \in \mathfrak{m}(\mathcal{F})$, then $\gamma^* \in \mathcal{F}^*$ and, if $\gamma \in \mathfrak{m}(\mathcal{G})$, then $\gamma^* \in \mathcal{G}^*$. Thus $\mathfrak{m}(\mathcal{F}) \cap \mathfrak{m}(\mathcal{G}) = \emptyset$ and it follows that $\varphi$ is not $\mathcal{F}$-$\mathcal{G}$-compatible.

Let $X, Y$ be semi-rectifiable compact plane sets, let $\mathcal{F}$ be an effective collection of paths in $X$, and let $\mathcal{G}$ be an effective collection of paths in $Y$. If $D^{(1)}(Y)$ is dense in $D^{(1)}_{\mathcal{G}}(Y)$ then, by Lemma 3.5, the collection $\mathfrak{m}(\mathcal{G})$ is the collection of all admissible paths in $Y$. So, by the comments following Definition 3.7, any function $\varphi \in D^{(1)}_{\mathcal{F}}(X)$ such that $\varphi(X) \subseteq Y$ is automatically $\mathcal{F}$-$\mathcal{G}$-compatible.

## 4. Composition of $\mathcal{F}$-differentiable functions

We now discuss an analogue of the chain rule for $\mathcal{F}$-differentiable functions. The following lemma is an $\mathcal{F}$-differentiability version of the usual change of variable formula.

**Lemma 4.1.** *Let $X$ be a semi-rectifiable, compact plane set and let $\mathcal{F}$ be an effective collection of paths in $X$. Let $\varphi \in D^{(1)}_{\mathcal{F}}(X)$ and let $\gamma : [a,b] \to X \in \mathcal{F}$. Then, for each $f \in C(\varphi(\gamma^*))$, we have*

$$\int_\gamma (f \circ \varphi)\varphi^{[1]}(z)\,\mathrm{d}z = \int_{\varphi\circ\gamma} f(z)\,\mathrm{d}z. \tag{1}$$

*Proof.* By Lemma 3.6, $\sigma := \varphi \circ \gamma$ is a rectifiable path so that the integral on the right-hand side of (1) exists. Fix $f \in C(\sigma^*)$ and let $\varepsilon > 0$. Set $M := |\varphi^{[1]}|_X \Lambda(\gamma)$ and let $h := f \circ \sigma : [a,b] \to \mathbb{C}$. Since $h$ is uniformly continuous, there exists $\delta > 0$ such that, for each $s, t \in [a,b]$ with $|s-t| < \delta$, we have $|h(t) - h(s)| < \varepsilon/(2M)$. Choose a partition $\mathcal{P}_0 = \{t_0^{(0)}, \dots, t_m^{(0)}\}$ of $[a,b]$ such that

$$\max_{0 \le j \le n-1} |t_{j+1} - t_j| < \delta.$$

For any partition $\mathcal{P} = \{t_0, \dots, t_n\}$ finer than $\mathcal{P}_0$ and, for each $j \in \{0, \dots, n-1\}$, let $\gamma_j^{(\mathcal{P})}$ be the restriction of $\gamma$ to $[t_j, t_{j+1}]$. We have

$$T(\mathcal{P}) := \left| \sum_{j=0}^{n-1} \int_{\gamma_j^{(\mathcal{P})}} (f(\varphi(z)) - h(s_j))\varphi^{[1]}(z)\,\mathrm{d}z \right| < \frac{\varepsilon}{2}, \tag{2}$$

for any $s_j \in [t_j, t_{j+1}]$ $(j = 0, 1, \dots, n-1)$.

Now fix a partition $\mathcal{P} = \{t_0, \ldots, t_n\}$ of $[a, b]$ finer than $\mathcal{P}_0$ such that, viewing the integral in the right-hand side of (1) as a Riemann-Stieltjes integral on $[a, b]$, we have

$$\left| \sum_{j=0}^{n-1} h(s_j)(\sigma(t_{j+1}) - \sigma(t_j)) - \int_\sigma f(z)\, \mathrm{d}z \right| < \frac{\varepsilon}{2}, \tag{3}$$

for any choice of $s_j \in [t_j, t_{j+1}]$ $(j = 0, 1, \ldots, n-1)$.

We now *claim* that, for this partition $\mathcal{P}$, we have

$$\left| \int_\gamma f(\varphi(z))\varphi^{[1]}(z)\, \mathrm{d}z - \sum_{j=0}^{n-1} h(s_j)(\sigma(t_{j+1}) - \sigma(t_j)) \right| < \frac{\varepsilon}{2} \tag{4}$$

for any choice of $s_j \in [t_j, t_{j+1}]$ $(j = 0, 1, \ldots, n-1)$.

For the remainder of the proof, for each $j \in \{0, \ldots, n-1\}$, fix $s_j \in [t_j, t_{j+1}]$ and let $S := \sum_{j=0}^{n-1} h(s_j)(\sigma(t_{j+1}) - \sigma(t_j))$.

By the definition of $\varphi^{[1]}$, we have

$$\sum_{j=0}^{m-1} h(s_j) \int_{\gamma_j^{(\mathcal{P})}} \varphi^{[1]}(z)\, \mathrm{d}z = S.$$

We also have

$$\left| \int_\gamma f(\varphi(z))\varphi^{[1]}(z)\, \mathrm{d}z - \sum_{j=0}^{n-1} h(s_j) \int_{\gamma_j^{(\mathcal{P})}} \varphi^{[1]}(z)\, \mathrm{d}z \right| = T(\mathcal{P}),$$

where $T(\mathcal{P}) < \varepsilon/2$ as in (2).

But now, by (2) and (3), we have

$$\left| \int_\gamma f(\varphi(z))\varphi^{[1]}(z)\, \mathrm{d}z - \int_\sigma f(w)\, \mathrm{d}w \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This holds for all $\varepsilon > 0$ and any choice of the $s_j$, so the result follows. $\square$

We can now state and prove a version of the chain rule for $\mathcal{F}$-differentiable functions.

**Theorem 4.2.** *Let $X, Y$ be semi-rectifiable, compact plane sets, let $\mathcal{F}$ be an effective collection of paths in $X$, and let $\mathcal{G}$ be an effective collection of paths on $Y$. Let $\varphi \in D_{\mathcal{F}}^{(1)}(X)$ with $\varphi(X) \subseteq Y$. Suppose that $\varphi$ is $\mathcal{F}$-$\mathcal{G}$-compatible. Then, for all $f \in D_{\mathcal{G}}^{(1)}(Y)$, $f \circ \varphi$ is $\mathcal{F}$-differentiable and $(f \circ \varphi)^{[1]} = (f^{[1]} \circ \varphi)\varphi^{[1]}$.*

*Proof.* Fix $f \in D^{(1)}_{\mathcal{G}}(Y)$ and $\gamma \in \mathcal{F}$. Then, by Lemma 4.1, we have

$$\int_\gamma (f^{[1]} \circ \varphi)(z)\varphi^{[1]}(z)\,\mathrm{d}z = \int_{\varphi\circ\gamma} f^{[1]}(z)\,\mathrm{d}z.$$

Since $\varphi$ is $\mathcal{F}$-$\mathcal{G}$-compatible, we have $\varphi \circ \gamma \in \mathfrak{m}(\mathcal{G})$ and so

$$\int_{\varphi\circ\gamma} f^{[1]}(z)\,\mathrm{d}z = f((\varphi\circ\gamma)^+) - f((\varphi\circ\gamma)^-).$$

But $(\varphi\circ\gamma)^+ = \varphi(\gamma^+)$ and $(\varphi\circ\gamma)^- = \varphi(\gamma^-)$. Thus $f \circ \varphi$ is $\mathcal{F}$-differentiable and has $\mathcal{F}$-derivative $(f^{[1]} \circ \varphi)\varphi^{[1]}$. This completes the proof. □

As a corollary we obtain the quotient rule for $\mathcal{F}$-differentiable functions. This was originally proved by means of repeated bisection in [4].

**Corollary 4.3.** *Let $X$ be a semi-rectifiable compact plane set, let $\mathcal{F}$ be an effective collection of paths in $X$, and let $f, g \in D^{(1)}_{\mathcal{F}}(X)$ such that $0 \notin g(X)$. Then we have $f/g \in D^{(1)}_{\mathcal{F}}(X)$ and*

$$(f/g)^{[1]} = (gf^{[1]} - fg^{[1]})/g^2.$$

*Proof.* We first show that $h := 1/g \in D^{(1)}_{\mathcal{F}}(X)$ and that we have $h^{[1]} = -g^{[1]}/g^2$. Since we have $0 \notin g(X)$, the function $\varphi(z) := 1/z$ ($z \in g(X)$) is continuous and complex-differentiable on $g(X)$, i.e. $\varphi \in D^{(1)}(g(X))$. Let $\mathcal{G}$ be the collection of all admissible paths in $g(X)$. Then we have $\varphi \in D^{(1)}(g(X)) \subseteq D^{(1)}_{\mathcal{G}}(g(X))$, and $g$ is $\mathcal{F}$-$\mathcal{G}$-compatible by the comments following Definition 3.7. By Theorem 4.2, $\varphi \circ g \in D^{(1)}_{\mathcal{F}}(X)$ with $(\varphi\circ g)^{[1]} = (\varphi^{[1]} \circ g)g^{[1]}$. However, $\varphi^{[1]}$ is just the ordinary complex derivative of $\varphi$, and so $\varphi^{[1]} \circ g = -1/g^2$. Thus $h^{[1]} = -g^{[1]}/g^2$. The result now follows from the product rule for $\mathcal{F}$-derivatives, Proposition 2.2(c). □

By combining Theorem 4.2 with our comments at the end of Section 3, we obtain the following corollary.

**Corollary 4.4.** *Let $X$ be a semi-rectifiable, compact plane set and let $\mathcal{F}$ be an effective collection of paths in $X$. Let $f, g \in D^{(1)}_{\mathcal{F}}(X)$ such that $g(X) \subseteq X$. Suppose that $D^{(1)}(X)$ is dense in $D^{(1)}_{\mathcal{F}}(X)$. Then $f \circ g \in D^{(1)}_{\mathcal{F}}(X)$ and $(f^{[1]} \circ g)g^{[1]}$ is the $\mathcal{F}$-derivative of $f \circ g$.*

This last result was proved in the first author's PhD thesis ([4]) using the quotient rule, under the apparently stronger condition that the set of rational functions with no poles on $X$ be dense in $D_{\mathcal{F}}^{(1)}(X)$. See the final section of this paper for an open problem related to this.

By applying Theorem 4.2 inductively, we obtain the Faá di Bruno formula for the composition of $n$-times $\mathcal{F}$-differentiable functions.

**Corollary 4.5.** *Let $X, Y$ be semi-rectifiable, compact plane sets, let $\mathcal{F}$ be an effective collection of paths in $X$, and let $\mathcal{G}$ be an effective collection of paths in $Y$. Let $n \in \mathbb{N}$, and let $\varphi \in D_{\mathcal{F}}^{(n)}(X)$ with $\varphi(X) \subseteq Y$. Suppose that $\varphi$ is $\mathcal{F}$-$\mathcal{G}$-compatible. Then, for all $f \in D_{\mathcal{G}}^{(n)}(Y)$, $f \circ \varphi \in D_{\mathcal{F}}^{(n)}(X)$ and, for each $k \in \{1, 2, \ldots, n\}$, we have*

$$(f \circ \varphi)^{[k]} = \sum_{i=0}^{k} (f^{[i]} \circ \varphi) \sum \frac{k!}{a_1! \cdots a_k!} \prod_{j=1}^{k} \left( \frac{\varphi^{[j]}}{j!} \right)^{a_j},$$

*where the inner sum is over all $a_1, \ldots, a_k \in \mathbb{N}_0$ such that*

$$a_1 + \cdots + a_k = i \quad \textit{and} \quad a_1 + 2a_2 + \cdots + ka_k = k.$$

## 5. Homomorphisms

We now discuss some algebras of infinitely $\mathcal{F}$-differentiable functions, analogous to the algebras $D(X, M)$ introduced by Dales and Davie in [6] (see also the introduction of the present paper). In particular, we describe some sufficient conditions under which a function can induce a homomorphism between these algebras. These conditions are similar to those discussed by Feinstein and Kamowitz in [9]. We begin with some definitions from [1, 6] and [9].

**Definition 5.1.** Let $M = (M_n)_{n=0}^{\infty}$ be a sequence of positive real numbers. We say that $M$ is an *algebra sequence* if $M_0 = 1$ and, for all $j, k \in \mathbb{N}_0$, we have

$$\binom{j+k}{j} \leq \frac{M_{j+k}}{M_j M_k}.$$

We define $d(M) := \lim_{n \to \infty} (n!/M_n)^{1/n}$ and we say that $M$ is *non-analytic* if $d(M) = 0$.

Let $X$ be a perfect compact plane set and let $M = (M_n)_{n=0}^{\infty}$ be an algebra sequence. Then the set of all rational functions with no poles on $X$ is contained in $D(X, M)$ if and only if $M$ is non-analytic.

We now discuss the algebras $D_{\mathcal{F}}(X, M)$ as introduced in [3].

**Definition 5.2.** Let $X$ be a semi-rectifiable, compact plane set and let $\mathcal{F}$ be an effective collection of paths in $X$. Let $M = (M_n)_{n=0}^{\infty}$ be an algebra sequence. We define the normed algebra

$$D_{\mathcal{F}}(X, M) := \left\{ f \in D_{\mathcal{F}}^{(\infty)}(X) : \sum_{j=0}^{\infty} \frac{|f^{[j]}|_X}{M_j} < \infty \right\}$$

with pointwise operations and the norm

$$\|f\| := \sum_{j=0}^{\infty} \frac{|f^{[j]}|_X}{M_j} \qquad (f \in D_{\mathcal{F}}(X, M)).$$

The proof that the $D_{\mathcal{F}}(X, M)$ are indeed algebras is similar to the proof of Theorem 1.6 of [6]. In fact, since $D_{\mathcal{F}}^{(1)}(X)$ is complete with the conditions above, it follows that $D_{\mathcal{F}}(X, M)$ is a Banach function algebra; this is noted in [3].

Unfortunately, it is not known in general whether the Banach function algebras $D_{\mathcal{F}}^{(n)}(X)$ and $D_{\mathcal{F}}(X, M)$ are natural on $X$, although some sufficient conditions are given in [4]. We note that a necessary condition for $D_{\mathcal{F}}(X, M)$ to be natural is that $M = (M_n)_{n=0}^{\infty}$ be a non-analytic algebra sequence.

**Definition 5.3.** Let $X$ be a semi-rectifiable, compact plane set, and let $\mathcal{F}$ be an effective collection of paths in $X$. Let $f \in D_{\mathcal{F}}^{(\infty)}(X)$. We say that $f$ is $\mathcal{F}$-*analytic* if

$$\limsup_{k \to \infty} \left( \frac{|\varphi^{[k]}|_X}{k!} \right)^{1/k} < \infty.$$

Note that a function $f \in D_{\mathcal{F}}^{(\infty)}(X)$ which is $\mathcal{F}$-analytic need not be analytic (in the sense of extending to be analytic on a neighbourhood of $X$). Let $X$ and $\mathcal{F}$ be as in Example 3.8, let $M = (M_n)_{n=0}^{\infty}$ be an algebra sequence, and let $f \in D_{\mathcal{F}}(X, M)$ with $f \notin D(X, M)$ such that $f$ is $\mathcal{F}$-analytic. Then $f$ is not analytic. For example, we may take take $f(z) = \mathrm{Im}(z)$ here, so that $f^{[1]}$ is identically 0.

We now give the main result of this section. No detailed proof is required, since once the Faá di Bruno formula is established the calculations are identical to those from [13] and [10] (see also [9]). Note that we do not assume the naturality of the algebras here.

**Theorem 5.4.** *Let $X, Y$ be semi-rectifiable, compact plane sets and let $n \in \mathbb{N}$. Let $\varphi \in D_{\mathcal{F}}^{(\infty)}(X)$ such that $\varphi(X) \subseteq Y$. Suppose that $\varphi$ is $\mathcal{F}$-analytic. Let $\mathcal{F}$ be an effective collection of paths on $X$ and let $\mathcal{G}$ be an effective collection of paths on $Y$ such that $\varphi$ is $\mathcal{F}$-$\mathcal{G}$-compatible. Let $M = (M_n)_{n=0}^{\infty}$ be a non-analytic algebra sequence.*

(a) *If $|\varphi^{[1]}|_X < 1$ then $\varphi$ induces a homomorphism from the algebra $D_{\mathcal{G}}(Y, M)$ into $D_{\mathcal{F}}(X, M)$.*
(b) *If the sequence $(n^2 M_{n-1}/M_n)$ is bounded and $|\varphi^{[1]}|_X \leq 1$ then $\varphi$ induces a homomorphism from $D_{\mathcal{G}}(Y, M)$ into $D_{\mathcal{F}}(X, M)$.*

Note that, in the above, $D_{\mathcal{F}}(X, M)$ and $D_{\mathcal{G}}(Y, M)$ are always Banach function algebras, so we do not need to make any additional completeness assumptions.

When $(n^2 M_{n-1}/M_n)$ is unbounded, the condition that $|f^{[1]}|_X \leq 1$ may no longer be sufficient for $\varphi$ to induce a homomorphism from $D_{\mathcal{G}}(Y, M)$ into $D_{\mathcal{F}}(X, M)$. The following example is from [11].

**Example 5.5.** Let $I = [0, 1]$ and let $\mathcal{F}$ be the collection of all admissible paths in $I$. For each $n \in \mathbb{N}$, let $M_n = (n!)^{3/2}$ and let $M = (M_n)_{n=0}^{\infty}$. Then $M$ is a non-analytic algebra sequence such that $n^2 M_{n-1}/M_n \to \infty$ as $n \to \infty$. Moreover, since $I$ is uniformly regular, we have $D_{\mathcal{F}}(I, M) = D(I, M)$, and $D_{\mathcal{F}}(I, M)$ is natural on $I$ by [5, Theorem 4.4.16]. Let $\varphi(t) := (1 + t^2)/2$ $(t \in I)$. Then $\varphi$ is an $\mathcal{F}$-analytic map from $I$ into $I$, $|\varphi'|_X \leq 1$, and $\varphi$ is $\mathcal{F}$-$\mathcal{F}$-compatible. However, by [9, Theorem 3.2], $\varphi$ does not induce a homomorphism from $D_{\mathcal{F}}(I, M)$ into $D_{\mathcal{F}}(I, M)$.

## 6. Open problems

We conclude with some open problems related to the content of this paper.

**Question 6.1.** *Can the assumption that $\varphi$ be $\mathcal{F}$-analytic in Theorem* 5.4 *be weakened or removed altogether*?

We next ask two questions about maximal collections of paths.

**Question 6.2.** *Let $X$ be a semi-rectifiable compact plane set, and let $\mathcal{F}$ be the collection of all Jordan paths in $X$. Is it necessarily true that $\mathfrak{m}(\mathcal{F})$ is the collection of all admissible paths in $X$*?

**Question 6.3.** *Let $\Gamma$ be an admissible path in $\mathbb{C}$, and set $X = \Gamma^*$. Let $\mathcal{F}$ be the collection of all subpaths of $\Gamma$. Does $\mathfrak{m}(\mathcal{F})$ necessarily include all Jordan paths in $X$?*

Let $X$ be a semi-rectifiable compact plane set. Our final questions concern the density of rational functions and differentiable functions in the algebras $D^{(1)}(X)$ and $D^{(1)}_{\mathcal{F}}(X)$, respectively. Some partial results were obtained in [3] and [7].

**Question 6.4.** *Is the set of all rational functions with no poles on $X$ always dense in $D^{(1)}(X)$?*

**Question 6.5.** *Is $D^{(1)}(X)$ always dense in $D^{(1)}_{\mathcal{F}}(X)$ when $\mathcal{F}$ is the collection of all admissible paths in $X$?*

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# A LEISURELY ELEMENTARY TREATMENT OF STIRLING'S FORMULA

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Abstract. A self-contained account of Stirling's formula for $n!$ is presented that is based on definitions of the constants $e$ and $\pi$ that appear in it, and uses only the rudiments of the analysis of the convergence of numerical sequences, infinite series, and infinite products.

## 1. Introduction

Stirling's formula tells us that $n!$ is asymptotically equal to $n^n e^{-n}\sqrt{2\pi n}$, implying that

$$\lim_{n\to\infty} \frac{n!}{n^{n+\frac{1}{2}}e^{-n}} = \sqrt{2\pi}. \tag{1}$$

This formula is remarkable because it provides an approximation of $n!$ that consists of a non-integral expression involving the irrational numbers $e$ and $\pi$. In this respect, to somebody seeing it for the first time, it must be just as amazing as Euler's identity linking the numbers $-1, i, e$ and $\pi$ *via* the equation $e^{i\pi} = -1$.

Over the years, many proofs of equation 1 have been published, and the literature is replete with ones of many different kinds. A sample of these can be found in [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. With one exception—and the present offering is no different in this regard—these articles contain *ad-hoc* proofs of equation 1. Reference [6] is the exception: it is Walter Hayman's most cited article, and his proof illustrates a powerful method for asymptotically estimating the coefficients of a class of power series of analytic functions, termed *admissible* by him.

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In this note we outline a derivation of equation 1 that could be taught at the end of a first course of analysis that includes a thorough treatment of the theory of convergence of sequences, series and infinite products, and the basic properties of the exponential and trigonometrical functions. It eschews integration completely.

We begin by establishing that there are positive constants $a, b$ such that

$$an^{n+\frac{1}{2}}e^{-n} \leq n! \leq bn^{n+\frac{1}{2}}e^{-n},\ n = 1, 2, \dots . \tag{2}$$

This is very often all that is needed for many applications. Next, we show that the positive sequence

$$S_n = \frac{n!}{n^{n+\frac{1}{2}}e^{-n}},\ n = 1, 2, \dots$$

is decreasing, and so convergent; thereafter, we concentrate on showing that $\sqrt{2\pi}$ is its limit. This is the heart of the matter, and, to deal with it, we establish Wallis's product formula for $\pi$ from first principles. Along the way we encounter the central binomial coefficients and derive an asymptotic expression for them; whence, as an application, we evaluate $\int_0^\infty e^{-s^2}\, ds$.

## 2. Proof of statement 2

A rigorous course on sequences and series would surely include a definition of the number $e$, and proofs that the sequences

$$u_n = (1 + 1/n)^n,\ v_n = (1 - 1/n)^n,\ n = 1, 2, \dots$$

are both strictly increasing, and converge to $e$ and $e^{-1}$, respectively, and hence that

$$(1 + \frac{1}{n})^n < e < (1 + \frac{1}{n})^{n+1},\ n = 1, 2, \dots$$

By considering the Geometric Means of the two lists of positive numbers $u_1, u_2, \dots, u_n$, and $v_2, v_3, \dots, v_n$, and utilising the last pair of displayed inequalities, it can be confirmed that the following crude estimates hold for $n!$, viz.,

$$(n + 1)^n e^{-n} < n! < en^{n+1}e^{-n},\ n = 1, 2, \dots,$$

which go someway towards explaining the presence of $e$ in equation 1. In particular, $\frac{1}{\sqrt{n}} < S_n < e\sqrt{n},\ n = 1, 2, \dots$.

To obtain a sharper lower bound for $S_n$, we use the power series expansion of the exponential function.

**Lemma 2.1.**

$$S_n > e^{-1}, \ n = 1, 2, \ldots.$$

*Proof.* Fix $n \geq 1$. Using the power series expansion for $\exp x$ with $x = n$, we have that, for any natural number $m$,

$$e^n > \sum_{k=n}^{n+m} \frac{n^k}{k!} = \frac{n^n}{n!} \left( 1 + \sum_{k=1}^{m} \frac{n^k}{\prod_{j=1}^{k}(n+j)} \right).$$

Now, if $1 \leq k \leq m$,

$$\begin{aligned} \frac{n^k}{\prod_{j=1}^{k}(n+j)} &= \frac{1}{\prod_{j=1}^{k}(1+\frac{j}{n})} \\ &\geq \prod_{j=1}^{k} \exp(-\frac{j}{n}) \\ &= \exp(-\frac{k(k+1)}{2n}) \\ &\geq \exp(-\frac{m(m+1)}{2n}). \end{aligned}$$

Hence, for any integer $m \geq 1$,

$$\frac{e^n \, n!}{n^n} > (m+1) \exp(-\frac{m(m+1)}{2n}).$$

The stated result now follows from this upon setting $m = \lfloor \sqrt{n} \rfloor$. ☐

**Theorem 2.2.** *The sequence $(S_n)$ is strictly monotonic decreasing and converges to a positive number.*

*Proof.* Evidently, $(S_n)$ is strictly monotonic decreasing if and only if

$$e < (1 + \frac{1}{n})^{n+\frac{1}{2}}, \ n = 1, 2, \ldots. \tag{3}$$

But, $2^{n-1} \leq n!, \ n = 1, 2, \ldots$, and so, if $0 \leq x < 1$, then

$$e^{2x} = 1 + 2\sum_{n=1}^{\infty} \frac{2^{n-1}}{n!} x^n \leq 1 + 2\sum_{n=1}^{\infty} x^n = \frac{1+x}{1-x},$$

with equality if and only if $x = 0$. Hence, inequality 3 follows on setting $x = 1/(2n+1)$. Thus, $(S_n)$ is monotonic decreasing. Since it is bounded below by $e^{-1}$, it converges to a positive number as claimed. ☐

**Corollary 2.3.**

$$e^{-1}n^{n+\frac{1}{2}}e^{-n} \leq n! \leq en^{n+\frac{1}{2}}e^{-n},\ n = 1, 2, \ldots. \tag{4}$$

*Proof.* This is an immediate consequence of Lemma 2.1, and the fact that $S_n \leq S_1 = e,\ n = 1, 2, \ldots$. □

This result implies statement 2, and suffices in many situations where one is interested in crude but tight estimates for sequences involving the factorial function. For instance, one can use it to obtain good bounds for the sequence of central binomial coefficients $\binom{2n}{n}$, $n = 1, 2, \ldots$. Indeed, it easily follows from statement 4 that

$$e^{-3}\sqrt{2}\,4^n n^{-\frac{1}{2}} \leq \binom{2n}{n} \leq e^{3}\sqrt{2}\,4^n n^{-\frac{1}{2}},\ n = 1, 2, \ldots.$$

In passing, we note that the sequence $(C_n)$ defined by

$$C_n = \frac{\sqrt{n}\binom{2n}{n}}{4^n},\ n = 1, 2, \ldots$$

is strictly increasing, and so converges to a positive number, which we denote by $C$; we'll determine its value in Section 4.

## 3. The limit of the sequence $(C_nS_n)$

It follows from Theorem 2.2 that the limit $S \equiv \lim_{n\to\infty} S_n$ exists, and $S \geq e^{-1}$. We proceed to identify the product $CS$.

**Theorem 3.1.** $CS = \sqrt{2}$.

*Proof.* To see this, note that

$$\begin{aligned} S_{2n} &= \frac{(2n)!}{(2n)^{2n}e^{-2n}\sqrt{2n}} \\ &= \frac{(2n)!\sqrt{n}}{(n!)^2 2^{2n+1/2}}\left(\frac{n!}{n^n e^{-n}\sqrt{n}}\right)^2 \\ &= \frac{C_n S_n^2}{\sqrt{2}}. \end{aligned}$$

Hence, by the product and subsequence rules for limits of sequences, $S = \frac{CS^2}{\sqrt{2}}$, and so $CS = \sqrt{2}$, as claimed. □

Thus, once we know $C$, we know $S$, and *vice versa.*

## 4. The value of $C$

Starting with the definition of $\pi$ as the smallest positive root of the sine function, in this section we determine $C$ by relating it to the value of the Wallis product.

### 4.1. **The Wallis product.**

Since, for every positive integer $n$,

$$\begin{aligned} C_n^2 &= \left(\frac{(2n)!\sqrt{n}}{(n!)^2 2^{2n}}\right)^2 \\ &= \frac{n}{2n+1}\frac{((2n)!)^2\,(2n+1)}{2^{4n}(n!)^4} \\ &= \frac{n}{2n+1}\prod_{k=1}^{n}(1-\frac{1}{4k^2}), \end{aligned}$$

we have that

$$2C^2 = \prod_{k=1}^{\infty}(1-\frac{1}{4k^2}).$$

We proceed to relate the value of this infinite product [1] to the limit of another sequence, which comes from a factorisation of $\sin(2n+1)x$. To obtain this, fix the positive integer $n$, and let $m = 2n+1$, for convenience. Let $\omega = e^{2\pi i/m}$. Then, for any complex number $z$,

$$\begin{aligned} z^{2m}-1 &= \prod_{r=1}^{m}(z^2-\omega^r) \\ &= (z^2-1)\prod_{r=1}^{n}(z^2-\omega^r)\prod_{r=n+1}^{2n}(z^2-\omega^r) \\ &= (z^2-1)\prod_{r=1}^{n}(z^2-\omega^r)(z^2-\bar{\omega}^r) \\ &= (z^2-1)\prod_{r=1}^{n}(z^4-2z^2\Re\omega^r+1). \end{aligned}$$

[1] Of course, if, at this point, we accept as known that this is $2/\pi$, we can delete the rest of this section. However, rather than do so, we prefer to prove this result *ab initio*.

Hence, letting $z = e^{ix}$, where $x$ is any real number, it follows that

$$\begin{aligned} \sin(mx) &= 2^n \sin x \prod_{r=1}^{n} (\cos(2x) - \cos(2r\pi/m)) \\ &= 2^{2n} \sin x \prod_{r=1}^{n} (\sin^2(r\pi/m) - \sin^2 x). \end{aligned}$$

In particular, we conclude from this that, first of all,

$$m = 2^{2n} \prod_{r=1}^{n} \sin^2(r\pi/m),$$

and thence that

$$\sin(mx) = m \sin x \prod_{r=1}^{n} \left(1 - \frac{\sin^2 x}{\sin^2(r\pi/m)}\right).$$

Choose $x = \pi/2m$, and consider the limiting behaviour of

$$\frac{1}{m \sin(\pi/2m)} = \prod_{r=1}^{n} \left(1 - \frac{\sin^2(\pi/2m)}{\sin^2(r\pi/m)}\right)$$

as $n \to \infty$. Clearly, the sequence on the left side converges to $2/\pi$. To handle the sequence on the right side, we need a variant of Tannery's theorem [1, 3], tailored to the situation in hand. Since this may not be familiar to many people, we include a proof for completeness.

**Lemma 4.1.** *Suppose $a_m(n)$, $m, n = 1, 2, \ldots$, is a double sequence of complex numbers, with the following properties:*

(1) *there is a sequence of positive numbers $m_r$, $r = 1, 2, \ldots$, such that $\prod_{r=1}^{\infty}(1 + m_r)$ is convergent, and*

$$\sup_{n \geq 1} |a_r(n)| \leq m_r, \; r = 1, 2, \ldots;$$

(2) *for each $m$, the sequence $a_m(n)$, $n = 1, 2, \ldots$, converges to $a_m$;*

(3) *the sequence $P_n$, defined by*

$$P_n = \prod_{m=1}^{n} (1 + a_m(n)), \; n = 1, 2, \ldots,$$

*converges to $P$, say.*

*Then*

$$P = \prod_{m=1}^{\infty}(1+a_m).$$

*Proof.* Fix $m$ and let $n > m$. Then

$$\begin{aligned}\left|P_n - \prod_{r=1}^{m}(1+a_r(n))\right| &= \left|\prod_{r=1}^{m}(1+a_r(n))\right|\left|\prod_{r=m+1}^{n}(1+a_r(n))-1\right|\\ &\le \prod_{r=1}^{m}(1+m_r)\left(\prod_{r=m+1}^{n}(1+m_r)-1\right)\\ &= Q_n - Q_m,\end{aligned}$$

where

$$Q_n = \prod_{r=1}^{n}(1+m_r),\ n = 1, 2, \ldots$$

Under our assumptions, the sequence $Q_n$ converges to the infinite product

$$Q = \prod_{m=1}^{\infty}(1+m_r),$$

while $\lim_{n\to\infty} a_m(n) = a_m$, and $\lim_{n\to\infty} P_n = P$. Hence, letting $n \to \infty$, we deduce from the inequality just established that

$$|P - \prod_{r=1}^{m}(1+a_r)| \le Q - Q_m,\ m = 1, 2, \ldots,$$

whence the result follows. □

We apply Lemma 4.1 with

$$a_r(n) = -\frac{\sin^2(\pi/2m)}{\sin^2(r\pi/m)},\ r = 1, 2, \ldots, n,\ n = 1, 2, \ldots.$$

Since $2x/\pi \le \sin x \le x$, if $0 \le x \le \pi/2$, it follows that

$$\sup\{|a_r(n)| : n = 1, 2, \ldots\} \le \frac{\pi^2}{16r^2},\ r = 1, 2, \ldots.$$

Also

$$\lim_{n\to\infty} a_r(n) = -\frac{1}{4r^2}.$$

Hence

$$\lim_{n\to\infty}\prod_{r=1}^{n}\left(1-\frac{\sin^2(\pi/2m)}{\sin^2(r\pi/m)}\right) = \prod_{r=1}^{\infty}(1-\frac{1}{4r^2}).$$

Combining our results we may conclude that

$$2C^2 = \prod_{r=1}^{\infty}(1 - \frac{1}{4r^2}) = \frac{2}{\pi},$$

and so $C = 1/\sqrt{\pi}$. It follows that $S = \sqrt{2\pi}$.

## 5. The value of the integral $\int_0^\infty e^{-s^2}\, ds$

As a quick look at the references reveals, many proofs of equation 1 rely on knowing the value of this integral. Here, we reverse matters and essentially derive its value from equation 1, by identifying it directly with $C\pi/2$. Conversely, of course, making this identification, we can dispense entirely with the previous section and obtain equation 1 very quickly.

Since $\binom{2n}{n}$ is the constant term in the trigonometrical polynomial $(e^{ix} + e^{-ix})^{2n}$, it's clear that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^{2n} x\, dx = 2^{-2n}\binom{2n}{n},\ \ n = 0, 1, \ldots.$$

Hence, for $n = 1, 2, \ldots,$

$$\begin{aligned} C_n &= \frac{\sqrt{n}\binom{2n}{n}}{2^{2n}} \\ &= \frac{\sqrt{n}}{2\pi}\int_{-\pi}^{\pi} \cos^{2n} x\, dx \\ &= \frac{2\sqrt{n}}{\pi}\int_0^{\pi/2} (1 - \sin^2 x)^n\, dx \\ &= \frac{2\sqrt{n}}{\pi}\int_0^1 (1 - t^2)^{n-\frac{1}{2}}\, dt \\ &= \frac{2}{\pi}\int_0^{\sqrt{n}} (1 - \frac{s^2}{n})^{n-\frac{1}{2}}\, ds. \end{aligned}$$

But, if $0 \leq s \leq \sqrt{n}$, then

$$(1 - \frac{s^2}{n})^{n-\frac{1}{2}} \leq e^{-(n-\frac{1}{2})s^2/n} = e^{-s^2}e^{s^2/2n} \leq e^{-s^2}\sqrt{e}.$$

Furthermore,

$$\lim_{n\to\infty}(1 - \frac{s^2}{n})^{n-\frac{1}{2}} = e^{-s^2}.$$

Hence, by the variant of Tannery's theorem for integrals, or by Lebesgue's dominated convergence theorem,

$$C = \lim_{n\to\infty} C_n = \frac{2}{\pi}\int_0^\infty e^{-s^2}\,ds.$$

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# A PROBLEM OF DIOPHANTUS MODULO A PRIME

JOSHUA HARRINGTON AND LENNY JONES

Abstract. A set $S$ of $k \geq 2$ positive rational numbers is called a *rational Diophantine $k$-tuple* if the product of any two elements of $S$ increased by 1 is a perfect square. Around the third century A.D., Diophantus found infinitely many rational Diophantine triples, and very recently, Dujella, Kazalicki, Mikić and Sziksza have proven the existence of infinitely many rational Diophantine sextuples. It is still unknown whether there exist any rational Diophantine septuples. In this note, we investigate this problem in $\mathbb{Z}_p$, the field of integers modulo a prime $p$, where the situation is quite different. We show, given any set $S$ of $k \geq 2$ positive integers, that there exist infinitely many primes $p$ such that all elements of $S$ are nonzero squares modulo $p$, and furthermore, that the product of any $t$ elements of $S$, where $1 \leq t \leq k$, increased by 1 is also a nonzero square modulo $p$.

## 1. Introduction

In Problem 19 of Book IV of the Arithmetica [7], Diophantus asked:

> *To find three numbers indeterminately such that the product of any two increased by 1 is a square.*

He provided the solution

$$\{x, \quad x+2, \quad 4x+4\}, \tag{1}$$

and he outlined a procedure using (1) to construct 4-element sets of positive rational numbers such that the product of any two increased by 1 is a square. As an example, he gave

$$\left\{\frac{1}{16}, \quad \frac{33}{16}, \quad \frac{17}{4}, \quad \frac{105}{16}\right\}.$$

In the literature, $k$-element sets $S$ of positive rational numbers, such that the product of any two increased by 1 is a square, are generally

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referred to as *rational Diophantine $k$-tuples*, or simply *Diophantine $k$-tuples* if all the elements of $S$ are integers. Since the time of Diophantus, many mathematicians have searched for rational Diophantine $k$-tuples and Diophantine $k$-tuples, where $k \geq 5$. Very recently, Dujella, Kazalicki, Mikić and Sziksza [4] have proven that there exist infinitely many rational Diophantine sextuples, but it is still unknown whether any rational Diophantine septuples exist. If a certain conjecture of Lang [1] is true, then there exists an upper bound on $k$ for the existence of a rational Diophantine $k$-tuple. Along these lines, Dujella [3] has shown unconditionally that no Diophantine sextuple exists, and that there are at most finitely many Diophantine quintuples. Although it is still unknown as to whether a single Diophantine quintuple exists, more recent work [5, 6, 10] suggests that the answer is most likely negative. In this note, we investigate this problem in the finite field of integers modulo a prime $p$, which we denote as $\mathbb{Z}_p$. We see that the situation is quite different in this setting. More precisely, we prove the following theorem.

**Theorem 1.1.** *Let $k \geq 2$ be a fixed integer, and let $S$ be any set of $k$ positive integers. Then there exist infinitely many primes $p$ for which each element of $S$ is a nonzero square modulo $p$, and furthermore, that 1 plus the product of any $t$ elements of $S$, where $1 \leq t \leq k$, is also a nonzero square modulo $p$.*

Throughout this note, we say that a set $S$ *has property $\mathcal{D}$* if $S$ satisfies the conditions in the statement of Theorem 1.1.

## 2. Preliminary Material

To help establish Theorem 1.1, we recall some ideas from number theory. A *quadratic residue* modulo the prime $p$ is a nonzero element $a \in \mathbb{Z}_p$ such that there exists $x \in \mathbb{Z}_p$ with $x^2 \equiv a \pmod{p}$. In other words, a quadratic residue is a nonzero square in $\mathbb{Z}_p$. For any integer $a$ and any prime $p$, we define the *Legendre symbol* as

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo the prime } p \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo the prime } p \\ 0 & \text{if } a \equiv 0 \pmod{p}. \end{cases} \tag{2}$$

It is easy to see from (2) that

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right), \tag{3}$$

for any integers $a$ and $b$. Although there are many celebrated theorems concerning the Legendre symbol, we require only two of the more well-known results, which we state without proof.

**Proposition 2.1.** [8] *Let $p$ be an odd prime. Then*

$$\left(\frac{2}{p}\right) = 1 \quad \textit{if and only if} \quad p \equiv \pm 1 \pmod 8.$$

The next remarkable theorem is due to Gauss [8].

**Theorem 2.2** (Law of Quadratic Reciprocity)**.** *Let $p$ and $q$ be odd primes. Then*

$$\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)(-1)^{(p-1)(q-1)/4}.$$

Finally, we need a theorem due to Dirichlet [8].

**Theorem 2.3** (Dirichlet's Theorem of Primes in an Arithmetic Progression)**.** *Let $r$ and $m$ be positive integers with* $\gcd(r, m) = 1$*. Then there exist infinitely many positive integers $n$ such that $mn + r$ is prime.*

**Remark 2.4.** In other words, according to Theorem 2.3, if $\gcd(r, m) = 1$, then there are infinitely many primes $p$ such that $p \equiv r \pmod m$.

## 3. Proof of Theorem 1.1

We are now in a position to give a proof of Theorem 1.1.

*Proof of Theorem 1.1.* Let $L$ be the largest element of $S$, and let $m = 8\prod_{q_i \in Q} q_i$, where $Q$ is the set of all odd primes $q_i \leq L! + 1$. By Theorem 2.3, there exist infinitely many primes $p \equiv 1 \pmod m$. Since $p \equiv 1 \pmod 8$, it follows from Proposition 2.1 and Theorem 2.2 that $\left(\frac{q_i}{p}\right) = 1$ for each $q_i \in Q$. Let $t$ be any integer with $1 \leq t \leq k$, and let $s$ be the product of any $t$ elements from $S$. Since $s < L! + 1$, all prime factors of $s$ and $s + 1$ are contained in $Q$. In particular, no element of $S$ is divisible by $p$. Hence, by (3), it follows that $\left(\frac{s}{p}\right) = \left(\frac{s+1}{p}\right) = 1$. Therefore, the set $S$ has property $\mathcal{D}$, and the proof is complete. $\square$

## 4. An Example and Some Open Questions

As an illustration of Theorem 1.1, we provide the following example.

**Example 4.1.** Let $k = 4$. Recall the sequence of Fibonacci numbers $(F_n)$ defined as

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2.$$

Let $S = \{F_2, F_3, F_4, F_5\} = \{1, 2, 3, 5\}$. According to the method described in the proof of Theorem 1.1, we get in this situation that

$$Q = \{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113\},$$

$$m = 126440218561670431152580825166174649973098747960,$$

and the smallest prime $p \equiv 1 \pmod{m}$ is

$$p = 1011521748493363449220646601329397199784789983681.$$

Then $S$ has property $\mathcal{D}$ since all elements of the set

$$\{1, 2, \ldots, 121 = F_5! + 1\}$$

are quadratic residues modulo $p$.

**Remark 4.2.** Call a Diophantine $t$-tuple $\mathcal{T}$ a *Fibonacci Diophantine $t$-tuple* if all elements of $\mathcal{T}$ are Fibonacci numbers. It is easy to show that the only Fibonacci Diophantine triple $\mathcal{T}$ containing $F_1 = F_2 = 1$ is $\mathcal{T} = \{1, 3, 8\}$ [9]. If the smallest index in $\mathcal{T}$ is larger than 1, then it is conjectured that all such Fibonacci Diophantine triples are of the form $\{F_{2k}, F_{2k+2}, F_{2k+4}\}$. Dujella [2] has shown that if $\{F_{2k}, F_{2k+2}, F_{2k+4}, d\}$ is a Diophantine 4-tuple, where $k \geq 1$, then $d$ cannot be a Fibonacci number. This fact, combined with the truth of the aforementioned conjecture, would establish that no Fibonacci Diophantine 4-tuples exist.

To conclude, we raise a couple of questions.

**Question 1.** *For each $k \geq 2$, what is the smallest prime $p$ such that a set $S$ with exactly $k$ elements exists with property $\mathcal{D}$?*

The answer to Question 1 for $k = 4$ is $p = 41$, which is achieved using $S = \{1, 4, 8, 9\}$.

**Question 2.** *For each $k \geq 2$, does there exist a natural number $N_k$ such that for all primes $p > N_k$, there exists a set $S$ with $k$ elements that has property $\mathcal{D}$?*

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# On Yamabe Soliton

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Abstract. The purpose of the paper is to prove that if the metric of a 3-dimensional $\alpha$-Sasakian structure on a complete Riemannian manifold is a Yamabe soliton then it is of constant curvature. We also derive some properties of the flow vector field $U$ of the Yamabe soliton together with an example of an $\alpha$–Sasakian manifold admitting Yamabe soliton.

## 1. INTRODUCTION

A well known question in differential geometry is whether a given compact connected Riemannian manifold is necessarily conformally equivalent to one having constant scalar curvature. This problem was formulated by Yamabe in 1960[13] and is known as the Yamabe problem. Yamabe gave a purported proof of the affirmative answer, but Trudinger in 1968[14] found an error, and then was able to correct the proof of Yamabe for the case when the scalar curvature is non-positive. Aubin improved Trudinger's result but the remaining cases were solved by Schoen using positive mass theorem.

Another motivation for considering the Yamabe problem is conformal geometry itself. Riemannian differential geometry attempted to generalize the highly successful theory of compact surfaces. From the earliest days, conformal changes of metric played an important role in surface theory. For instance, it is a consequence of the uniformization theorem of complex analysis that one can find a conformal change of metric which makes the scalar curvature constant. This led to the Yamabe problem.

Several years ago, the notion of Yamabe flow was introduced by Richard Hamilton (see [5],[6]) as a tool for constructing metrics of constant scalar curvature in a given conformal class of Riemannian

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metrics on $(M^n, g)(n \geq 3)$. It can also be explained as negative $L^2$-gradient flow of the (normalized) total scalar curvature, restricted to a given conformal class it can be interpreted as deforming a Riemannian metric to a conformal metric of constant scalar curvature, when this flow converges. Complete shrinking gradient Yamabe solitons under the assumptions of suitable scalar curvature (resp. Ricci tensor) have finite topological type. On a smooth Riemannian manifold, Yamabe flow can be defined as the evolution of the Riemannian metric $g_0$ in time $t$ to $g = g(t)$ by means of the equation

$$\frac{\partial}{\partial t} g = -rg, \quad g(0) = g_0,$$

where $r$ denotes the scalar curvature corresponds to $g$. Yamabe soliton can be defined on a Riemannian manifold by a vector field $U$(known as Flow Vector field) satisfying:

$$\pounds_U g = (c - r)g, \tag{1.1}$$

where $\pounds_U$ denotes the Lie-derivative operator along the direction of $U$ and the constant $c = -\dot{\sigma}(g_0)$ (see Chow et al. [3]). Similar to Ricci Soliton, Yamabe Soliton can be considered as a special solution of the Yamabe flow. In Mathematical Physics, Yamabe flow relates to the fast diffusion case of the plasma equation. Recently, Sharma (refer to [11]) have studied Yamabe soliton on 3-dimensional Sasakian metric on a complete manifold. Since $\alpha$-Sasakian is a generalization of Sasakian manifold, we are interested to study 3-dimensional $\alpha$-Sasakian manifold when its metric is a Yamabe soliton. We deduce some properties of the flow vector field $U$ of the Yamabe soliton.

## 2. PRELIMINARIES

An odd-dimensional differentiable manifold $(M^n, g)$ may admit an almost contact metric structure $(\Phi, \xi, \eta, g)$ consisting of a Reeb vector field $\xi$, a (1,1)-tensor field $\Phi$ and a Riemannian metric $g$ satisfying

$$\Phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \forall\, X, Y \in \chi(M), \tag{2.2}$$

where $\chi(M)$ represents the collection of all smooth vector fields on $M$.

Moreover, if the relation

$$d\eta(X, Y) = g(\Phi X, Y),$$

holds for arbitrary smooth vector fields $X$ and $Y$, then we call such a structure a contact metric structure and the manifold with that structure is said to be contact metric manifold. As a consequence of this, the following relations hold:

$$\Phi\xi = 0, \quad \eta \circ \Phi = 0, \quad d\eta(\xi, X) = 0, \quad g(\Phi X, Y) = -g(X, \Phi Y), \tag{2.3}$$

$\forall\, X \in \chi(M)$. For details we refer to Blair [1].
An almost contact structure on $M$ is said to be an $\alpha$-Sasakian manifold, $\alpha$ being a non-zero constant, if

$$(\nabla_X \Phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X), \ \forall\, X, Y \in \chi(M) \tag{2.4}$$

holds. As a consequence, it follows that:

$$\nabla_X \xi = -\alpha\Phi X, \tag{2.5}$$

$$(\nabla_X \eta)Y = -\alpha g(\Phi X, Y), \ \forall\, X, Y \in \chi(M). \tag{2.6}$$

If $\alpha = 1$, then the $\alpha$-Sasakian structure reduces to Sasakian manifold, thus $\alpha$-Sasakian structure may be considered as a generalization of Sasakian one. In other words, Sasakian manifold is a particular case of $\alpha$-Sasakian manifold. Also in a 3-dimensional $\alpha$-Sasakian manifold the following relations are true:

$$R(X, Y)\xi = \alpha^2\{\eta(Y)X - \eta(X)Y\}, \tag{2.7}$$

$$S(X, \xi) = 2\alpha^2\eta(X), \tag{2.8}$$

$$Q\xi = 2\alpha^2\xi, \qquad \forall X, Y \in \chi(M), \tag{2.9}$$

where $R$ is the Riemannian curvature tensor and $Q$ is the Ricci operator associated with the $(0, 2)$ Ricci tensor $S$. For details we refer to [7].

**Definition 2.1.** *([1])In an almost contact Riemannian manifold, if an infinitesimal transformation $U$ satisfies*

$$(\pounds_U \eta)(X) = \sigma\eta(X), \quad \forall\, X \in \chi(M) \tag{2.10}$$

*for a scalar function $\sigma$, then we call it an infinitesimal contact transformation. If $\sigma$ vanishes identically, then it is called an infinitesimal strict transformation.*

**Definition 2.2.** *A vector field $U$ in an $n$-dimensional Riemannian manifold $(M, g)$ is said to be conformal if*

$$\pounds_U g = 2\nu g, \tag{2.11}$$

*for some smooth function* $\nu$ *on* $M$. *Moreover, a conformal vector field satisfies*

$$(\pounds_U S)(X,Y) = -(n-2)g(\nabla_X D\nu, Y) + (\Delta\nu)g(X,Y) \tag{2.12}$$

$$\pounds_U r = -2\nu r + 2(n-1)\Delta\nu, \tag{2.13}$$

*where* $D$ *is the gradient operator and* $\Delta = -div.D$ *is the Laplacian operator of* $g$. *For details, we refer to Yano* [12].

Before proceeding for the main results let us state the following lemmas:

**Lemma 2.1.** *In an* $\alpha$*-Sasakian manifold, the following relations are valid:*

**(i):** $\eta(\pounds_U \xi) = \frac{r-c}{2}$,
**(ii):** $(\pounds_U \eta)(\xi) = \frac{c-r}{2}$.
**(iii):** $\nu = \frac{c-r}{2}$.

*Proof.* The proof (i) and (ii) readily follows from the definition of the Yamabe soliton on a Riemannian manifold. Since $U$ is a conformal vector field, therefore using (2.11) and (i), we have the desired result (iii). □

**Lemma 2.2.** *In an* $\alpha$*-Sasakian 3-Metric, the Ricci tensor* $S$ *is given by*

$$S = \left(\frac{r}{2} - \alpha^2\right) g + \left(3\alpha^2 - \frac{r}{2}\right) \eta \otimes \eta. \tag{2.14}$$

*Proof.* We recall that the Riemannian curvature tensor in a 3-dimensional Riemannian manifold is given by

$$\begin{aligned} R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{2.15}$$

where $r$ is the Ricci scalar curvature and $X, Y, Z \in \chi(M)$.
Replacing $Z$ with $\xi$ in (3.1) and recalling (2.8), yields

$$\eta(Y)QX - \eta(X)QY + \left(\alpha^2 - \frac{r}{2}\right)\{\eta(Y)X - \eta(X)Y\} = 0.$$

Again, replacing $Y$ with $\xi$ and thereby using (2.9), we get the desired result. □

## 3. MAIN RESULTS

Putting the value of $\nu$ in (2.12) and (2.13), one obtains

$$(\pounds_U S)(X,Y) = \frac{1}{2}\big[g(\nabla_X Dr, Y) - (\Delta r)g(X,Y)\big], \tag{3.1}$$

and

$$\pounds_U r = -2\Delta r - r(c-r). \tag{3.2}$$

Since $g$ is an Yamabe Soliton, taking the Lie-derivative of (2.14) in the direction of $U$ and using (3.1) and (3.2), yields

$$\begin{aligned} g(\nabla_X Dr, Y) = &-\big[\Delta r + 2\alpha^2(c-r)\big]g(X,Y) \\ &+ \big[2\Delta r + r(c-r)\big]\eta(X)\eta(Y) \\ &+ (6\alpha^2 - r)\big[(\pounds_U \eta)(X)\eta(Y) + (\pounds_U \eta)(Y)\eta(X)\big]. \end{aligned} \tag{3.3}$$

Since $\xi$ is killing, therefore $\xi r = 0$. Differentiating covariantly along the direction of an arbitrary vector field $X$, one obtains $g(\nabla_X Dr, \xi) = (\alpha \Phi X)r$. Replacing $Y$ with $\xi$ in (3.3) and using the foregoing equation together with the lemma provides

$$\alpha(\Phi X)r = \left\{\Delta r + (c-r)\left(\alpha^2 + \frac{r}{2}\right)\right\}\eta(X) + (6\alpha^2 - r)(\pounds_U \eta)X. \tag{3.4}$$

Putting $X = \xi$, in above and using the lemma(ii) we obtain

$$\Delta r = -4\alpha^2(c-r). \tag{3.5}$$

From (3.4) and (3.5), we can deduce

$$(6\alpha^2 - r)(\pounds_U \eta)X = \alpha(\Phi X)r + \frac{1}{2}(r-c)(r-6\alpha^2)\eta(X). \tag{3.6}$$

Feeding (3.5) and (3.6) in (3.3), yields

$$\nabla_X Dr = 2\alpha^2(c-r)\big(X - \eta(X)\xi\big) - \alpha\big((\phi X)r\big)\xi - \alpha\eta(X)(\Phi Dr). \tag{3.7}$$

Differentiating covariantly along the direction of $Y$, we obtain

$$\begin{aligned}
\nabla_Y \nabla_X Dr = &\\
&- 2\alpha^2(Yr)\big(X - \eta(X)\xi\big)\\
&+ 2\alpha^2(c-r)\big\{\nabla_Y X + \alpha g(\Phi Y, X)\xi - \eta(\nabla_Y X)\xi + \alpha\eta(X)(\Phi Y)\big\}\\
&+ \alpha\big(\nabla_Y \nabla_{\Phi X} r\big)\xi\\
&- \alpha^2\big((\Phi X)r\big)\Phi Y\\
&- \alpha\eta(X)\big\{\alpha g(Y, Dr)\xi - \alpha\eta(X)\Phi Dr + \Phi(\nabla_Y Dr)\big\}\\
&- \alpha\big\{-\alpha g(\Phi Y, X) + \eta(\nabla_Y X)\big\}(\Phi Dr). \qquad (3.8)
\end{aligned}$$

Replacing $X$ and $Y$ in above, one obtains

$$\begin{aligned}
\nabla_X \nabla_Y Dr = &-2\alpha^2(Xr)\big(Y - \eta(Y)\xi\big)\\
&+ 2\alpha^2(c-r)\big\{\nabla_X Y + \alpha g(\Phi X, Y)\xi\\
&\quad - \eta(\nabla_X Y)\xi + \alpha\eta(Y)(\Phi X)\big\}\\
&+ \alpha\big(\nabla_X \nabla_{\Phi Y} r\big)\xi - \alpha^2\big((\Phi Y)r\big)\Phi X\\
&- \alpha\eta(Y)\big\{\alpha g(X, Dr)\xi - \alpha\eta(Y)\Phi Dr + \Phi(\nabla_X Dr)\big\}\\
&- \alpha\big\{-\alpha g(\Phi X, Y) + \eta(\nabla_X Y)\big\}(\Phi Dr). \qquad (3.9)
\end{aligned}$$

Again from (3.7), we have

$$\begin{aligned}
\nabla_{[Y,X]} Dr = &\, 2\alpha^2(c-r)\big\{\nabla_Y X - \nabla_X Y - \eta(\nabla_Y X)\xi + \eta(\nabla_X Y)\xi\big\}\\
&+ \alpha\big\{(\Phi\nabla_Y X)r - (\Phi\nabla_X Y)r\big\}\xi\\
&- \alpha\big\{\eta(\nabla_Y X) - \eta(\nabla_X Y)\big\}(\Phi Dr). \qquad (3.10)
\end{aligned}$$

Also, the Riemannian curvature tensor $R$ is given by,

$$R(X,Y)Dr = [\nabla_X, \nabla_Y]Dr - \nabla_{[X,Y]}Dr.$$

Feeding the equations (3.8), (3.9) and (3.10) in the foregoing formula, then we obtain on contracting the above over $X$ and recalling the skew-symmetric property of $\Phi$ together with $\xi r = 0$ and (3.7),

$$S(X, Dr) = -\alpha g(\Phi\nabla_{e_i} Dr, e_i)\eta(X). \qquad (3.11)$$

Combining the above with Lemma(2.1) and recalling Lemma(2.3), one obtains

$$(r - 2\alpha^2)Xr = 0.$$

which shows the scalar curvature $r$ is constant. Hence appealing to (3.5), yields $r = c$. Thus we can state:

**Theorem 3.1.** *If the metric of a 3-dimensional $\alpha$-Sasakian manifold is a Yamabe soliton, then it is of constant scalar curvature c.*

Hence (1.1) reduces to $\pounds_U g = 0$, i.e. $U$ is killing. Differentiating covariantly along an arbitrary vector field $X$, we have $\nabla_X \pounds_U g = 0$. The identity

$$(\nabla_X \pounds_U g)(Y, Z) = g((\pounds_U \nabla)(X, Y), Z) + g((\pounds_U \nabla)(X, Z), Y),$$

can be deduced from the formula [for details we refer to [12]],

$$(\pounds_U \nabla_X g - \nabla_X \pounds_U g - \nabla_{[U,X]} g)(Y, Z)$$
$$= -g((\pounds_U \nabla)(X, Y), Z) - g((\pounds_U \nabla)(X, Z), Y),$$

which implies,

$$g((\pounds_U \nabla)(Z, X), Y) + g((\pounds_U \nabla)(Z, Y), X) = 0.$$

By the combinatorial combination of the above together with the skew-symmetric property of $\phi$, yields

$$(\pounds_U \nabla)(Y, Z) = 0.$$

Taking $Y = Z = \xi$, we obtain $(\pounds_U \nabla)(\xi, \xi) = 0$. Hence, with the geodesic property of $\xi$, the following identity

$$(\pounds_U \nabla)(X, Y) = \nabla_X \nabla_Y U - \nabla_{\nabla_X Y} U + R(U, X)Y,$$

yields $R(U, \xi)\xi + \nabla_\xi \nabla_\xi U = 0$, which concludes that $U$ is Jacobi along the direction of $\xi$. Since $r$ is constant(= c), from (3.4), we can say that either $r = 6\alpha^2$ or $r \neq 6\alpha^2$. In the former case, from Lemma(2.2) we can conclude that $S = 2\alpha^2 g$ i.e. $M$ is an Einstein manifold and being of dimension 3 it is of constant curvature $\alpha^2$. Thus we can state as follows:

**Theorem 3.2.** *If the metric of a 3-dimensional $\alpha$-Sasakian manifold is a Yamabe soliton, then the flow vector field $U$ is killing and is Jacobi along the direction of $\xi$. In particular if $c = 6\alpha^2$, the manifold reduces to an Einstein manifold.*

If $c(= r) \neq 6\alpha^2$, then recalling equation (3.6) we conclude $\pounds_U \eta = 0$. Thus from the definition of an infinitesimal contact transformation, the scalar function $\sigma$ vanishes identically and hence we can state:

**Theorem 3.3.** *If the metric of a 3-dimensional $\alpha$-Sasakian manifold is a Yamabe soliton, then the infinitesimal contact transformation of the conformal vector field is strict.*

A brief computation on using the result of for a 3-dimensional $\alpha$-Sasakian manifold that the $\phi$-sectional curvature equals $\frac{1}{2}(r-4\alpha^2)$. By virtue of theorem (3.1), we get the following:

**Theorem 3.4.** *For a 3-dimensional $\alpha$-Sasakian manifold the $\phi$-sectional curvature(sectional curvature with respect to a plane orthogonal to $\xi$) is constant and equals to $-\alpha^2$.*

## 4. EXAMPLE OF AN $\alpha$-SASAKIAN 3-METRIC AS YAMABE SOLITON

Let us consider the 3-dimensional Riemannian manifold $M = \mathbb{R}^3$ with a rectangular
cartesian coordinate system $(x_i)$.
Let us choose the vector fields $\{E_1, E_2, E_3\}$ as

$$E_1 = \frac{\partial}{\partial x_1}, \quad E_2 = -2\alpha\frac{\partial}{\partial x_2}, \quad E_3 = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_3}, \tag{4.1}$$

$\alpha$ a non-zero constant. Thus, $\{E_1, E_2, E_3\}$ forms a basis of $\chi(M) = \chi(\mathbb{R}^3)$.
Let $g$ be the Riemannian metric on $\chi(\mathbb{R}^3)$ defined by

$$\begin{cases} g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1, \\ g(E_1, E_2) = g(E_1, E_3) = g(E_2, E_3) = 0. \end{cases} \tag{4.2}$$

Let $\xi = E_1$ be the vector field associated with the 1-form $\eta$. The (1,1)-tensor field $\phi$ be defined by,

$$\phi(E_1) = 0, \quad \phi(E_2) = -E_3, \quad \phi(E_3) = E_2. \tag{4.3}$$

Since, $\{E_1, E_2, E_3\}$ is a basis, any vector fields $X$ and $Y$ in $M$ can be uniquely expressed as

$$X = X^1E_1 + X^2E_2 + X^3E_3 \quad \text{and} \quad Y = Y^1E_1 + Y^2E_2 + Y^3E_3,$$

where $X^i, Y^i (i = 1, 2, 3)$ are smooth functions over $M$.
Now using the linearity of $\phi$ and $g$, and taking $\xi = E_1$ we have,

$$\eta(\xi) = 1, \quad \phi^2X = -X+\eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y)-\eta(X)\eta(Y),$$

for any vector fields $X$ and $Y$ in $M$. Thus $(\phi, \xi, \eta, g)$ defines an almost contact metric structure on $M$.
Let $\nabla$ be the Levi-Civita connection with respect to the Riemannian metric $g$. Then we have,

$$[E_2, E_3] = -2\alpha e_1, \quad [E_1, E_2] = 0, \quad [E_1, E_3] = 0.$$

By using Koszul's formulae (see [9]), we have

$$\begin{aligned} &\nabla_{E_1} E_3 = -\alpha E_2, && \nabla_{E_1} E_2 = \alpha E_3, && \nabla_{E_1} E_1 = 0,\\ &\nabla_{E_2} E_3 = -\alpha E_1, && \nabla_{E_2} E_1 = \alpha E_3, && \nabla_{E_2} E_2 = 0,\\ &\nabla_{E_3} E_1 = -\alpha E_2, && \nabla_{E_3} E_2 = \alpha E_1, && \nabla_{E_3} E_3 = 0. \end{aligned}$$

Also, the Riemannian curvature tensor $R$ is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Then,

$$\begin{aligned} &R(E_1,E_2)E_2 = \alpha^2 E_1, && R(E_1,E_3)E_3 = \alpha^2 E_1,\\ &R(E_2,E_1)E_1 = \alpha^2 E_2, && R(E_2,E_3)E_3 = -3\alpha^2 E_2,\\ &R(E_3,E_1)E_1 = \alpha^2 E_3, && R(E_3,E_2)E_2 = -3\alpha^2 E_3, \end{aligned}$$

and

$$R(E_1,E_2)E_3 = R(E_2,E_3)E_1 = R(E_3,E_1)E_2 = 0.$$

Then, the Ricci tensor $S$ is given by

$$\begin{aligned} &S(E_1,E_1) = 2\alpha^2, && S(E_2,E_2) = -2\alpha^2, && S(E_3,E_3) = -2\alpha^2,\\ &S(E_1,E_2) = 0, && S(E_1,E_3) = 0, && S(E_2,E_3) = 0. \end{aligned}$$

It is easy to verify that the above structure satisfies the conditions of $\alpha$–Sasakian manifold. Also, the constructed metric of the $\alpha$–Sasakian manifold is Yamabe soliton. It is seen that the scalar curvature $r = -2\alpha^2$, implies that the infinitesimal contact transformation of the flow vector field is strict.

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# Counting Commutativities in Finite Algebraic Systems

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Abstract. We examine the total possible number of commutativities in a finite algebraic system, concentrating on groups, but also examining rings and semigroups. Numerical restrictions are found and bounds for the total number of commutativities in subgroups and factor groups are derived. Finally, a curious connection with group representations is explored.

## 1. Introduction

Consider the Cayley table of a finite group $G$. For $a, b, \in G$, if $ab = ba$, we place a 1 in each of the boxes corresponding to $ab$ and $ba$. This is called a commutativity in $G$. Otherwise we put a 0 in each of these boxes, indicating a non-commutativity in $G$. If $G$ is an abelian group, there will be a 1 in each box, so we disregard this uninteresting case.

We call this matrix of 1's and 0's the commutativity chart for $G$. Here for example is the commutativity chart for $S_3$, the group of all permutations on the set $\{1, 2, 3\}$ under composition. $S_3$ is in fact the smallest non-abelian group.

| | $e$ | (123) | (132) | (12) | (13) | (23) |
|---|---|---|---|---|---|---|
| $e$ | 1 | 1 | 1 | 1 | 1 | 1 |
| (123) | 1 | 1 | 1 | 0 | 0 | 0 |
| (132) | 1 | 1 | 1 | 0 | 0 | 0 |
| (12) | 1 | 0 | 0 | 1 | 0 | 0 |
| (13) | 1 | 0 | 0 | 0 | 1 | 0 |
| (23) | 1 | 0 | 0 | 0 | 0 | 1 |

We denote by $I(G)$ the number of times that 1 appears in the commutativity chart and by $O(G)$ the number of times that 0 appears. Thus $I(S_3) = 18$ and $O(S_3) = 18$ also.

In general we see that $I(G) + O(G) = |G|^2$ and $O(G) > 0$ since we are assuming $G$ is non-abelian. Also we have $I(G) > 0$ since for

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example $xx = xx$ for all $x \in G$. One of our objectives of this note will be to discuss the possible values of $I(G)$ and $O(G)$, where $G$ is a finite non-abelian group and to investigate the values of $I(S)$ and $O(S)$ for other non-commutative algebraic systems $S$.

Since if $ab \neq ba$ then $ba \neq ab$ and $xx = xx$ for all $x$, we see that $O(G)$ is always an even number, but there are examples to show that $I(G)$ can be either even or odd. For example, $I(A_4) = 48$, where $A_4$ is the alternating group of order 12, while $I(G(21)) = 105$, where $G(21)$ is the non-abelian group of order 21. We emphasise that throughout, $G$ denotes a finite non-abelian group.

## 2. Some Elementary Results

Let us recall some facts from elementary group theory. Two elements $x$ and $y$ in $G$ are said to be conjugate if there exists $w \in G$ with $y = w^{-1}xw$. The relation of conjugacy is easily seen to be an equivalence relation on $G$, under which $G$ is partitioned into disjoint conjugacy classes. For example, in the group $S_3$, the conjugacy classes are $\{e\}, \{(123), (132)\}$ and $\{(12), (13), (23)\}$.

In general, let $G$ have exactly $k(G)$ conjugacy classes and let $Cl(x)$ be the class containing $x$. Let $C_G(x)$, the centralizer of $x$ in $G$, be the subgroup of $G$ given by $C_G(x) = \{a \in G \mid ax = xa\}$. There is a nice connection between conjugacy classes and centralizers viz. $|Cl(x)| = (G : C_G(x))$, i.e. the number of cosets of $C_G(x)$ in $G$, and both these numbers are divisors of $|G|$.

From the definition, we have that

$$I(G) = \sum_{x \in G} |C_G(x)| = \sum_{x \in G} \frac{|G|}{|Cl(x)|}$$

$$= |G| \sum_{x \in G} \frac{1}{|Cl(x)|} = |G|k(G). \text{ See [5].}$$

It follows that $O(G) = |G|^2 - I(G) = |G|(|G| - k(G))$. Thus in the case of $S_3$, since $k(S_3) = 3$, we have $I(G) = 6 \cdot 3 = 18$ and $O(G) = 6 \cdot (6 - 3) = 18$, in agreement with our previous calculations.

**Theorem 2.1.** *If $|G|$ is odd, then $k(G)$ is odd.*

*Proof.* If $|G|$ is odd, since $O(G)$ is even and $O(G) = |G|(|G| - k(G))$, we see that $|G| - k(G)$ must be even, so $k(G)$ is odd. □

We note that the converse of this result is not true; $k(S_3) = 3$, but $|S_3| = 6$.

**Theorem 2.2.** *$I(G)$ is odd if and only if $|G|$ is odd.*

*Proof.* If $|G|$ is odd then by Theorem 2.1 $k(G)$ is odd, so $I(G) = |G|k(G)$ is odd. Conversely, if $I(G)$ is odd then $|G|$ clearly must be odd. □

In fact the smallest possible odd value of $I(G) = 105 = 21 \cdot 5$, arising from $G(21)$, which is the smallest odd-order non-abelian group. We remark that Theorem 2.1, which says that if $|G|$ is odd, then $|G| - k(G) \equiv 0 \pmod 2$, can be improved upon considerably using the theory of matrix group representations. A lovely theorem of Burnside [3] states that if $|G|$ is odd, then $|G| - k(G) \equiv 0 \pmod{16}$.

Again $G(21)$ shows that this result is the best possible. Since $O(G) = |G|(|G| - k(G))$ we have

**Theorem 2.3.** *If $|G|$ is odd, then $O(G)$ is a multiple of $16|G|$.*

Again, $O(G(21)) = 336 = 16 \cdot 21$, shows that this result is the best possible.

We now investigate the possible values of $I(G)$ and $O(G)$ as $G$ ranges over all finite non-abelian groups. For a given group $G$ it is easy, if tedious, to calculate the value of $k(G)$, and for certain classes of groups, and for groups of small order, this information is readily available from a variety of sources.

In particular let $D_n$ be the dihedral group of order $2n$ $(n > 2)$ given by

$$< a, b \mid a^n = 1 = b^2; b^{-1}ab = a^{-1} >$$

Then if $n(= 2m)$ is even, we have $k(D_{2m}) = m+3$, making $I(D_{2m}) = 4m(m+3) = 4m^2 + 12m$.

If $n(= 2m+1)$ is odd, then $k(D_{2m+1}) = m+2$, so $I(D_{2m+1}) = (4m+2)(m+2) = 4m^2 + 10m + 4$.

The values of $O(D_n)$ can be found from $O(G) = |G|^2 - I(G)$.

The symmetric group $S_n$ of order $n!$ has exactly $p(n)$ conjugacy classes, where $p(n)$ is the (integer) partition function, so $I(S_n) = n!p(n)$ and $O(S_n) = n!(n! - p(n))$.

For distinct odd primes $p$ and $q$, with $p < q$ where $p|(q-1)$, there is a unique non-abelian group $G(pq)$ of order pq. Easy calculations show that $G(pq)$ has exactly $p + \frac{q-1}{p}$ conjugacy classes, so that $I(G(pq)) = q(p^2 + q - 1)$ and $O(G(pq)) = p^2q^2 - I(G) = q(q-1)(p^2-1)$.

We now present a chart with three columns. In the first column are the possible orders of a finite non-abelian group $G$. In the second and third columns we give the values of $I(G)$ and $O(G)$ for each non-abelian group of order $|G|$. Since it is known that there are only finitely many groups with a given order and also only finitely many groups with a given number of conjugacy classes ([6], [9]) , we see that there are just finitely many (maybe zero) groups with a given $I(G)$ or a given $O(G)$. Note that there may be several different groups of order $|G|$ with the same $k(G)$ and hence the same $I(G)$ and $O(G)$.

| $\|G\|$ | $I(G)$ | $O(G)$ |
|---|---|---|
| 6 | 18 | 18 |
| 8 | 40 | 24 |
| 10 | 40 | 60 |
| 12 | 48 | 96 |
| 12 | 72 | 72 |
| 14 | 70 | 126 |
| 16 | 112 | 144 |
| 16 | 160 | 96 |
| 18 | 108 | 216 |
| 18 | 162 | 162 |
| 20 | 100 | 300 |
| 20 | 160 | 240 |
| 21 | 105 | 336 |
| 22 | 154 | 330 |
| 24 | 120 | 456 |
| 24 | 168 | 408 |
| 24 | 192 | 384 |
| 24 | 216 | 360 |
| 24 | 288 | 288 |
| 24 | 360 | 216 |
| 26 | 208 | 468 |
| 27 | 297 | 432 |
| 28 | 280 | 504 |
| 30 | 270 | 630 |
| 30 | 360 | 540 |
| 30 | 450 | 450 |
| 32 | 352 | 672 |
| 32 | 448 | 576 |

| $\|G\|$ | $I(G)$ | $O(G)$ |
|---|---|---|
| 32 | 544 | 480 |
| 34 | 340 | 816 |
| 36 | 216 | 1080 |
| 36 | 324 | 972 |
| 36 | 360 | 936 |
| 36 | 432 | 864 |
| 36 | 648 | 648 |
| 38 | 418 | 1026 |
| 39 | 273 | 1248 |
| 40 | 400 | 1200 |
| 40 | 520 | 1080 |
| 40 | 640 | 960 |
| 40 | 1000 | 600 |
| 42 | 294 | 1470 |
| 42 | 420 | 1344 |
| 42 | 504 | 1260 |
| 42 | 630 | 1134 |
| 42 | 882 | 882 |
| 44 | 616 | 1320 |
| 46 | 598 | 1518 |
| 48 | 384 | 1920 |
| 48 | 480 | 1824 |
| 48 | 576 | 1728 |
| 48 | 672 | 1632 |
| 48 | 720 | 1584 |
| 48 | 768 | 1536 |
| 48 | 864 | 1440 |
| 48 | 1008 | 1296 |

| $\|G\|$ | $I(G)$ | $O(G)$ |
|---|---|---|
| 48 | 1152 | 1152 |
| 48 | 1440 | 864 |
| 50 | 700 | 1800 |
| 50 | 1000 | 1500 |
| 52 | 364 | 2340 |
| 52 | 832 | 1872 |
| 54 | 540 | 2376 |
| 54 | 810 | 2106 |
| 54 | 972 | 1944 |
| 54 | 1188 | 1728 |
| 54 | 1458 | 1458 |
| 55 | 385 | 2640 |
| 56 | 448 | 2688 |
| 56 | 952 | 2184 |
| 56 | 1120 | 2016 |
| 56 | 1960 | 1176 |
| 57 | 513 | 2736 |
| 58 | 928 | 2436 |
| 60 | 300 | 3300 |
| 60 | 540 | 3060 |
| 60 | 720 | 2880 |
| 60 | 900 | 2700 |
| 60 | 1080 | 2520 |
| 60 | 1200 | 2400 |
| 60 | 1440 | 2160 |
| 60 | 1800 | 1800 |

We note that for direct products of groups $G_1$ and $G_2$, $I(G_1 \times G_2) = I(G_1)I(G_2)$ and $k(G_1 \times G_2) = k(G_1)k(G_2)$. However, $O(S_3)O(S_3) = 18 \cdot 18 = 324 \neq 972 = O(S_3 \times S_3)$.

By [7] we have $\frac{k(G)}{|G|} \leq \frac{5}{8}$ so $I(G) \leq \frac{5}{8}|G|^2$, and $O(G) \geq \frac{3}{8}|G|^2$.

Also, by examining Cayley tables, it is clear that $I(G) \geq 3|G| - 2$, so that $O(G) \leq |G|^2 - 3|G| + 2$.

Thus, consulting the above charts, we see that the allowable values for $I(G)$ are: 18, 40, 48, 70, 72, 100, 105, 108, 112, 120, 154, 160, 162, 168, 192, 208, 216, 270, 273, 280, 288, 294, 297, 300, 324, 340, 352, 360, 364, 384, 385, 400, 418, 432,. . .

Similarly the allowable values for $O(G)$ are: 18, 24, 60, 72, 96, 126, 144, 162, 216, 240, 288, 300, 330, 336, 360, 384, 408, 432, 450, 456, 468, 480, 504, 540, 576, 600, 630, 648, 672,. . .

We mention that the function $|G| - k(G)$ is examined in considerable detail in [1] . Also, one can show that $I(G) = O(G)$ if and only if $G/Z(G) = S_3$, where $Z(G)$ is the centre of $G$.

## 3. Subgroups and Factor Groups

Gallagher [4] gives elementary proofs of the following results for all finite groups $G$, where $H$ is a subgroup of $G$ and $N$ is a normal subgroup of $G$.

(i) $k(H) < (G : H)k(G)$, for $H \neq G$;
(ii) $k(G) \leq (G : H)k(H)$;
(iii) $k(G) \leq k(G/N)k(N)$.

In our notation, these results immediately become

**Theorem 3.1.** *(i)* $I(H) < I(G)$ *if* $H \neq G$*;*
*(ii)* $I(G) \leq (G : H)^2 I(H)$*;*
*(iii)* $I(G/N) \geq I(G)/I(N)$.

## 4. Other Algebraic Systems

Let $S = \{a, b\}$ be a set of cardinality 2. Define a binary operation $*$ on $S$ as follows

| $*$ | $a$ | $b$ |
|---|---|---|
| $a$ | $a$ | $b$ |
| $b$ | $a$ | $b$ |

Easy calculations show that $S$ is a non-commutative semigroup with $I(S) = 2 = O(S)$, so the sequences of allowable value of $I(S)$ and $O(S)$ for semigroups are different from those of $I(G)$ and $O(G)$ for groups.

The reader is invited to determine the sequences of allowable values of $I(S)$ and $O(S)$ for non-commutative semigroups.

Moving on to rings, consider the following set of $2 \times 2$ matrices over $\mathbb{Z}_2$ under matrix addition and multiplication mod 2:

$$R = \{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)\}$$

It is easy to see that $\{R, +, \cdot\}$ is a non-commutative ring of order 4. The commutativity chart for $\{R, \cdot\}$ looks as follows:

| | $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ |
|---|---|---|---|---|
| $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ | 1 | 1 | 1 | 1 |
| $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ | 1 | 1 | 0 | 0 |
| $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ | 1 | 0 | 1 | 0 |
| $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ | 1 | 0 | 0 | 1 |

Thus $I(R) = 10$ and $O(R) = 6$. This single example shows that the sequences of allowable values of $I(R)$ and $O(R)$ for finite rings are different from those for finite groups.

Again the reader is invited to investigate this problem for other algebraic systems such as near–rings, loops, quasigroups etc.

We remark that if $S$ is a set with $|S| = n$ we can always choose closed binary operations $*$ and $\circ$ on $S$ such that $I(S, *) = n$ $(n > 1)$, and $O(S, \circ) = 2n$ ($n$ arbitrary).

For example, if $S = \{a, b, c\}$ define $*$ by

| $*$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $a$ | $c$ |
| $b$ | $b$ | $b$ | $b$ |
| $c$ | $a$ | $c$ | $c$ |

to achieve $I(S, *) = 3$ and similarly for the general case.

| $\circ$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ |
| $b$ | $b$ | $a$ | $a$ |
| $c$ | $b$ | $b$ | $c$ |

Also in the second example $O(S, \circ) = 6$ and similarly for the general case.

## 5. A Connection with Matrix Representations of Groups

There is a surprising connection between $I(G)$ and matrix representations of $G$. For definitions we refer the interested reader to [5].

Let $d_i, 1 \le i \le k$, be the degrees of the irreducible complex matrix representations of a finite group $G$ i.e. the sizes of the square matrices involved. There are $k(G)$ of these where $G$ has $k(G)$ conjugacy classes.

$$\text{Let } T(G) = \sum_{i=1}^{k(G)} d_i.$$

[For example, for $S_3, (d_1, d_2, d_3) = (1, 1, 2)$ so $T(S_3) = 4$.]

Using the Cauchy-Schwarz inequality on $(1, 1, 1, \ldots, 1)$ and $(d_1, d_2, d_3, \ldots, d_k)$ as in [8] , and remembering that $\sum_{i=1}^{k} d_i^2 = |G|$, we find that

$$(T(G))^2 < k(G)|G| = I(G). \text{ (G non–abelian)}$$

Let us see how this inequality looks for some specific groups of small order.

[We use the notation $Q_n$ for the dicyclic group of order $4n$ for $n > 1$ where $Q_n = \langle a, b | a^{2n} = 1; b^2 = a^n, b^{-1}ab = a^{-1} \rangle$].

| **Group** | $\boldsymbol{(T(G))^2}$ | $\boldsymbol{I(G)}$ | |
|---|---|---|---|
| $S_3$ | 16 | 18 | |
| $D_4$ | 36 | 40 | |
| $Q_2$ | 36 | 40 | (quaternion group) |
| $D_5$ | 36 | 40 | |
| $D_6$ | 64 | 72 | |
| $Q_3$ | 64 | 72 | |
| $A_4$ | 36 | 48 | |
| $D_7$ | 64 | 70 | |
| $S_4$ | 100 | 120 | |

When we write $T(G) < \sqrt{I(G)}$ in a particular case such as $D_4$, we get $T_4 < \sqrt{I(D_4)} = \sqrt{40} = 6.3245$. Now $T(D_4)$ is an integer so $T(D_4) \leq 6$ and 6 is actually the correct answer!

Similarly in the case of $S_4$, we get $T(S_4) < \sqrt{120} = 10.95445$. Again $T(S_4)$ is an integer, so $T(S_4) \leq 10$ which gives the correct value of $T(S_4) = 10$.

It is remarkable that such a basic function as $I(G)$, whose values can be read from the Cayley table, can be used to find information about $T(G)$, which would appear to be a much more advanced group theoretic concept.

## 6. Analogues of $I(G)$ and $O(G)$

There are so many analogies between $k(G)$ and $T(G)$ (as defined in section 5) that we make the following definitions:

For a finite non–abelian group $G$, let $N(G) = |G|T(G)$ and $M(G) = |G|(|G| - T(G))$.

It is not immediately clear what the interpretations of $N(G)$ and $M(G)$ are, but these functions have many properties analogous to $I(G)$ and $O(G)$. To save space we state results only, but methods of proof are very similar to those for results concerning $I(G)$ and $O(G)$. We remark that the properties of $|G| - T(G)$ are examined in some detail in [2] .

**Theorem 6.1.** *$I(G) < N(G)$ and $O(G) > M(G)$.*

**Theorem 6.2.** *There are only finitely many groups $G$ (maybe zero) with a given $N(G)$ or a given $M(G)$.*

**Theorem 6.3.** *$N(G)$ is odd if and only if $|G|$ is odd.*

**Theorem 6.4.** *If $|G|$ is odd, $M(G)$ is a multiple of $4|G|$.*

**Theorem 6.5.** *If $H$ is a proper subgroup of $G$, then $N(H) < N(G)$.*

**Theorem 6.6.** *$M(G)$ is always even.*

**Theorem 6.7.** *$N(G) < |G|^{\frac{3}{2}}(k(G))^{\frac{1}{2}}$.*

**Theorem 6.8.** *$N(G_1 \times G_2) = N(G_1) \cdot N(G_2)$.*

**Theorem 6.9.** *For the non–abelian group $G(pq)$, we have $N(G) = pq(p+q-1)$ and $M(G) = pq(p-1)(q-1)$, where $p$ and $q$ are distinct odd primes.*

**Theorem 6.10.** *$N(G) \leq \frac{3}{4}|G|^2$ and $M(G) \geq \frac{1}{4}|G|^2$.*

Finally, we give a chart of values of $N(G)$ and $M(G)$ for non–abelian groups $G$ of small order which leads to information about the sequences of allowable values of $N(G)$ and $M(G)$.

| $\|G\|$ | $N(G)$ | $M(G)$ |
|---|---|---|
| 6 | 24 | 12 |
| 8 | 48 | 16 |
| 10 | 60 | 40 |
| 12 | 72 | 72 |
| 12 | 96 | 48 |
| 14 | 112 | 84 |
| 16 | 120 | 136 |
| 16 | 192 | 64 |
| 18 | 120 | 204 |
| 20 | 160 | 240 |
| 20 | 240 | 160 |
| 21 | 189 | 252 |

| $\|G\|$ | $N(G)$ | $M(G)$ |
|---|---|---|
| 22 | 264 | 220 |
| 24 | 240 | 336 |
| 24 | 288 | 288 |
| 24 | 336 | 240 |
| 24 | 384 | 292 |
| 24 | 432 | 144 |
| 26 | 364 | 312 |
| 27 | 405 | 324 |
| 28 | 448 | 336 |
| 30 | 480 | 420 |
| 30 | 540 | 360 |
| 30 | 600 | 300 |

The sequence of allowable values of $N(G)$ thus begins $24, 48, 60, 72, 96, 112, 120, 160, 189, 192, 240, 264, 288, \ldots$

The sequence of allowable values of $M(G)$ thus begins $12, 16, 40, 48, 64, 72, 84, 136, 144, \ldots$

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# Irish Dancing Groups

JILL E. TYSSE

Abstract. There is so much symmetry involved in the arrangements of the dancers in traditional Irish céilí dancing that it seems natural to try to describe some of these dances using group theory. We describe a connection between progressive dances such as the Siege of Ennis and the calculation of the cosets of the dihedral group $D_n$ in the symmetric group $S_n$ and offer suggestions for using this application as a classroom activity. This article may be of interest to students taking a first course in group theory and their professors.

## 1. Introduction

Did you know that every time you dance the Siege of Ennis at a wedding, you are calculating cosets of a dihedral group as a subgroup of a symmetric group? We describe how to do this and offer suggestions for ways to include this as an engaging activity in a group theory course. Apart from the obvious purpose of illustrating the concept of group cosets, this example can be used to illustrate many of the standard theorems concerning cosets, as well as Lagrange's Theorem. It aids in a review of the symmetric groups and the dihedral groups and can serve as a starting point for the discussion of normal subgroups and group presentations.

## 2. Preliminaries

Two families of groups we will see in this paper are the symmetric groups and the dihedral groups. We let $S_n$ denote the symmetric group on $n$ letters. It has a presentation with generators $s_i$, $1 \leq i \leq$

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$n-1$ and relations

$$s_i^2 = 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ s_i s_j = s_j s_i \text{ for } |i-j| > 1.$$

The generators $s_i$ can be identified with the transpositions $(i, i+1)$ for each $1 \leq i \leq n-1$. We let $D_n$ denote the dihedral group of order $2n$, the symmetry group of a regular $n$-gon. It has a presentation with generators $a$ and $b$ together with the relations

$$a^2 = b^2 = (ab)^n = 1.$$

We can identify the elements of $D_n$ with permutations of the vertices of a regular $n$-gon, written using cycle notation. We will compose cycles from right to left so that, for example, $(123)(13) = (23)$ and not $(12)$.

## 3. Cosets and the Siege of Ennis

To dance the Siege of Ennis, dancers line up in teams of four down the dance hall. The teams stand side by side holding hands and facing another team as in Figure 1.

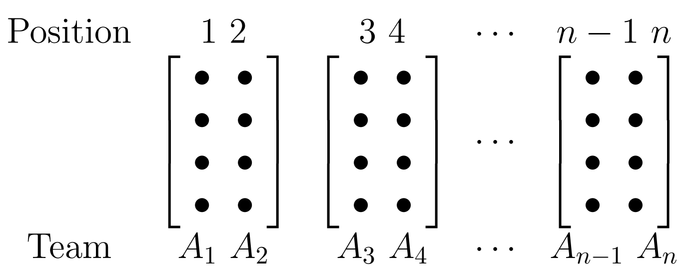


Figure 1. Siege of Ennis

See [7] for a list of the dance moves for the Siege of Ennis or [8] for a video of the dance. Here, we are not interested in the dance moves themselves, rather we will look solely at the progression of the dancers from one round of the dance to the next.

We will begin by considering $n = 4$ teams and in Section 4 we will generalize to $n$ teams. Suppose we have four teams, $A$, $B$, $C$, and $D$, each consisting of four people, dancing the Siege of Ennis. In the first round of the dance, team $A$ is dancing with team $B$ and team $C$ is dancing with team $D$, as illustrated below in Figure 2. We number the positions of the teams 1 to 4 from left to right.

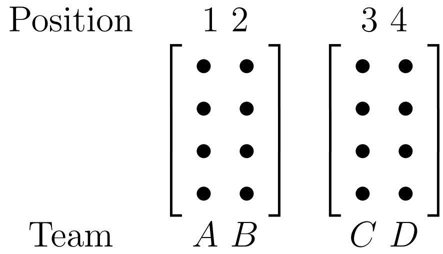


FIGURE 2. Round 1

At the end of the first round of the dance, (as you will remember from all the weddings you've been to), teams $A$ and $B$ switch places with one another: the dancers from team $A$ lift their arms up to form arches, and the dancers from team $B$ proceed through those arches to switch places with team $A$. Similarly, teams $C$ and $D$ switch places. Then in the second round of the dance, teams $A$ and $D$ dance with one other while teams $B$ and $C$ wait out the second round, as in the left-hand illustration of Figure 3.

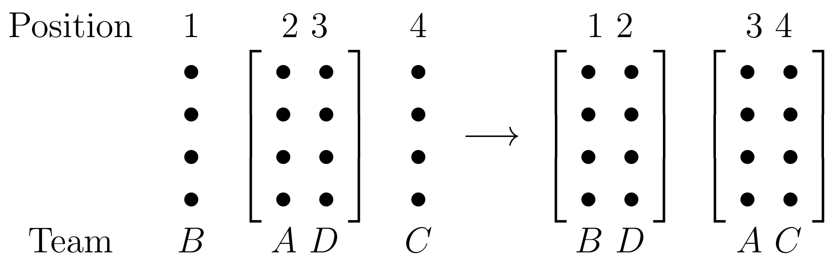


FIGURE 3. Round 2 to Round 3

When the second round of the dance has finished, teams $A$ and $D$ switch places again via arches and the dance starts all over again for round 3, now with teams $B$ and $D$ dancing together, and $A$ and $C$ dancing together, as in the right-hand side of Figure 3.

From here, the progressions continue similarly, with the outer two pairs of teams switching places according to the permutation $a = (12)(34)$, followed by the inner two teams switching places according to $b = (23)$, and so on. In the second column of Figure 4 we list the permutations of the teams at each round of the dance, stopping right before the teams arrive back to their starting positions whence the pattern would repeat itself if the dance were to continue.

We might, at this point, wonder what happens if we start again from a permutation of the teams we have not yet encountered, say $BACD$. And then, once we have figured out the possible permutations that result from this starting position, we could try a third starting position, say $CBAD$. The resulting permutations are listed in Figure 4.

| Round | Prog. 1 | Prog. 2 | Prog. 3 |
|---|---|---|---|
| 1 | $ABCD$ | $BACD$ | $CBAD$ |
| 2 | $BADC$ | $ABDC$ | $BCDA$ |
| 3 | $BDAC$ | $ADBC$ | $BDCA$ |
| 4 | $DBCA$ | $DACB$ | $DBAC$ |
| 5 | $DCBA$ | $DCAB$ | $DABC$ |
| 6 | $CDAB$ | $CDBA$ | $ADCB$ |
| 7 | $CADB$ | $CBDA$ | $ACDB$ |
| 8 | $ACBD$ | $BCAD$ | $CABD$ |

FIGURE 4. All permutations

Thus we have listed all $24 = 4!$ possible permutations of the four teams. Each permutation corresponds to an element of the symmetric group $S_4$. We take $ABCD$ to be the identity position. The list of permutations in the second column of Figure 4 is generated by the permutations $a = (12)(34)$ and $b = (23)$. Since their product $ab = (1243)$ is of order 4 this second column corresponds to the group $D_4$ following the presentation of the dihedral group given in Section 2. Since $BACD$ is obtained from $ABCD$ by applying the permutation (12), the third column corresponds to the right coset $D_4(12)$, of $D_4$ in $S_4$. And since $CBAD$ is obtained from the identity position by applying the permutation (13), the fourth column is the right coset $D_4(13)$ of $D_4$ in $S_4$. See Section 5 for suggestions on how to use these observations in a classroom setting.

## 4. GENERALIZATION TO $n$ TEAMS

Suppose now, for the more general case, that we have $n$ teams of dancers dancing the Siege of Ennis. If $n = 2m$ is even, we will start with the situation in Figure 5.

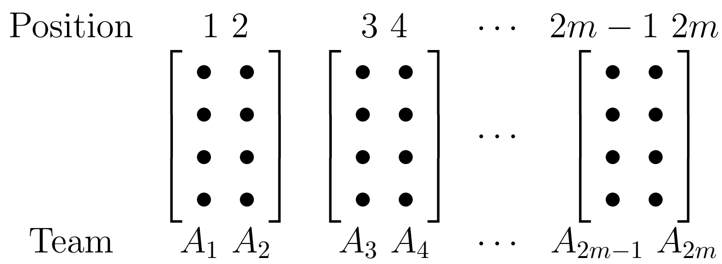


Figure 5. $n$ even

And if $n = 2m + 1$ is odd, we will start with the situation in Figure 6.

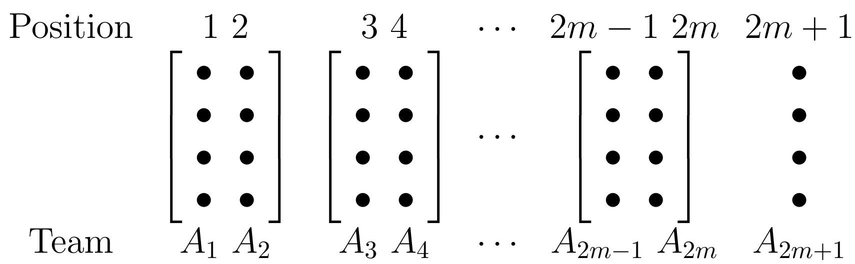


Figure 6. $n$ odd

As before, the dancers progress from one round of the dance to the next via arches, exchanging places with the team facing them. The permutations

$$r = \prod_{i=1}^{m} (2i - 1, 2i) \text{ and } s = \prod_{j=1}^{m-1} (2j, 2j + 1)$$

generate the list of possible team permutations from the starting position when $n = 2m$ is even and the permutations

$$t = \prod_{i=1}^{m} (2i - 1, 2i) \text{ and } u = \prod_{j=1}^{m} (2j, 2j + 1)$$

generate the list of permutations in the case where $n = 2m + 1$ is odd.

We can see by a direct calculation that the products $rs$ and $tu$ can each be written as cycles of length $n$. We get

$$rs = (2, 4, 6, \ldots 2m - 2, 2m, 2m - 1, 2m - 3, \ldots, 5, 3, 1)$$

and

$$tu = (2, 4, 6, \ldots 2m, 2m + 1, 2m - 1, \ldots, 5, 3, 1).$$

Therefore, recalling the presentation of $D_n$ from Section 2, the elements $r$ and $s$, being two elements of order 2 such that their product $rs$ has order $n = 2m$ generate the dihedral group $D_n$ in the case that $n$ is even. And the elements $t$ and $u$, being two elements of order 2 such that their product, $tu$ has order $n = 2m + 1$ generate the dihedral group $D_n$ in the case that $n$ is odd.
As in the $n = 4$ case, we have an algorithm for calculating all of the right cosets of $D_n$ in $S_n$ via the Siege of Ennis. We choose a new starting position - one that we have not already encountered - and then apply the generators repeatedly to this new starting position. We then repeat this procedure, choosing another starting permutation we have not yet seen. We continue until we have listed all $n!$ permutations of $S_n$.

## 5. Classroom notes

We offer some suggestions for using these ideas as part of a classroom activity once the symmetric and dihedral groups have been studied and right after the notion of a coset has been introduced. The $n = 4$ example is easy to work through in class, either having 8 students demonstrate the progressions (with 2 students in each team, for simplicity), or involving the whole class if that is feasible, perhaps for some extra credit! As the students are producing the permutations that are listed in Figure 4, one may wish to have the the students "translate" them into cycle notation – for example, $CDBA$ would become (1423). Here are some exercises that could be asked of the students at this point:

(1) Are there any repeats in your table of permutations?
(2) All 24 permutations form which group?
(3) Taking $ABCD$ to be the identity permutation, prove that the set of permutations in the second column of Figure 4 is isomorphic to the group $D_4$. (Either set up a mapping relating the permutations in the second column to the permutations of the vertices of a square, or tackle this via generators and relations.)
(4) Neither the third nor the fourth columns are subgroups of $S_4$. Why not?
(5) Show that the third and the fourth columns are right cosets of $D_4$ in $S_4$. Are they also left cosets of $D_4$ in $S_4$?

(6) Consider the way we produced the permutations in the second column of Figure 4. Identify the generators of $D_4$ that are being used here.

It may be helpful for the students to keep this example in mind when proving some of the standard results about cosets: any element of a coset can be used as its representative, cosets are either identical or disjoint, cosets are all of the same size and partition the group, as well as Lagrange's Theorem. In fact, before these theorems are proved formally, students could be asked to explain why each of these statements holds in the context of the Siege of Ennis, perhaps as homework exercises following this activity. In addition, Exercise (5) above could be used as a set-up for discussing normal subgroups, and Exercises (3) and (6) could be used to introduce group presentations.

## 6. Further Reading

This is but one example of group theory in Irish dance and there are many more if we care to look for them. The observation of group theory in dance is not new. For example, [1], [2], [5], and [6] all discuss the symmetries of contra dancing. And at least one other author has written about group theory in Irish dance: Andrea Hawksley, in [4], discusses the appearance of braid groups in the céilí dance *The Waves of Tory*.

## References


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[5] I. Peterson: *Contra dances, matrices, and groups*, `https://www.sciencenews.org/article/contra-dances-matrices-and-groups` (accessed 16-5-2016).

[6] C. von Renesse with V. Ecke, J.F. Fleron, and P.K. Hotchkiss: *Discovering the Art of Mathematics: Dance*, `https://www.artofmathematics.org/books/dance` (accessed 16-5-2016).

[7] Siege of Ennis dance instructions: `https://danceminder.com/dance/show/siegen` (accessed 16-5-2016).

[8] Video of the Siege of Ennis: `https://www.youtube.com/watch?v=AWmgwJkNPCQ` (accessed 14-5-2016).

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# PROBLEMS

IAN SHORT

## Problems

The first problem was contributed by Finbarr Holland of University College Cork.

**Problem 77.1.** Suppose that $f : [0,1] \to \mathbb{R}$ is a convex function and $\int_0^1 f(t)\,dt = 0$. Prove that

$$\int_0^1 t(1-t)f(t)\,dt \leqslant 0,$$

with equality if and only if $f(t) = a(2t-1)$ for some real number $a$.

I learned the second problem from Tony Barnard of Kings College London some years ago.

**Problem 77.2.** Each member of a group of $n$ people writes his or her name on a slip of paper, and places the slip in a hat. One by one the members of the group then draw a slip from the hat, without looking. What is the probability that they all end up with a different person's name?

For the third problem, interpret 'evaluate' to mean that you should express the given quantity in a simple formula in terms of integers and known constants using standard functions and the usual operations of arithmetic.

**Problem 76.3.** Evaluate

$$1+\cfrac{1^2}{1+\cfrac{2^2}{1+\cfrac{3^2}{1+\cdots}}}.$$

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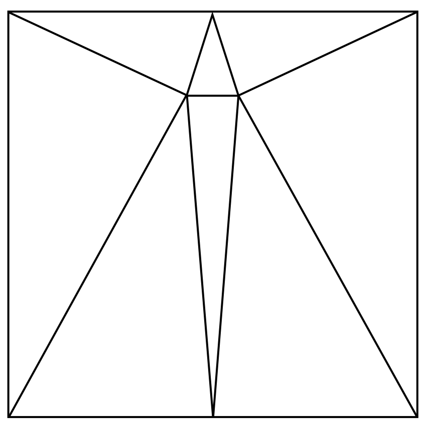
## Solutions

Here are solutions to the problems from *Bulletin* Number 75.

The well-known picture in the solution to the first problem was supplied by both Ángel Plaza and the North Kildare Mathematics Problem Club. The argument that there must be at least 8 triangles was supplied by Henry Ricardo of the New York Math Circle, USA. It was also supplied as a sketch by the North Kildare Mathematics Problem Club. The club point out that this proof can be found in work of Charles Cassidy and Graham Lord (J. Rec. Math. 13, 1980/81). Cassidy and Lord attribute the problem of finding a tessellation using 8 triangles to Martin Gardner (Sci. Amer. 202, 1960), and refer to an earlier proof that 8 is minimal by H. Lindgren (Austral. Math. Teacher 18, 1962).

*Problem 75.1*. What is the least positive integer $n$ for which a square can be tessellated by $n$ acute-angled triangles?

*Solution 75.1.* The figure below shows that a square can be tessellated by 8 acute-angled triangles.



The following sequence of observations proves (in brief) that any tessellation comprises at least 8 triangles.

(i) Any vertex interior to the square must be incident to at least 5 triangles because the sum of 4 acute angles is less than $2\pi$.

(ii) Any vertex on a side of the square must be incident to at least 3 triangles as the sum of 2 acute angles is less than $\pi$.

(iii) Each corner of the square must be incident to at least 2 triangles because $\pi/2$ is not acute.

(iv) Suppose there are no interior vertices. Choose a corner of the square. By (iii), there is an edge from this corner to a side of the square (or possibly the opposite corner). This edge 'traps' another corner, preventing it from being connected to a side of the square (or its opposite corner) by an edge, in contradiction with (iii).

(v) Suppose there is a unique interior vertex $u$. By a similar argument to (iv), $u$ is connected by 4 edges to each of the 4 corners of the square. By (i), there must be another edge of the tessellation incident to $u$; by assumption, this edge must be incident to a vertex $v$ on a side $\ell$ of the square. By (ii), there must be another edge of the tessellation that is incident to $v$ and another vertex $w$, where $w$ does not lie on $\ell$ or one of the corners incident to $\ell$. The vertex $w$ cannot lie in the interior of the square, so it must lie on one of the remaining sides or corners of the square. However, this is impossible without two edges intersecting.

(vi) By (v), there are at least two interior vertices. Each interior vertex must be incident to at least 5 triangles. Any 2 interior vertices can be incident to at most 2 common triangles, as they share at most 1 edge. Therefore, the tessellation has at least $5 + 5 - 2 = 8$ triangles. □

The second problem was solved by the North Kildare Mathematics Problem Club and the proposer, Finbarr Holland.

*Problem 75.2.* Let

$$s_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}, \quad n = 0, 1, 2, \dots .$$

Suppose $0 < \alpha < 1$. Prove that when $n \geqslant 1$,

$$e^x \leqslant \frac{s_n(x) - \alpha x s_{n-1}(x)}{1 - \alpha x} \quad \text{for all } x \in [0, 1/\alpha)$$

if and only if $\alpha \geqslant 1/(n + 1)$.

*Solution 75.2.* Suppose that the first inequality involving $e^x$ is true. Rearranging this inequality we obtain

$$\frac{e^x - s_n(x)}{x(e^x - s_{n-1}(x))} \leqslant \alpha,$$

for $x \in (0, 1/\alpha)$. Now

$$\lim_{x \to 0^+} \frac{e^x - s_n(x)}{x^{n+1}} = \frac{1}{(n + 1)!} \quad \text{and} \quad \lim_{x \to 0^+} \frac{x(e^x - s_{n-1}(x))}{x^{n+1}} = \frac{1}{n!},$$

so

$$\lim_{x\to 0^+} \frac{e^x - s_n(x)}{x(e^x - s_{n-1}(x))} = \frac{1}{n+1}.$$

Therefore $\alpha \geqslant 1/(n+1)$.

Conversely, suppose that $\alpha \geqslant 1/(n+1)$. Then, if $m \geqslant n$, we have

$$\frac{1}{m!} \leqslant \frac{1}{n!(n+1)^{m-n}} \leqslant \frac{\alpha^{m-n}}{n!}.$$

Hence

$$\begin{aligned}\frac{s_n(x) - \alpha x s_{n-1}(x)}{1-\alpha x} &= s_{n-1}(x) + \frac{x^n}{n!(1-\alpha x)}\\ &= s_{n-1}(x) + \frac{1}{n!}\sum_{k=0}^{\infty} \alpha^k x^{k+n}\\ &\geqslant s_{n-1}(x) + \sum_{k=0}^{\infty} \frac{1}{(n+k)!}x^{k+n}\\ &= e^x,\end{aligned}$$

for $x \in [0, 1/\alpha)$, with equality if and only if $x = 0$. □

The third problem was solved by Ángel Plaza, Eugene Gath of the University of Limerick, Henry Ricardo, and the North Kildare Mathematics Problem Club. We present the solution of Henry Ricardo, which was similar to some of the others.

*Problem 75.3***.** Given positive real numbers $a$, $b$, and $c$, prove that

$$a + b + c \leqslant \sqrt[3]{abc}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right).$$

*Solution 75.3.* Using the AM–GM inequality, we have

$$\begin{aligned}\frac{a+b+c}{\sqrt[3]{abc}} &= \sqrt[3]{\frac{a^2}{bc}} + \sqrt[3]{\frac{b^2}{ca}} + \sqrt[3]{\frac{c^2}{ab}}\\ &= \sqrt[3]{\frac{a\cdot a\cdot b}{b\cdot b\cdot c}} + \sqrt[3]{\frac{b\cdot b\cdot c}{c\cdot c\cdot a}} + \sqrt[3]{\frac{c\cdot c\cdot a}{a\cdot a\cdot b}}\\ &\leqslant \frac{1}{3}\left(\frac{a}{b} + \frac{a}{b} + \frac{b}{c}\right) + \frac{1}{3}\left(\frac{b}{c} + \frac{b}{c} + \frac{c}{a}\right) + \frac{1}{3}\left(\frac{c}{a} + \frac{c}{a} + \frac{a}{b}\right)\\ &= \frac{a}{b} + \frac{b}{c} + \frac{c}{a},\end{aligned}$$

and the desired inequality follows immediately. □

We invite readers to submit problems and solutions. Please email submissions to imsproblems@gmail.com in any format (we prefer LaTeX). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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