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MACCOOL'S SECOND PROOF OF MORLEY'S MIRACLE

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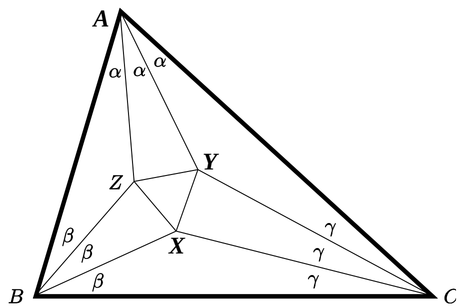
In memory of Kenneth Beales and Trevor West

Abstract. Here is a traditional proof of Morley's miracle that is unrivalled for brevity and simplicity. It stems from a sadly neglected mathematical gem published in 1914.

1. Introduction

That the triangle XYZ in the figure below is always equilateral is formally known as *Morley's trisector theorem* and informally as *Morley's miracle*. Its modern discovery dates back to 1899 and since then it has been proved many times by a wide variety of methods. The website [1] tracks developments and plays host to roughly twenty proofs including MacCool's original effort.



However as [3] explained, the proof there was based entirely on straight line geometry and similar triangles. It opined that MacCool's second notebook which was marked "Advanced" and contained diagrams of circles might hold an alternative proof. And so indeed it has proved, although it has taken me a very long time to decipher the Ogham. So whilst I have yet to find any evidence that MacCool was familiar with Pythagoras, the result that we know today as the inscribed angle theorem [Euclid: Book 3, Prop 22] does

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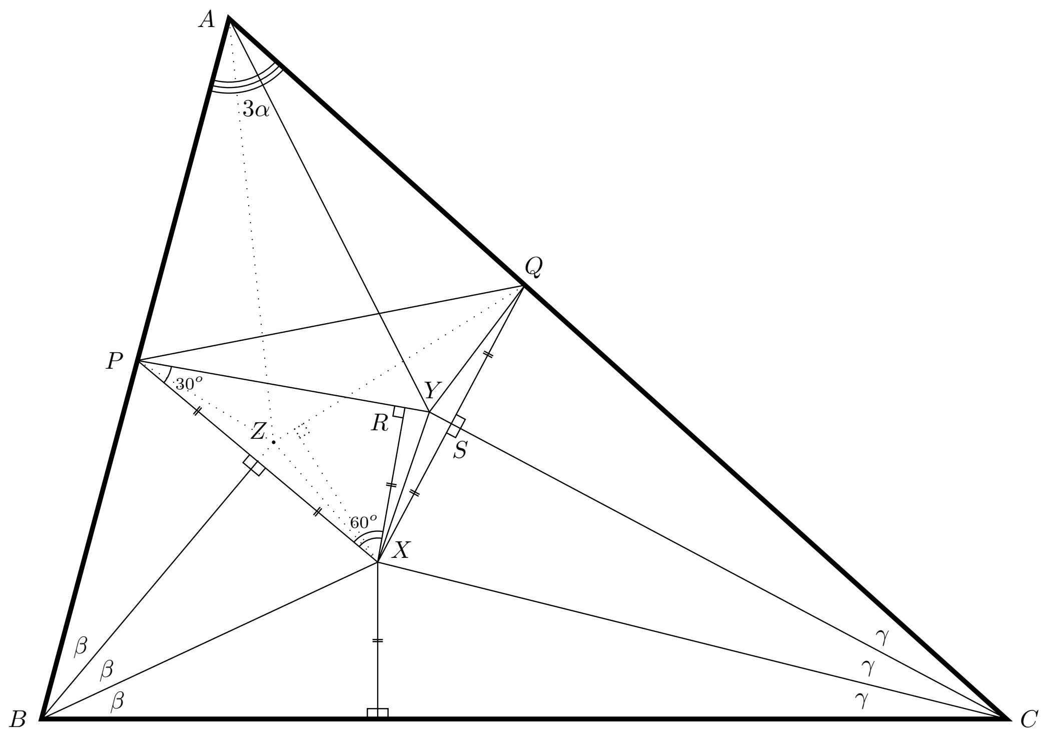
indeed appear in his “Advanced” notebook, and soon afterwards comes the second proof of Morley’s theorem. This is even shorter and easier than his “Basic” one, and completely debunks the urban myth that all purely geometric proofs must necessarily be longer and more complex than the “backward” ones.

2. Proof

His proof runs as follows. In any triangle ABC let X be the Morley vertex adjacent to BC. First construct the points P and Q on AB and AC respectively such that $|BP| = |BX|$ and $|CQ| = |CX|$. Then construct the right-angled triangle PRX with hypotenuse PX and $\angle XPR = 30^\circ$ as shown below. The six marked segments will all have equal length. Produce PR and the trisector CS to meet in Y. Note that the three right-angled triangles, ΔRXY and ΔSXY and ΔSQY, (which MacCool calls *wedges*) have equal hypotenuses and an equal (marked) side therefore they are congruent. Evidently $\alpha + \beta + \gamma = 60^\circ$.



Now $\angle QXP = 360^\circ - 2(90^\circ - \beta) - 2(90^\circ - \gamma) = 120^\circ - 2\alpha$. So $\angle YXR = \angle SXY = \angle YQS = \frac{1}{2}(\angle QXP - 60^\circ) = 30^\circ - \alpha$. As ΔPQX is isosceles its base angles $\angle XPQ$ and $\angle PQX$ are both $30^\circ + \alpha$ so $\angle YPQ = \alpha$ and $\angle PQY = (30^\circ + \alpha) - (30^\circ - \alpha) = 2\alpha$.

Therefore $\angle QYP = 180° - 3\alpha$. And now for the advanced bit. Finn spots this is supplementary to $\angle BAC$ making $APYQ$ a cyclic quadrilateral. Consequently $\angle YAQ = \angle YPQ = \alpha$ which fixes Y as the Morley vertex adjacent to AC. Next he performs a similar construction (shown in outline) starting from a right-angled triangle on hypotenuse QX and angles 30° and 60° at Q and X respectively. This generates three more wedges which are clearly congruent to the first three, plus the Morley vertex Z adjacent to AB. In particular $|XY| = |XZ|$ and $\angle YXZ = 60°$ from which he deduces that ΔXYZ is equilateral. ∎

3. Conclusion

After scanning numerous proofs the only "modern" one I've seen that is remotely like this is given in [2] and attributed to W. E. Philip. William Edward Philip was Third Wrangler at Cambridge in 1894, but despite many references to [2] in the literature the beauty of his proof seems to have been strangely overlooked. Indeed [2] also contains a version of Leon Bankoff's 1962 trigonometric proof, long regarded as the easiest non-backward approach to the theorem. As years passed without anyone finding a short, simple, non-backward geometric proof a mistaken belief has proliferated that no such proof exists. So, as the centenary of its publication approaches, the time seems ripe to call attention to [2] and bring it back centre stage.

References

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